

The free energy of the quantum Heisenberg
ferromagnet:
validity of the spin wave approximation

Alessandro Giuliani

Based on joint work with
M. Correggi and R. Seiringer

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- 1 Introduction: continuous symmetry breaking and spin waves
- 2 Main results: free energy at low temperatures
- 3 Sketch of the proof: upper and lower bounds

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General question: rigorous understanding of the phenomenon of **spontaneous breaking** of a **continuous symmetry**.

Easier case: **abelian continuous symmetry**.

Several rigorous results based on:

- reflection positivity,
- vortex loop representation
- cluster and spin-wave expansions,

by Fröhlich-Simon-Spencer, Dyson-Lieb-Simon, Bricmont-Fontaine-Lebowitz-Lieb-Spencer, Fröhlich-Spencer, Kennedy-King, ...

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- classical Heisenberg (Fröhlich-Simon-Spencer by RP)
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The simplest quantum model for the spontaneous symmetry breaking of a continuous symmetry:

$$H_\Lambda := \sum_{\langle x,y \rangle \subset \Lambda} (S^2 - \vec{S}_x \cdot \vec{S}_y)$$

where:

- Λ is a cubic subset of \mathbb{Z}^3 with (say) periodic b.c.
- $\vec{S}_x = (S_x^1, S_x^2, S_x^3)$ and S_x^i are the generators of a $(2S + 1)$ -dim representation of $SU(2)$, with $S = \frac{1}{2}, 1, \frac{3}{2}, \dots$:

$$[S_x^i, S_y^j] = i\epsilon_{ijk} S_x^k \delta_{x,y}$$

- The energy is normalized s.t. $\inf \text{spec}(H_\Lambda) = 0$.

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One special ground state is

$$|\Omega\rangle := \otimes_{x \in \Lambda} |S_x^3 = -S\rangle$$

All the other ground states have the form

$$(S_T^+)^n |\Omega\rangle, \quad n = 1, \dots, 2S|\Lambda|$$

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A special class of excited states (**spin waves**) is obtained by raising a spin in a coherent way:

$$|1_k\rangle := \frac{1}{\sqrt{2S|\Lambda|}} \sum_{x \in \Lambda} e^{ikx} S_x^+ |\Omega\rangle \equiv \frac{1}{\sqrt{2S}} \hat{S}_k^+ |\Omega\rangle$$

where $k \in \frac{2\pi}{L}\mathbb{Z}^3$. They are such that

$$H_\Lambda |1_k\rangle = S\epsilon(k) |1_k\rangle$$

where $\epsilon(k) = 2 \sum_{i=1}^3 (1 - \cos k_i)$.

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They can be looked for *in the vicinity* of

$$|\{n_k\}\rangle = \prod_k (2S)^{-n_k/2} \frac{(\hat{S}_k^+)^{n_k}}{\sqrt{n_k!}} |\Omega\rangle$$

If $N = \sum_k n_k > 1$, these are not eigenstates.

They are neither normalized nor orthogonal.

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In 3D, it predicts

$$f(\beta) \simeq \frac{1}{\beta} \int \frac{d^3k}{(2\pi)^3} \log(1 - e^{-\beta S \epsilon(k)})$$
$$m(\beta) \simeq S - \int \frac{d^3k}{(2\pi)^3} \frac{1}{e^{\beta S \epsilon(k)} - 1}$$

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$$f(\beta) \underset{\beta \rightarrow \infty}{\simeq} \beta^{-5/2} S^{-3/2} \int \frac{d^3 k}{(2\pi)^3} \log(1 - e^{-k^2})$$

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$$S_x^+ = \sqrt{2S} a_x^+ \sqrt{1 - \frac{a_x^+ a_x}{2S}}, \quad S_x^3 = a_x^+ a_x - S,$$

where $[a_x, a_y^+] = \delta_{x,y}$ are **bosonic operators**.

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In the bosonic language

$$\begin{aligned}
 H_{\Lambda} &= S \sum_{\langle x,y \rangle} \left(-a_x^+ \sqrt{1 - \frac{n_x}{2S}} \sqrt{1 - \frac{n_y}{2S}} a_y \right. \\
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 &\equiv S \sum_{\langle x,y \rangle} (a_x^+ - a_y^+) (a_x - a_y) - K \equiv T - K
 \end{aligned}$$

The spin wave approximation consists in neglecting K and the on-site hard-core constraint.

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$$H_\Lambda = S \sum_{\langle x,y \rangle} (a_x^+ - a_y^+)(a_x - a_y) - K$$

For large S , the interaction K is of relative size $O(1/S)$ as compared to the hopping term.

Easier case: $S \rightarrow \infty$ with βS constant (CG 2012)

[The classical limit is $S \rightarrow \infty$ with βS^2 constant (Lieb 1973).

See also Conlon-Solovej (1990-1991).]

Harder case: fixed S , say $S = 1/2$. So far, not even a sharp upper bound on the free energy was known. Rigorous upper bounds, off by a constant, were given by Conlon-Solovej and Toth in the early 90s.

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Side remark: the Hamiltonian can be rewritten as

$$H_{\Lambda} = S \sum_{\langle x,y \rangle} \left(a_x^+ \sqrt{1 - \frac{n_y}{2S}} - a_y^+ \sqrt{1 - \frac{n_x}{2S}} \right) \cdot \left(a_x \sqrt{1 - \frac{n_y}{2S}} - a_y \sqrt{1 - \frac{n_x}{2S}} \right)$$

i.e., it describes a weighted hopping process of bosons on the lattice. The hopping on an occupied site is discouraged (or not allowed).

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Theorem [Correggi-G-Seiringer 2013]
(free energy at low temperature).

For any $S \geq 1/2$,

$$\lim_{\beta \rightarrow \infty} f(S, \beta) \beta^{5/2} S^{3/2} = \int \log \left(1 - e^{-k^2} \right) \frac{d^3 k}{(2\pi)^3} .$$

- The proof is based on upper and lower bounds. It comes with explicit estimates on the remainder.

Relative errors: • $O((\beta S)^{-3/8})$ (upper bound)

• $O((\beta S)^{-1/40+\epsilon})$ (lower bound)

- We do not really need S fixed. Our bounds are uniform in S , provided that $\beta S \rightarrow \infty$.
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We sketch the proof for $S = 1/2$ only.

In this case the Hamiltonian takes the form:

$$H_{\Lambda} = \frac{1}{2} \sum_{\langle x,y \rangle} \left[(a_x^+ - a_y^+)(a_x - a_y) - 2n_x n_y \right] \equiv T - K$$

or, in the “random hopping” language,

$$H_{\Lambda} = \frac{1}{2} \sum_{\langle x,y \rangle} (a_x^+(1-n_y) - a_y^+(1-n_x))(a_x(1-n_y) - a_y(1-n_x))$$

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We localize in Dirichlet boxes B of side ℓ :

$$f(\beta, \Lambda) \leq (1 + \ell^{-1})^{-3} f^D(\beta, B)$$

In each box, we use the Gibbs variational principle:

$$f^D(\beta, B) = \frac{1}{\ell^3} \inf_{\Gamma} \left[\text{Tr} H_B^D \Gamma + \frac{1}{\beta} \text{Tr} \Gamma \ln \Gamma \right]$$

For an upper bound we use as trial state

$$\Gamma_0 = \frac{P e^{-\beta T^D} P}{\text{Tr}(P e^{-\beta T^D} P)},$$

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To bound the effect of the projector, we use

$$1 - P \leq \sum_x (1 - P_x) \leq \frac{1}{2} \sum_x n_x (n_x - 1)$$

Therefore, $\langle 1 - P \rangle$ can be bounded via the Wick's rule: using $\langle a_x a_x^+ \rangle \simeq (\text{const.}) \beta^{-3/2}$ we find

$$\frac{\text{Tre}^{-\beta T^D} (1 - P)}{\text{Tre}^{-\beta T^D}} \leq (\text{const.}) \ell^3 \beta^{-3}$$

Optimizing, we find $\ell \propto \beta^{7/8}$, which implies

$$f(\beta) \leq C_0 \beta^{-5/2} \left(1 - O(\beta^{-3/8}) \right).$$

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Therefore, $\langle 1 - P \rangle$ can be bounded via the Wick's rule: using $\langle a_x a_x^+ \rangle \simeq (\text{const.}) \beta^{-3/2}$ we find

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Optimizing, we find $\ell \propto \beta^{7/8}$, which implies

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$$f(\beta, \Lambda) \geq f(\beta, B).$$

Key ingredient for a preliminary lower bound:

Lemma 1.

$$H_B \geq c\ell^{-2}\left(\frac{1}{2}\ell^3 - S_T\right).$$

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Lemma 1 \Rightarrow a priori bound on the particle number:
in fact, since H_B commutes with \vec{S}_T ,

$$\mathrm{Tr}(e^{-\beta H_B}) = \sum_{S_T=0}^{\ell^3/2} (2S_T + 1) \mathrm{Tr}_{S_T^3 = -S_T} (e^{-\beta H_B})$$

By Lemma 1, the r.h.s. is bounded from above by

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Lower bound. Steps 1 and 2.

Optimizing over ℓ we find

$$f(\beta, \Lambda) \geq -(\text{const.})\beta^{-5/2}(\log \beta)^{5/2}.$$

We can now cut off the “high” energies:

$$\text{Tr} P_{H_B \geq E_0} e^{-\beta H_B} \leq e^{-\beta E_0/2} e^{-\frac{\beta}{2} \ell^3 f(\beta/2, B)} \leq 1,$$

$$\text{if } E_0 \simeq \ell^3 \beta^{-5/2} (\log \beta)^{5/2}.$$

We are left with the trace on $H_B \leq E_0$, which we compute on the sector $S_T^3 = -S_T$.

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If $\rho_E(x, y)$ is the two-particle density matrix,

$$\langle E|K|E\rangle = \sum_{\langle x, y \rangle} \langle E|n_x n_y|E\rangle \leq 3\ell^3 \|\rho_E\|_\infty$$

Key estimate:

Lemma 2. For all $E \leq E_0$

$$\|\rho_E\|_\infty \leq (\text{const.})E_0^3$$

Now: $\ell = \beta^{1/2+\epsilon} \Rightarrow E_0 \simeq \ell^{-2+O(\epsilon)} \Rightarrow \|\rho_E\|_\infty \leq \ell^{-6+O(\epsilon)}$

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Key observation: the eigenvalue equation implies

$$-\tilde{\Delta}\rho_E(x, y) \leq 4E\rho_E(x, y),$$

where $\tilde{\Delta}$ is the Neumann Laplacian on

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We extend ρ on \mathbb{Z}^6 by Neumann reflections and find

$$-\Delta\rho_E(z) \leq 4E\rho_E(z) + 2\rho_E(z)\chi_1^R(z)$$

where $\chi_1^R(z_1, z_2)$ is equal to 1 if z_1 is at distance 1 from one of the images of z_2 , and 0 otherwise.

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$$\rho_E(z) \leq (1 - E/3)^{-1} \left(\langle \rho_E \rangle_z + \frac{1}{6} \|\rho_E\|_\infty \chi_1^R(z) \right)$$

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Iterating,

$$\rho_E(z) \leq \left(1 - \frac{E}{3}\right)^{-n} \left((P_n * \rho_E)(z) + \frac{1}{6} \|\rho_E\|_\infty \sum_{j=0}^{n-1} P_j * \chi_1^R(z) \right)$$

where $P_n(z, z')$ is the probability that a SSRW on \mathbb{Z}^6 starting at z ends up at z' in n steps. For large n :

$$P_n(z, z') \simeq \left(\frac{3}{\pi n}\right)^3 e^{-3|z-z'|^2/n}.$$

Moreover, if G is the Green function on \mathbb{Z}^6 ,

$$\sum_{j=0}^{n-1} P_j(z, z') \leq \sum_{j=0}^{\infty} P_j(z, z') = 12G(z - z')$$

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Lower bound. Step 3: Proof of Lemma 2.

Let us now pretend for simplicity that χ_1^R is equal to χ_1 . In this simplified case we find:

$$\rho(z) \leq \frac{1}{\left(1 - \frac{E}{3}\right)^n} \left(\frac{27}{\pi^3 n^3} \sum_{w \in \mathbb{Z}^6} e^{-\frac{3}{n}|z-w|^2} \rho(w) + 2\|\rho\|_\infty G * \chi_1(z) \right)$$

Picking $n \sim E^{-1}$ we get:

$$\rho(z) \leq (\text{const.}) \max\{E^3, \ell^{-6}\} + (1+\delta) \times 2 \times 0.258 \times \|\rho\|_\infty$$

where we used the fact that

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