Maxima and entropic repulsion of Gaussian free field: Going beyond $\mathbb{Z}^d$

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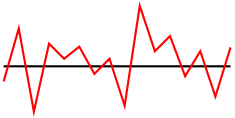
Gaussian free field (GFF)

- $\mathcal{G} = (V, E)$: connected graph, containing a distinguished set of vertices $B \subset V$. Assume $(V \backslash B, E)$ remains connected.
- A free field $\varphi = \{\varphi_x\}_{x \in V}$ on $\mathcal{G}$ is a collection of centered Gaussian random variables with covariance $\mathbb{E}[\varphi_x \varphi_y] = G(x, y)$, where G is the Green's function for (symmetric) random walk on $\mathcal{G}$ killed upon hitting B.
  The law of the free field is (formally) given by the Gibbs measure
  $$\mathbb{P} = \frac{1}{\mathcal{Z}} e^{-\frac{1}{2}\mathcal{E}(\varphi)} \prod_{x \in V \backslash B} d\varphi_x \prod_{y \in B} \delta_0(\varphi_y), \quad \text{where } \mathcal{E}(\varphi) = \frac{1}{2} \sum_{\langle xy \rangle \in E} (\varphi_x - \varphi_y)^2$$
  is the Dirichlet energy on G, and $\mathcal{Z}$ is a normalization factor.
- For this talk, it is helpful to imagine φ as a random interface in $\mathcal{G} \times \mathbb{R}$ separating two phases (water/oil, $(+)$-spin/$(-)$-spin).

Stochastic geometry of the free field (I)

- Let $\mathcal{G}_n = (V_n, E_n)$ be an increasing nested sequence of graphs which tends to an infinite graph $\mathcal{G}_\infty = (V_\infty, E_\infty)$.
- Let $\varphi^{(n)}$ be the free field on $\mathcal{G}_n$ (with "wired" boundary condition by gluing $(V_\infty \backslash V_n)$ into one vertex).

Maxima of the (unconditioned) free field

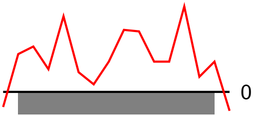


$$\varphi_*^{(n)} = \max_{x \in V_n} (\varphi^{(n)})_x$$

Question I: Find the asymptotics of $\varphi_*^{(n)}$ as $n \to \infty$.
In particular, identify the leading-order term $\mathbb{E}[\varphi_*^{(n)}]$, as well as the recentered fluctuations about the mean $[\varphi_*^{(n)}] - \mathbb{E}[\varphi_*^{(n)}]$.

Stochastic geometry of the free field (II)

Entropic repulsion under the "hard wall" condition



Let φ be a free field on $\mathcal{G}_\infty$, with law $\mathbb{P}$. Define the "hard wall" event

$$\Omega_n^+ = \{\varphi_x \geq 0 \text{ for all } x \in V_n\}.$$

We want to look at

$$\varphi_x \text{ under } \mathbb{P}(\cdot|\Omega_n^+)$$

Due to the loss of volume on V_n, the field φ needs to gain space above the hard wall in order to accommodate local fluctuations (an entropic effect).

Question II: Identify the asymptotics of the height of the free field under Ω_n^+ as $n \to \infty$.

For both Question I and Question II:

- Naively, the leading-order asymptotics in both situations grow at the same order of n.
- The asymptotics differ qualitatively depending on whether $\mathcal{G}_\infty$ supports strongly recurrent random walk ('subcritical regime') or transient recurrent random walk ('supercritical regime').

The case of $\mathbb{Z}^d$

Finite box $\Lambda_n = ([-n,n]\cap\mathbb{Z})^d$. Let $\varphi^{(n)}$ be the free field on Λ_n.
Maxima: $\varphi_*^{(n)} = \max_{x\in V_n}\left(\varphi^{(n)}\right)_x$

d	$\mathbb{E}[\varphi_*^{(n)}]$	$\varphi_*^{(n)} - \mathbb{E}[\varphi_*^{(n)}]$
1	$\mathcal{O}(\sqrt{n})$	$\mathcal{O}(\sqrt{n})$
2	$\mathcal{O}(\log(n))$	$\mathcal{O}(1)$
≥ 3	$\mathcal{O}(\sqrt{\log(n)})$	$\mathcal{O}(1)$

The sequence of recentered maxima is tight when $d=2$ [Bramson-Zeitouni '12] and $d\geq 3$ [via Borell-TIS ineq].

Entropic repulsion

- $d\geq 3$ [Bolthausen-Deuschel-Zeitouni '95]: For every $x\in\mathbb{Z}^d$,

$$\frac{\varphi_x}{\sqrt{\log(n)}} \text{ under } \mathbb{P}(\cdot|\Omega_n^+) \xrightarrow[n\to\infty]{P} 2\sqrt{G_{\mathbb{Z}^d}(0,0)}.$$

- $d=2$ [BDZ '01]: For every $x\in\mathbb{Z}^2$,

$$\frac{\varphi_x}{\log(n)} \text{ under } \mathbb{P}(\cdot|\Omega_n^+) \text{ tends to } 2\sqrt{G_{\mathbb{Z}^2}(0,0)},$$

  the mode of convergence being more delicate.

Going beyond $\mathbb{Z}^d$: fractal-like graphs

Sequence of approximating graphs $\mathcal{G}_n = (V_n, E_n)$ tending to $\mathcal{G}_\infty = (V_\infty, E_\infty)$.
Assume there exist positive constants ℓ, m, and ρ such that for all $x \in V_\infty$,

$$|V_n| \asymp m^n, \quad R_{\text{eff}}\left(x, (B(x, \ell^n))^c\right) \asymp \rho^n.$$

Here $B(x,r)$ is the ball of radius r in the graph distance centered at x, and $R_{\text{eff}}(A_1, A_2)$ is the effective resistance between sets $A_1, A_2 \subset V_\infty$.
When $\rho > 1$, random walk on graph is strongly recurrent; if $\rho < 1$, RW is transient.

In the strongly recurrent case ($\rho > 1$), the maxima of the unconditioned free field has asymptotics

$$\mathbb{E}[\varphi_*^{(n)}] = \mathcal{O}(\rho^{n/2}), \quad \varphi_*^{(n)} - \mathbb{E}[\varphi_*^{(n)}] = \mathcal{O}(\rho^{n/2}).$$

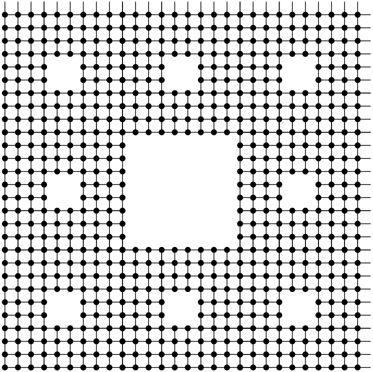
The latter [Kumagai-Zeitouni '13] shows the absence of tightness in the recentered fluctuations, which generalizes the case of $\mathbb{Z}$.

In the transient case ($\rho < 1$), the leading-order asymptotics for entropic repulsion is demonstrated for (highly symmetric) generalized Sierpinski carpet graphs [C.-Ugurcan '13]. For every $x \in V_\infty$,

$$\frac{(\text{local sample mean of } \varphi \text{ at } x)}{\sqrt{\log((m\rho)^n)}} \text{ under } \mathbb{P}(\cdot|\Omega_n^+) \xrightarrow[n\to\infty]{P} \sqrt{2\underline{G}},$$

where $\underline{G} = \inf_{x \in V_\infty} G_{\mathcal{G}_\infty}(x,x)$. This generalizes the case of $\mathbb{Z}^d$, $d \geq 3$ treated in [BDZ '95].

Generalized Sierpinski carpet graphs



This is the graph associated with the standard two-dimensional Sierpinski carpet, which has $\rho > 1$. Higher-dimensional analogs (such as the Menger sponge) may have $\rho < 1$.

More aboue the supercritical regime

$\mathbb{Z}^d,\ d \geq 3$

Maximum of the unconditioned free field: $\sqrt{2d\, G_{\mathbb{Z}^d}(0,0)\log(n)}$.
Height of the free field under entropic repulsion: $2\sqrt{G_{\mathbb{Z}^d}(0,0)\log(n)}$

For fractal-like graphs in the supercritical regime

Maximum of the unconditioned free field: $\sqrt{2c_d\, G_*\log((m\rho)^n)}$ (?).
Height of the free field under entropic repulsion: $2\sqrt{\underline{G}\log((m\rho)^n)}$

To be resolved: Is $G_* = \underline{G} := \inf_{x\in V_\infty} G_{\mathcal{G}_\infty}(x,x)$? What is the dimensional constant c_d?

Resolving this question will allow us to find sharp asymptotics for the expected cover times of random walk on fractal-like graphs, building on the results of [Ding-Lee-Peres '12, Ding '12].

[Note that we expect 'concentration' of cover times to the mean.]