The Altshuler-Shklovskii Formulas for Random Band Matrices

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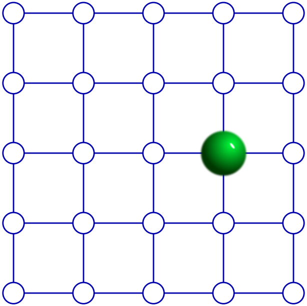
Quantum particle on a lattice

Define the d-dimensional lattice of side length L,

$$\mathbb{T} := ([-L/2, L/2) \cap \mathbb{Z})^d .$$

Always consider the limit $L \to \infty$.



The model is defined by a Hamiltonian, a self-adjoint matrix $H = (H_{xy})_{x,y \in \mathbb{T}}$.

Models of quantum disorder

Disorder can be modelled by introducing randomness in H.

Two famous random models:

Wigner matrix. The entries of H are i.i.d. up to the constraint $H = H^*$.
Mean-field model with no spatial structure.

Microscopic spectral statistics governed by sine kernel of random matrix theory (Erdős-Schlein-Yau-… [2009–2012], Tao-Vu [2009–2012]).

Random Schrödinger operator. On-site randomness + short-range hopping:

$$H = -\Delta + V ,$$

where $V = (v_x)_{x \in \mathbb{T}}$ is a diagonal matrix with i.i.d. entries.

For $d = 1$: Microscopic spectral statistics are Poisson (Goldscheid-Molchanov-Pastur [1977], Minami [1996]).

For $d > 1$: complicated phase diagram, only partially understood (Fröhlich–Spencer [1983], Aizenman–Molchanov [1993]).

For $d = 1$ we have the explicit matrix representations

Wigner matrix: $$\begin{pmatrix} H_{11} & \cdots & H_{1L} \\ \vdots & & \vdots \\ H_{L1} & \cdots & H_{LL} \end{pmatrix}$$

Random Schrödinger operator: $$\begin{pmatrix} v_1 & 1 & & & \\ 1 & v_2 & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & v_{L-1} & 1 \\ & & & 1 & v_L \end{pmatrix}$$

Band matrices

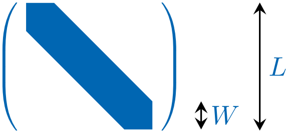
Model of quantum transport in disordered media, interpolates between Wigner matrices and Random Schrödinger operators.

Let f be an even probability density on $\mathbb{R}^d$, and $W \in [1, L]$.

H is a d-dimensional band matrix with band width W and band profile f if:

- H has mean-zero entries independent up to the constraint $H = H^*$.
- $\mathbb{E}|H_{xy}|^2 = S_{xy} := \frac{1}{W^d} f\left(\frac{x-y}{W}\right)$.

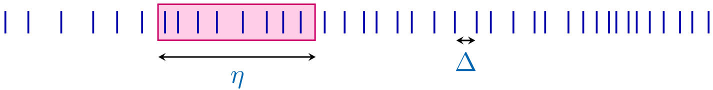
For $d = 1$ and $f = \frac{1}{2}\mathbf{1}_{[-1,1]}$ the band matrix H is of the form

$H =$ 

Eigenvalue statistics on different scales

Goal: statistics of the eigenvalue process $\sum_i \delta_{\lambda_i}$; dependence on energy scale?

Let $\Delta = L^{-d}$ denote the typical level spacing.



Scales:

microscopic	mesoscopic	macroscopic
$\eta \sim \Delta$	$\Delta \ll \eta \ll 1$	$\eta \sim 1$
Poisson / sine kernel	universalities	model-dependent

More generally, consider linear statistics

$$Y_\phi^\eta(E) := \sum_i \phi^\eta(\lambda_i - E)\,, \qquad \phi^\eta(e) := \eta^{-1}\phi(e/\eta)\,,$$

where λ_i are eigenvalues of H, ϕ is a fixed test function, and E a fixed energy inside the spectrum.

Physical motivation (Thouless): conductance directly related to number of eigenvalues in a mesoscopic energy window around the Fermi energy E.

Correlations of $\{Y_\phi^\eta(E_i)\}$ may be expressed using the truncated correlation functions $p^{(k)}$: for instance

$$\langle Y_\phi^\eta(E_1)\,;Y_\phi^\eta(E_2)\rangle \;=\; \int \mathrm{d}x\,\mathrm{d}y\,\phi^\eta(x-E_1)\,\phi^\eta(y-E_2)\,p^{(2)}(x,y)\,.$$

If the sine kernel held on all mesoscopic scales, we would get, with $\omega := |E_2 - E_1|$,

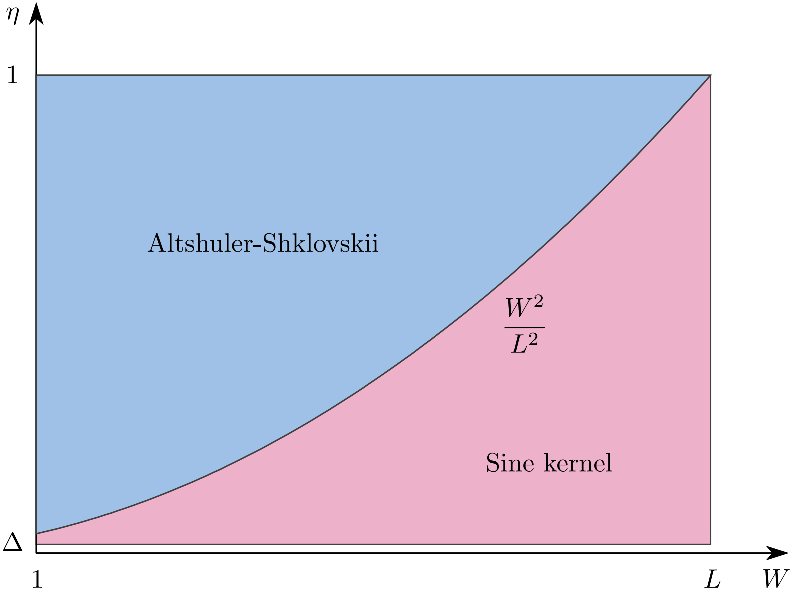
$$\int_{|e-\omega|\leqslant\eta}\left(\frac{\sin(e/\Delta)}{e/\Delta}\right)^2 \mathrm{d}e \;\sim\; \frac{1}{\omega^2} \qquad (\Delta \ll \eta \ll \omega \ll 1)\,. \tag{1}$$

Extrapolation from $\eta \sim \Delta$ to $\eta \gg \Delta$ looks easy. In fact, (1) was proved for GUE by Boutet de Monvel–Khorunzhy [1999].

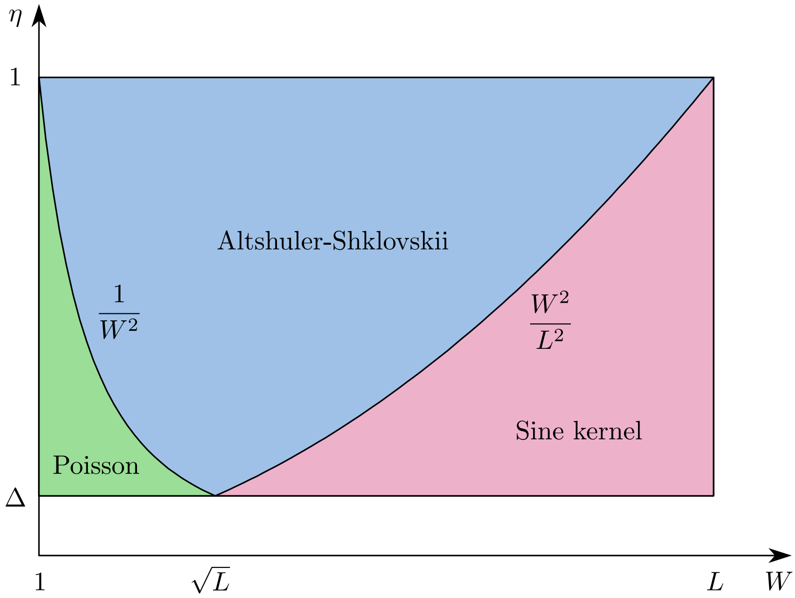
However, (1) is in general wrong.

- The sine kernel may fail on mesoscopic scales. Correct behaviour given by Altshuler-Shklovskii formulas. Previously predicted in physics literature.
- Even for Wigner matrices, the sine kernel fails to predict the correct subleading terms. New observation, contradicting several physics predictions.

The expected phase diagram for $d = 3$



The expected phase diagram for $d = 1$



Altshuler-Shklovskii (AS) formulas

A transition in mesoscopic statistics occurs at the Thouless energy

$$\eta_0 = \big(\text{time for diffusion to reach the boundary of } \mathbb{T}\big)^{-1} .$$

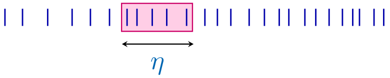
For random band matrices the diffusion coefficient is W^2 (Erdős-K [2011]), so that $\eta_0 \sim W^2/L^2$. For $\eta \gg \eta_0$ boundary effects are irrelevant. For $\eta \ll \eta_0$ the statistics are mean-field.

AS formulas, derived in physics literature by Altshuler and Shklovskii [1986]:

(1) Behaviour in diffusion regime, $\eta_0 \ll \eta \ll 1$:

For $d = 1, 2, 3$ we have

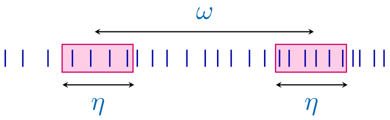
$$\operatorname{Var} Y_\phi^\eta(E) \;\sim\; (\eta/\eta_0)^{d/2-2} .$$



For $d = 1, 3$ and $\eta \ll \omega \ll 1$ we have

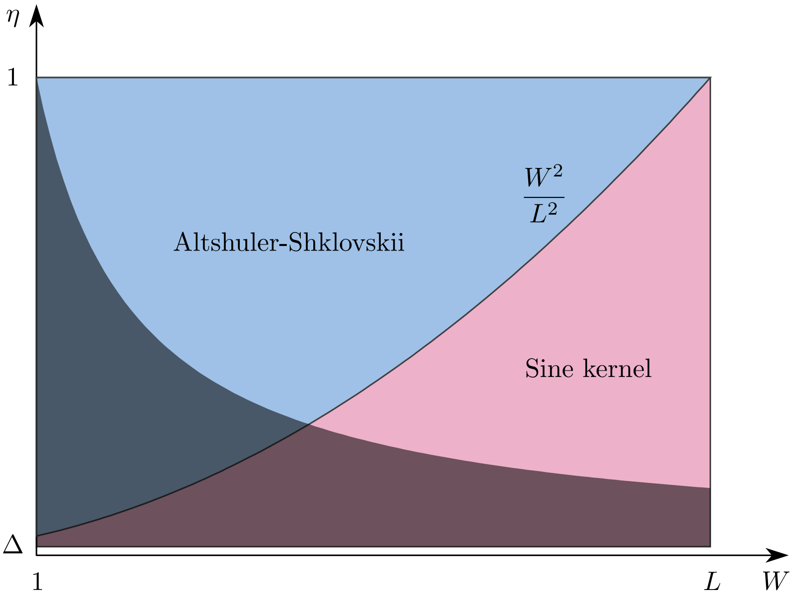
$$\langle Y_\phi^\eta(E+\omega/2)\,; Y_\phi^\eta(E-\omega/2)\rangle \;\sim\; \omega^{d/2-2} .$$

$d = 2$ is critical, leading term vanishes.



(2) Behaviour in mean-field regime, $\eta \ll \eta_0$: same formulas with $d = 0$.

Results [Erdős-K, 2013]: domain of validity (e.g. for $d = 3$)



Results [Erdős-K, 2013]: outline

(a) Proof of the AS formulas for $d = 1, 2, 3, 4$: mesoscopic universality.

(b) For $d \geqslant 5$ universality breaks down.

(c) For $d = 2$ the correlations are governed by so-called weak localization corrections. Our result differs substantially from the prediction of Kravtsov–Lerner [1995].

(d) Critical band matrix model for $d = 1$ with $S_{xy} = \mathbb{E}|H_{xy}|^2 \sim |x - y|^{-2}$. Describes the system at metal-insulator transition. Our result agrees with prediction of Chalker-Kravtsov-Lerner [1996] on the multifractality of the eigenvectors.

(e) We introduce a large family of random band matrices that interpolates between the real ($\beta = 1$) and complex ($\beta = 2$) symmetry classes, and track the crossover in the mesoscopic eigenvalue statistics.

(f) CLT: Mesoscopic densities $\{Y_\phi^\eta(E)\}_{\phi,E}$ converge to Gaussian process whose covariance given by the AS formulas.

The main result

Theorem (Erdős-K [2013])

Let ϕ_1 and ϕ_2 be smooth with sufficient decay and $\eta = W^{-\rho d}$ for some $\rho < 1/3$.

Suppose that $L \leqslant W^C$.

Then for E_1 and E_2 away from the spectral edges ± 1 we have

$$\frac{\langle Y_{\phi_1}^{\eta}(E_1)\,;Y_{\phi_2}^{\eta}(E_2)\rangle}{\langle Y_{\phi_1}^{\eta}(E_1)\rangle\langle Y_{\phi_2}^{\eta}(E_2)\rangle} \;=\; \Theta_{\phi_1,\phi_2}^{\eta}(E_1, E_2)\,(1 + O(W^{-c}))\,,$$

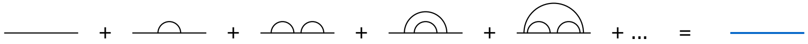
where $\Theta_{\phi_1,\phi_2}^{\eta}(E_1, E_2)$ is an explicit (but complicated) deterministic expression.

$\Theta_{\phi_1,\phi_2}^{\eta}(E_1, E_2)$ can be explicitly analysed in the regimes $\eta \gg \eta_0$ and $\eta \ll \eta_0$.

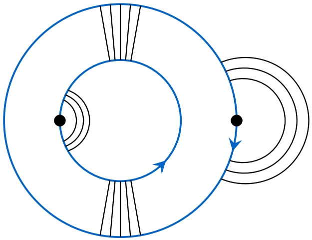
The leading term Θ

The proof is based on a renormalized expansion scheme that is organized using graphs (more later).

Renormalized propagator:



Leading term Θ: one-loop diagram with two intraparticle and two interparticle ladders.



Behaviour of Θ for $\eta \gg \eta_0$ (sample)

Let D be the covariance matrix of f. Let $\omega := |E_2 - E_1|$.

- For $d = 1, 2, 3$ and $\omega = 0$ we have

$$\Theta \;=\; \frac{C_d}{\beta\sqrt{\det D}(LW)^d}\,\eta^{d/2-2}\left(V_d(\phi_1,\phi_2) + O(W^{-c})\right),$$

  where

$$V_d(\phi_1,\phi_2) \;:=\; \int_{\mathbb{R}} \mathrm{d}t\,|t|^{1-d/2}\,\overline{\widehat{\phi}_1(t)}\,\widehat{\phi}_2(t)\,.$$

- If $d = 1, 2, 3$ and $\omega \gg \eta$ then

$$\Theta \;=\; \frac{1}{\beta\sqrt{\det D}(LW)^d}\,\omega^{d/2-2}\left(K_d + O(W^{-c})\right)$$

  where $K_1 < 0$, $K_2 = 0$, and $K_3 > 0$.

Similar results hold for $d = 4$.

The weak localization correction for $d = 2$

For $d = 2$ and $\omega \gg \eta$ we have $K_2 = 0$, and the largest nonzero contribution is given by the weak localization correction

$$\Theta = \frac{C_2}{\beta\sqrt{\det D}(LW)^d}\big((Q-1)|\log\omega| + O(1)\big),$$

where $Q := \frac{1}{32}\int |D^{-1/2}x|^4 f(x)\,\mathrm{d}x$.

At odds with prediction of Kravtsov-Lerner [1995]

$$\Theta \sim \frac{1}{(LW)^d}\begin{cases} W^{-2}\omega^{-1} & \text{if } \beta = 1 \\ W^{-4}\omega^{-1} & \text{if } \beta = 2\,. \end{cases}$$

(Arises from the so-called two-loop diagrams.)

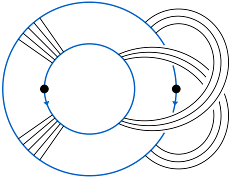
Our result:

$$\Theta \sim \frac{1}{\beta(LW)^d}|\log\omega|\,.$$

(Arises from one-loop diagrams.)

Corrections for Wigner matrices

Computation of two-loop diagrams shows that physics predictions, coinciding with microscopic Wigner-Dyson statistics, are wrong even for Wigner matrices.



For $L \times L$ Wigner matrices, with $\omega \gg \eta \gg L^{-1/2}$ and $\omega = s\Delta$, we get

$$\frac{\langle Y^{\eta}_{\phi_1}(E_1)\,; Y^{\eta}_{\phi_2}(E_2)\rangle}{\langle Y^{\eta}_{\phi_1}(E_1)\rangle\langle Y^{\eta}_{\phi_2}(E_2)\rangle} = \frac{1}{\beta(\mathrm{i}s)^2}\left(1 + \frac{(L\eta)^2}{s^2} + \frac{s^2}{L^2} + \cdots + \frac{L}{s^2}\delta_{\beta,1} + \cdots\right).$$

Red: Corrections to the one-loop diagrams.

Blue: Uncancelled term from two-loop diagrams. Physics folklore: two-loop diagrams cancel out within a so-called Hikami box. In fact, for $\eta \gg L^{-1/2}$ there is no cancellation.

Critical band matrix model

Set $d = 1$ and $S_{xy} \sim |x - y|^{-2}$. This behaves like the case $d = 2$ and describes a system at the Anderson transition.

We prove that the number of eigenvalues $\mathcal{N}(I)$ in $I \subset \mathbb{R}$ satisfies

$$\operatorname{Var} \mathcal{N}(I) \ \sim \ W^{-d}\, \mathbb{E}\mathcal{N}(I)\,.$$

For disjoint I and I', the numbers $\mathcal{N}(I)$ and $\mathcal{N}(I')$ are asymptotically independent.

This relation was predicted by Chalker-Kravtsov-Lerner [1996], and characterizes multifractality of the eigenvectors. The coefficient W^{-d} (spectral compressibility) is in accordance with predictions for multifractality exponents.

Sketch of proof

Expand

$$Y_{\phi}^{\eta}(E) \;=\; \mathrm{Tr}\,\phi^{\eta}(H-E) \;=\; 2\,\mathrm{Re}\int_0^{\infty} \widehat{\phi}(\eta t)\,\mathrm{e}^{\mathrm{i}tE}\mathrm{e}^{-\mathrm{i}tH}\,,$$

and expand the exponential as a power series in H. Need to control it for times $t \lesssim \eta^{-1}$.

Main difficulty: terms are highly oscillating.

Need a systematic resummation procedure. We use a two-step resummation.

Step 1. Chebyshev-Fourier expansion in $\{U_n(H)\}_{n\in\mathbb{N}}$. More stable than Taylor expansion, corresponds to an algebraic self-energy renormalization.

Step 2. Organize algebra using graphs. Systematically bundle together oscillatory sums arising of specific families of subgraphs and compute them with high precision. Up to here everything is algebra: no estimates allowed.

After this step we perform a term-by-term estimate using pointwise bounds on the resolvent of $S = (S_{xy})$ (local central limit theorems).

Conclusion

- Proof of the Altshuler-Shklovskii formulas: mesoscopic universality.
- Weak localization corrections differ substantially from predictions.
- Mesoscopic densities $\{Y_\phi^\eta(E)\}_{\phi,E}$ converge to Gaussian process, covariance given by the Altshuler-Shklovskii formulas.
- Proof uses a variety of algebraic resummations to control highly oscillating sums.

Open questions:

- Extend analysis to rest of phase diagram, $\Delta \ll \eta \leqslant W^{-d/3}$.
- Do the same for random Schrödinger operator.

General random band matrix model

Set

$$\mathbb{E}|H_{xy}|^2 = W^{-d} f(u), \qquad u := \frac{x-y}{W},$$

and

$$\mathbb{E}H_{xy}^2 = W^{-d} f(u)\,(1-h(u))\,\mathrm{e}^{\mathrm{i}g(u)}.$$

Here $f \geqslant 0$ and $0 \leqslant h \leqslant 1$ are even and g is odd.

Our main theorem remains valid for this model.

The changes in Θ are governed by the quantity

$$\sigma := \inf_{q \in \mathbb{R}^d} \int (x \cdot q - g(x))^2 f(x)\,\mathrm{d}x + \int h(x) f(x)\,\mathrm{d}x.$$

In particular, there is a continuous crossover in mesoscopic statistics from $\beta = 1$ (small σ) to $\beta = 2$ (large σ).