Excitation spectrum of interacting bosons in the mean-field infinite-volume limit

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Model

- We consider an interacting, homogeneous Bose gas.
- The Hamiltonian of such an N-particle system is given by

$$H_N = -\sum_{i=1}^{N} \Delta_i + \lambda \sum_{1 \le i < j \le N} v(\mathbf{x}_i - \mathbf{x}_j)$$

defined on the Hilbert space $L^2_{\mathrm{sym}}((\mathbb{R}^d)^N)$.

- $\lambda \ge 0$ is a coupling constant.
- We assume v is a real and symmetric function such that

$$v \in L^1(\mathbb{R}^d), \quad \hat{v} \in L^1(\mathbb{R}^d)$$
$$v(\mathbf{x}) \ge 0, \;\; \mathbf{x} \in \mathbb{R}^d, \;\; \hat{v}(\mathbf{p}) \ge 0, \;\; \mathbf{p} \in \mathbb{R}^d.$$

- We want to describe a physical system of *positive* density in a large volume limit.
- To this end we replace $\mathbb{R}^d$ by the torus $(-L/2, L/2]^d$ and the potential by its *periodized* version

$$v^L(\mathbf{x}) = \frac{1}{L^d} \sum_{\mathbf{p} \in (2\pi/L)\mathbb{Z}^d} \hat{v}(\mathbf{p}) \mathrm{e}^{\mathrm{i}\mathbf{p}\mathbf{x}}.$$

- The Hamiltonian in the box reads

$$H_N^L = -\sum_{i=1}^{N} \Delta_i^L + \lambda \sum_{1 \leq i < j \leq N} v^L(\mathbf{x}_i - \mathbf{x}_j).$$

- The *total momentum* operator

$$P_N^L = \sum_{i=1}^{N} -\mathrm{i}\partial_{\mathbf{x}_i}^L.$$

The excitation spectrum

- H_N and P_N commute, thus we can consider the joint energy-momentum spectrum $\mathrm{sp}(H_N, P_N) \subset \mathbb{R}^{d+1}$.

- Let E_N denote the *ground state energy* of H_N. Then

$$\boxed{\text{Excitation spectrum} := \mathrm{sp}(H_N - E_N, P_N).}$$

Bogoliubov excitation spectrum

The diagonalised Bogoliubov Hamiltonian

$$H_{\text{Bog}} = E_{\text{Bog}} + \sum_{\mathbf{p}\neq 0} e(\mathbf{p}) b_{\mathbf{p}}^{\dagger} b_{\mathbf{p}}$$

with $e(\mathbf{p}) = \sqrt{\mathbf{p}^4 + 2\lambda\rho\hat{v}(\mathbf{p})\mathbf{p}^2}$ and

$$E_{\text{Bog}} := -\frac{1}{2} \sum_{\mathbf{p}\in\frac{2\pi}{L}\mathbb{Z}^d\setminus\{0\}} \left(|\mathbf{p}|^2 + \hat{v}(\mathbf{p}) - \sqrt{|\mathbf{p}|^4 + 2\lambda\rho\hat{v}(\mathbf{p})|\mathbf{p}|^2} \right).$$

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$\Rightarrow$ Choice of mean-field scaling $\lambda = 1/\rho$.

For $\mathbf{p} \in \frac{2\pi}{L}\mathbb{Z}^d$ define the Bogoliubov elementary excitation spectrum $e(\mathbf{p}) := \sqrt{\mathbf{p}^4 + 2\hat{v}(\mathbf{p})\mathbf{p}^2}$.
For any $\mathbf{p} \in \frac{2\pi}{L}\mathbb{Z}^d$ we consider the Bogoliubov excitation energies with total momentum $\mathbf{p}$:

$$\left\{ \sum_{i=1}^{j} e(\mathbf{k_i}) \;:\; \mathbf{k_1}, \ldots, \mathbf{k_j} \in \frac{\mathbf{2\pi}}{\mathbf{L}}\mathbb{Z}^{\mathbf{d}}, \, \mathbf{k_1} + \ldots + \mathbf{k_j} = \mathbf{p}, \, \mathbf{j} = \mathbf{1}, \mathbf{2}, \ldots \right\}$$

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Let $K^1_{\text{Bog}}(\mathbf{p}), K^2_{\text{Bog}}(\mathbf{p}), \ldots$ be these energies in the increasing order.

Similarly, let $K^1_N(\mathbf{p}), K^2_N(\mathbf{p}), \ldots$ be the corresponding excitation energies of H_N, that is, the eigenvalues of $H_N - E_N$ of total momentum $\mathbf{p}$ in the increasing order.

Theorem

Lower bound

Let $c > 0$. Then there exists C such that

1. if

$$L^{2d+2} \leq cN,$$

then

$$E_N \geq \frac{1}{2}\hat{v}(\mathbf{0})(N-1) + E_{\text{Bog}} - CN^{-1/2}L^{2d+3};$$

2. if in addition

$$K_N^j(\mathbf{p}) \leq cNL^{-d-2},$$

then

$$\begin{aligned} E_N + K_N^j(\mathbf{p}) \quad \geq \quad & \frac{1}{2}\hat{v}(\mathbf{0})(N-1) + E_{\text{Bog}} + K_{\text{Bog}}^j(\mathbf{p}) \\ & -CN^{-1/2}L^{d/2+3}\left(K_N^j(\mathbf{p}) + L^d\right)^{3/2}. \end{aligned}$$

Upper bound

Let $c > 0$. Then there exists $c_1 > 0$ and C such that

1. if

$$L^{2d+1} \leq cN \quad \text{and} \quad L^{d+1} \leq c_1 N,$$

then

$$E_N \leq \frac{1}{2}\hat{v}(\mathbf{0})(N-1) + E_{\text{Bog}} + CN^{-1/2}L^{2d+3/2};$$

2. if in addition

$$\begin{aligned} K^j_{\text{Bog}}(\mathbf{p}) &\leq cNL^{-d-2} \\ \text{and} \quad K^j_{\text{Bog}}(\mathbf{p}) &\leq c_1 NL^{-2}, \end{aligned}$$

then

$$\begin{aligned} E_N + K^j_N(\mathbf{p}) \leq\ & \frac{1}{2}\hat{v}(\mathbf{0})(N-1) + E_{\text{Bog}} + K^j_{\text{Bog}}(\mathbf{p}) \\ & + CN^{-1/2}L^{d/2+3}(K^j_{\text{Bog}}(\mathbf{p}) + L^{d-1})^{3/2}. \end{aligned}$$

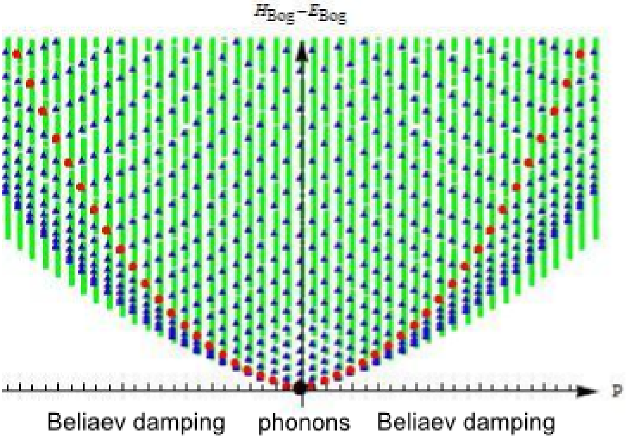
- By the exponential property of Fock spaces we have the identification

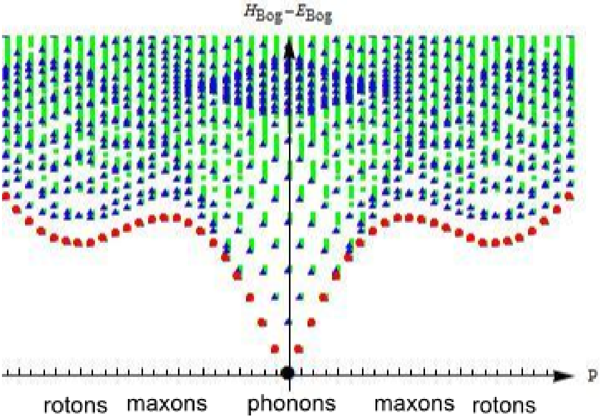
$$\mathcal{H} = \Gamma_{\rm s}\left(l^2(\frac{2\pi}{L}\mathbb{Z}^d)\right) = \Gamma_{\rm s}\left(\mathbb{C} \oplus l^2(\frac{2\pi}{L}\mathbb{Z}^d \setminus \{\mathbf{0}\})\right)$$
$$\simeq \Gamma_{\mathsf{s}}(\mathbb{C}) \otimes \Gamma_{\mathsf{s}}\Big(l^2(\frac{2\pi}{L}\mathbb{Z}^d\setminus\{\mathbf{0}\})\Big).$$

- We embed the space of the zeroth mode $\Gamma_{\mathsf{s}}(\mathbb{C}) = l^2(\{0,1,\dots\})$ in a larger space $l^2(\mathbb{Z})$. The extended space

$$\mathcal{H}^{\rm ext} := l^2(\mathbb{Z}) \otimes \Gamma_{\mathsf{s}}\Big(l^2(\frac{2\pi}{L}\mathbb{Z}^d\setminus\{\mathbf{0}\})\Big).$$

- We have also a unitary operator $U|n_0\rangle \otimes \Psi^{>} = |n_0 - 1\rangle \otimes \Psi^{>}$. For $\mathbf{p} \neq \mathbf{0}$ we define $b_{\mathbf{p}} := a_{\mathbf{p}} U^{\dagger}$ (on $\mathcal{H}^{\rm ext}$).


$H_{Bog} - E_{Bog}$
p
Beliaev damping
phonons
Beliaev damping



Thank you for your attention!