Approximate Dynamics in Slowly Modulated Photonic Crystals for Physical States

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in collarboration with Giuseppe De Nittis

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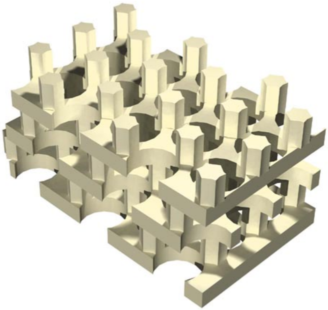
2014.03.18@Warwick

Talk based on

Collaboration with **Giuseppe De Nittis**

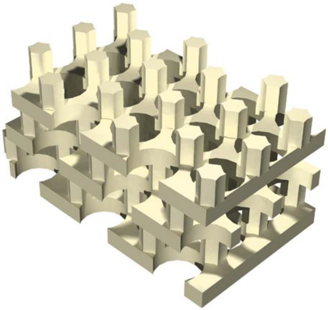
- *The Perturbed Maxwell Operator as Pseudodifferential Operator*
  Documenta Mathematica **19**, 2014
- *Effective Light Dynamics in Perturbed Photonic Crystals*
  to appear in Comm. Math. Phys.
- *On the Role of Symmetries in the Theory of Photonic Crystals*
  in preparation
- *Semiclassical dynamics in Photonic Crystals*
  in preparation



Assumption

Material weights $w = (\varepsilon, \mu)$

1. $\varepsilon, \mu \in L^{\infty}\big(\mathbb{R}^3, \mathrm{Mat}_{\mathbb{C}}(3)\big)$
2. $\varepsilon^* = \varepsilon, \mu^* = \mu$
3. $0 < c \leq \varepsilon, \mu \leq C < \infty$
4. ε, μ *periodic wrt* $\Gamma \cong \mathbb{Z}^3$
5. $\mathrm{Im}\, \varepsilon, \mathrm{Im}\, \mu = 0$



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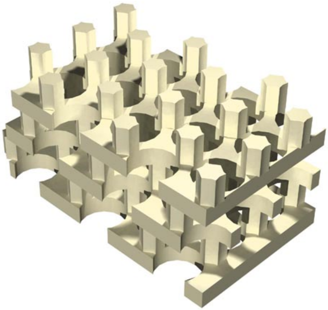

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Maxwell equations as Schrödinger equation

1. *Field energy*

$$\mathcal{E}\big(\mathbf{E},\mathbf{H}\big) = \frac{1}{2}\int_{\mathbb{R}^3} \mathrm{d}x \, \big(\mathbf{E}(x)\cdot\varepsilon(x)\,\mathbf{E}(x) + \mathbf{H}(x)\cdot\mu(x)\,\mathbf{H}(x)\big)$$

2. *Dynamical equations*

$$\begin{aligned} \varepsilon\,\partial_t\mathbf{E}(t) &= -\nabla_x \times \mathbf{H}(t), & \mathbf{E}(0) &= \mathbf{E} \\ \mu\,\partial_t\mathbf{H}(t) &= +\nabla_x \times \mathbf{E}(t), & \mathbf{H}(0) &= \mathbf{H} \end{aligned}$$

3. *No sources*

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1. *Field energy* $(\mathbf{E}, \mathbf{H}) \in L^2_w(\mathbb{R}^3, \mathbb{C}^6)$ with norm

$$\big\|(\mathbf{E}, \mathbf{H})\big\|^2_{L^2_w} := \int_{\mathbb{R}^3} \mathrm{d}x \, \big(\mathbf{E}(x) \cdot \varepsilon(x)\, \mathbf{E}(x) + \mathbf{H}(x) \cdot \mu(x)\, \mathbf{H}(x)\big)$$

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$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \mathbf{E}(t) \\ \mathbf{H}(t) \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & +\varepsilon^{-1}\,(-\mathrm{i}\nabla_x)^{\times} \\ -\mu^{-1}\,(-\mathrm{i}\nabla_x)^{\times} & 0 \end{pmatrix}}_{=M} \begin{pmatrix} \mathbf{E}(t) \\ \mathbf{H}(t) \end{pmatrix}$$

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$$J := \Big\{(\mathbf{E}, \mathbf{H}) \in L^2_w(\mathbb{R}^3, \mathbb{C}^6) \;\big|\; \nabla_x \cdot \varepsilon \mathbf{E} = 0 \wedge \nabla_x \cdot \mu \mathbf{H} = 0\Big\}$$

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The frequency band picture

$$M \cong M^{\mathcal{Z}} = \int_{\mathbb{B}}^{\oplus} \mathrm{d}k \, M(k)$$
$$= \int_{\mathbb{B}}^{\oplus} \mathrm{d}k \begin{pmatrix} 0 & +\varepsilon^{-1} \, (-\mathrm{i}\nabla_y + k)^{\times} \\ -\mu^{-1} \, (-\mathrm{i}\nabla_y + k)^{\times} & 0 \end{pmatrix}$$

$$\mathfrak{D}(M^{\mathcal{Z}}) = \bigsqcup_{k \in \mathbb{B}} \mathfrak{D}\big(M(k)\big) \subset L^2(\mathbb{B}) \otimes L^2_w(\mathbb{T}^3, \mathbb{C}^6)$$

$M(k)|_{G(k)} = 0 \Rightarrow$ focus on $M(k)|_{J(k)}$

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$$\mathfrak{D}\big(M(k)\big) = H^1(\mathbb{T}^3, \mathbb{C}^6) \cap J(k) \oplus G(k) \subset L^2_w(\mathbb{T}^3, \mathbb{C}^6)$$

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The frequency band picture

Physical bands

$$M(k)\varphi_n(k) = \omega_n(k)\,\varphi_n(k)$$

- Frequency band functions $k \mapsto \omega_n(k)$
- Bloch functions $k \mapsto \varphi_n(k)$
- both $\in \mathcal{C}$, analytic away from band crossings

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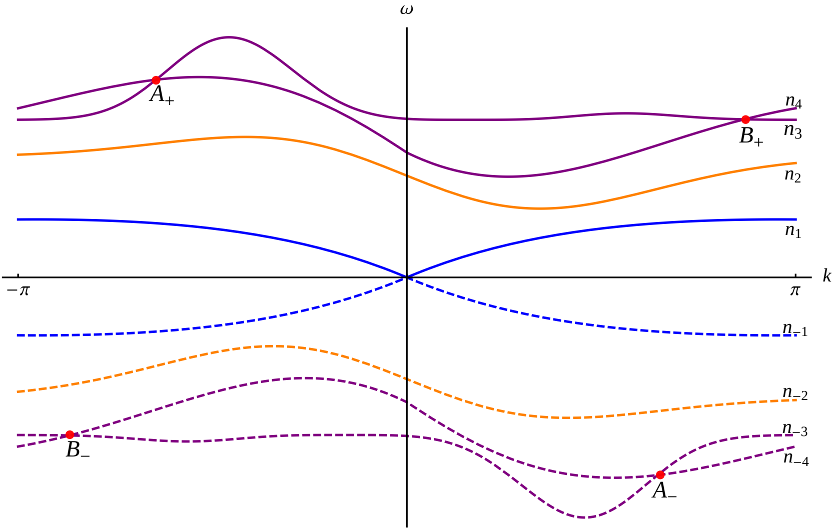

Complex conjugation induces a symmetry

Complex conjugation

$$C : L^2_w(\mathbb{R}^3, \mathbb{C}^6) \longrightarrow L^2_w(\mathbb{R}^3, \mathbb{C}^6),$$
$$(C\Psi)(x) = \overline{\Psi(x)}$$

$$C^{\mathcal{Z}} : L^2(\mathbb{B}) \otimes L^2_w(\mathbb{T}^3, \mathbb{C}^6) \longrightarrow L^2(\mathbb{B}) \otimes L^2_w(\mathbb{T}^3, \mathbb{C}^6),$$
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Complex conjugation induces a symmetry

Particle-hole-type symmetry

$$\left.\begin{aligned} C\varepsilon C &= \varepsilon \\ C\mu C &= \mu \end{aligned}\right\} \Longrightarrow \; C\,M\,C = -M$$

$\rightsquigarrow$ C **not** a time-reversal symmetry

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Symmetrically related bands

$$M(k)\varphi_n(k) = \omega_n(k)\,\varphi_n(k)$$
$$\Longleftrightarrow$$
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Symmetrically related bands

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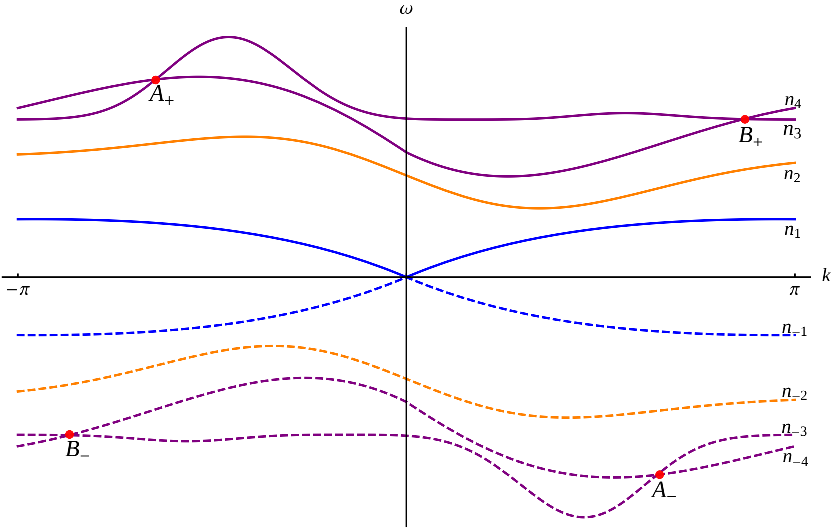

Physical states & their geometry

Definition (Physical states from a narrow range of frequencies)

$$\Pi_0 = \int_{\mathbb{B}}^{\oplus} \mathrm{d}k \, 1_{\sigma_{\mathrm{rel}}(k)}\big(M(k)\big) \text{ so that}$$

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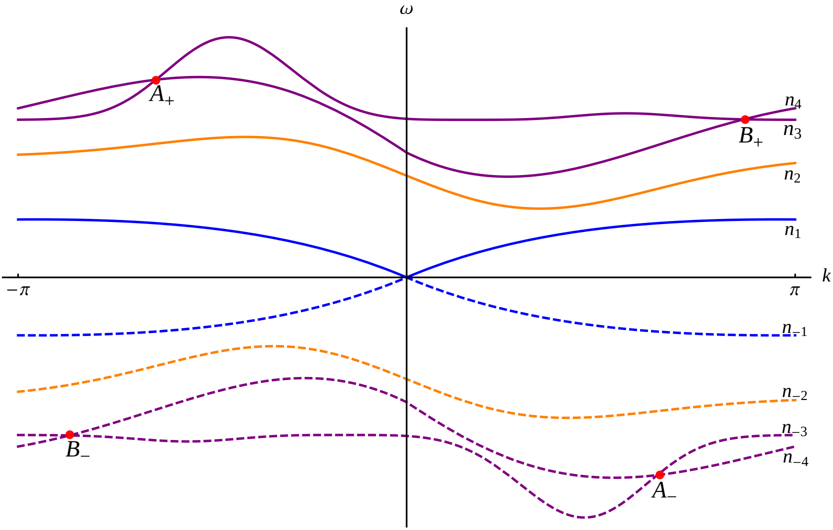

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Physical states & their geometry

Corollary (De Nittis-L. 2013)

Physical states cannot be supported by a single band.

Proof.

$$\psi_{\mathrm{Re}} = \mathrm{Re}^{\,\mathcal{Z}} \varphi_n = \tfrac{1}{2}\big(\varphi_n + C^{\mathcal{Z}} \varphi_n\big),$$
$$\psi_{\mathrm{Im}} = \mathrm{Im}^{\,\mathcal{Z}} \varphi_n = \tfrac{1}{2\mathrm{i}}\big(\varphi_n - C^{\mathcal{Z}} \varphi_n\big).$$

linear combination of Bloch functions to $\omega_n(k)$ and $-\omega_n(-k)$. □

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Physical states & their geometry

Corollary (De Nittis-L. 2013)

$c_1 \big(\lvert \psi_{\mathrm{Re}\,,\mathrm{Im}} \rangle \langle \psi_{\mathrm{Re}\,,\mathrm{Im}} \rvert \big) = 0$ *even though* $c_1 \big(\lvert \varphi_n \rangle \langle \varphi_n \rvert \big) \neq 0$

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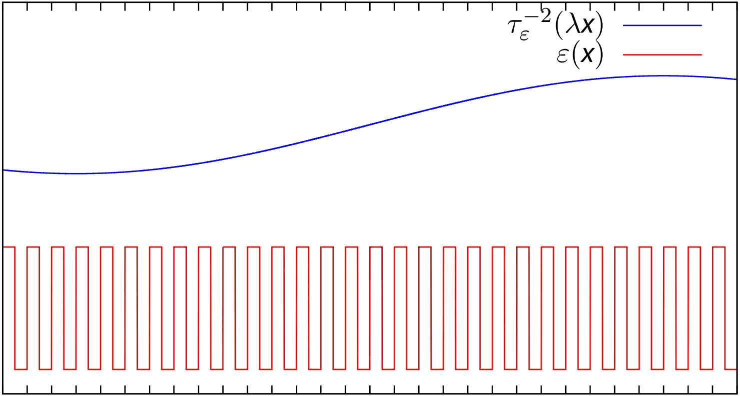
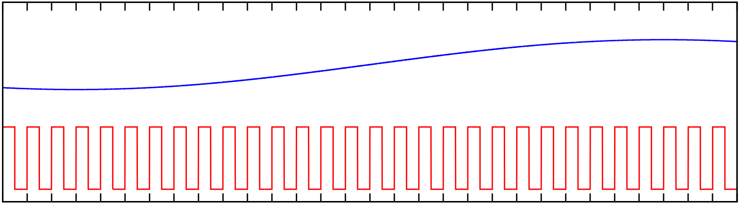
Perturbed photonic crystals



x [lattice constants]

Perturbed photonic crystals

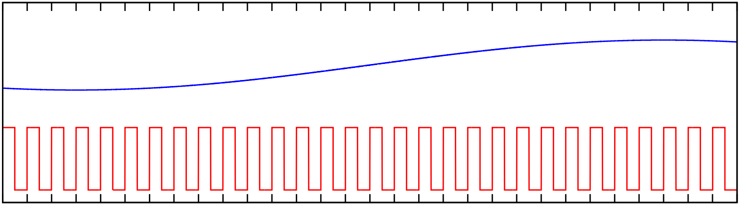


Perturbation of material constants

$$\lambda = \frac{\text{[lattice spacing]}}{\text{[length scale of modulation]}} \ll 1$$

$$\varepsilon(x) \rightsquigarrow \varepsilon_\lambda(x) := \tau_\varepsilon^{-2}(\lambda x)\, \varepsilon(x), \quad \mu(x) \rightsquigarrow \mu_\lambda(x) := \tau_\mu^{-2}(\lambda x)\, \mu(x)$$

Perturbed photonic crystals



Assumption (Slow modulation)

$\tau_\varepsilon, \tau_\mu \in \mathcal{C}^\infty_{\mathsf{b}}(\mathbb{R}^3)$, $\tau_\varepsilon, \tau_\mu \geq c > 0$

Adiabatically perturbed Maxwell operator

$$
\begin{aligned}
M_\lambda &= S_\lambda^{-2} \, M \\
&= \begin{pmatrix} \tau_\varepsilon^2(\lambda x) & 0 \\ 0 & \tau_\mu^2(\lambda x) \end{pmatrix} \begin{pmatrix} 0 & +\varepsilon^{-1}(x) \, (-\mathrm{i}\nabla_x)^\times \\ -\mu^{-1}(x) \, (-\mathrm{i}\nabla_x)^\times & 0 \end{pmatrix}
\end{aligned}
$$

slow modulation & periodic Maxwell operator

existence of **physical states**
in the presence of perturbations

Existence of physical states

Definition (Physical states: unperturbed)

$$\Pi_0 = \int_{\mathbb{B}}^{\oplus} \mathrm{d}k \; 1_{\sigma_{\mathrm{rel}}(k)}\big(M(k)\big) \text{ so that}$$

1. $\sigma_{\mathrm{rel}}(k) = \sigma_{\mathrm{rel}}(-k) = \bigcup_{n \in \mathcal{I}} \{\omega_n(k)\}$ **isolated** family of bands
2. **source free**: $\operatorname{ran} \Pi_0 \subset \mathcal{Z} J$
3. **real**: $C^{\mathcal{Z}} \, \Pi_0 \, C^{\mathcal{Z}} = \Pi_0$

Existence of physical states

Definition (Physical states: perturbed)

$$\Pi_\lambda \cong \int_{\mathbb{B}}^{\oplus} \mathrm{d}k \; 1_{\sigma_{\mathrm{rel}}(k)}\big(M(k)\big) + \mathcal{O}(\lambda) \text{ so that}$$

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Existence of physical states

Theorem (De Nittis-L. 2013)

Suppose the bands $\sigma_{\mathrm{rel}}(k) = \sigma_{\mathrm{rel}}(-k)$ *are isolated and* $0 \notin \sigma_{\mathrm{rel}}(0)$*. Then there exist orthogonal projections*

$$\Pi_\lambda = \Pi_{+,\lambda} + \Pi_{-,\lambda} + \mathcal{O}(\lambda^\infty)$$

so that

$$\big[M_\lambda, \Pi_{\pm,\lambda}\big] = \mathcal{O}(\lambda^\infty)$$

whose ***range*** *supports* ***physical states****.*

Goal

Approximate $\mathrm{e}^{-\mathrm{i}tM_\lambda}$ for **physical** states from a **narrow range of frequencies** up to $\mathcal{O}(\lambda^n)$.

Reduction in complexity: only **few** bands contribute to dynamics

Goal

Approximate $e^{-it M_\lambda}$ for **physical** states from a **narrow range of frequencies** up to $\mathcal{O}(\lambda^n)$.

Reduction in complexity: only **few** bands contribute to dynamics

Effective light dynamics

Theorem (De Nittis-L. 2013)

*Suppose the bands $\sigma_{\mathrm{rel}}(k) = \sigma_{\mathrm{rel}}(-k)$ are isolated and $0 \notin \sigma_{\mathrm{rel}}(0)$. Then there exist a unitary V_λ and an **effective Maxwell operator***

$$M_{\mathrm{eff}} = V_\lambda^{-1} \, \mathrm{Op}_\lambda(\mathcal{M}_{\mathrm{eff}}) \, V_\lambda$$

which approximates the full light dynamics,

$$\mathrm{e}^{-\mathrm{i}t M_\lambda^{\mathcal{Z}}} \, \Pi_\lambda \, \mathrm{Re}\,^{\mathcal{Z}} = \mathrm{Re}\,^{\mathcal{Z}} \, \mathrm{e}^{-\mathrm{i}t M_{\mathrm{eff}}} \, \Pi_\lambda \, \mathrm{Re}\,^{\mathcal{Z}} + \mathcal{O}(\lambda^\infty),$$
$$\mathrm{e}^{-\mathrm{i}t M_\lambda^{\mathcal{Z}}} \, \Pi_\lambda \, \mathrm{Im}\,^{\mathcal{Z}} = \mathrm{Im}\,^{\mathcal{Z}} \, \mathrm{e}^{-\mathrm{i}t M_{\mathrm{eff}}} \, \Pi_\lambda \, \mathrm{Im}\,^{\mathcal{Z}} + \mathcal{O}(\lambda^\infty),$$

and leaves $\mathrm{ran}\,\Pi_\lambda$ invariant up to $\mathcal{O}(\lambda^\infty)$.

Π_λ, V_λ and $\mathcal{M}_{\mathrm{eff}}$ can be computed explicitly order-by-order in λ

Effective light dynamics

Theorem (De Nittis-L. 2013)

*Suppose the bands $\sigma_{\mathrm{rel}}(k) = \sigma_{\mathrm{rel}}(-k)$ are isolated and $0 \notin \sigma_{\mathrm{rel}}(0)$. Then there exist a unitary V_λ and an **effective Maxwell operator***

$$M_{\mathrm{eff}} = V_\lambda^{-1} \, \mathrm{Op}_\lambda(\mathcal{M}_{\mathrm{eff}}) \, V_\lambda$$

which approximates the full light dynamics,

$$\mathrm{e}^{-itM_\lambda^{\mathcal{Z}}} \, \Pi_\lambda \, \mathrm{Re}\,^{\mathcal{Z}} = \mathrm{Re}\,^{\mathcal{Z}} \, \mathrm{e}^{-itM_{\mathrm{eff}}} \, \Pi_\lambda \, \mathrm{Re}\,^{\mathcal{Z}} + \mathcal{O}(\lambda^\infty),$$
$$\mathrm{e}^{-itM_\lambda^{\mathcal{Z}}} \, \Pi_\lambda \, \mathrm{Im}\,^{\mathcal{Z}} = \mathrm{Im}\,^{\mathcal{Z}} \, \mathrm{e}^{-itM_{\mathrm{eff}}} \, \Pi_\lambda \, \mathrm{Im}\,^{\mathcal{Z}} + \mathcal{O}(\lambda^\infty),$$

and leaves ran Π_λ invariant up to $\mathcal{O}(\lambda^\infty)$.

Π_λ, V_λ and $\mathcal{M}_{\mathrm{eff}}$ can be computed explicitly order-by-order in λ

Effective light dynamics

Corollary ("Peierl's substitution", De Nittis-L. 2013)

In case $\sigma_{\mathrm{rel}}(k) = \bigl\{ \omega(k), -\omega(-k) \bigr\}$ *consists of two* topologically trivial *bands, then the symbol to the pseudodifferential operator is*

$$\mathcal{M}_{\mathrm{eff}}(r,k) = \tau_{\varepsilon}(r)\,\tau_{\mu}(r) \begin{pmatrix} \omega(k) & 0 \\ 0 & -\omega(-k) \end{pmatrix} + \mathcal{O}(\lambda).$$

$\Rightarrow$ motivates the definition of a **Maxwell-Harper operator**

$\rightsquigarrow$ Hofstadter butterfly?

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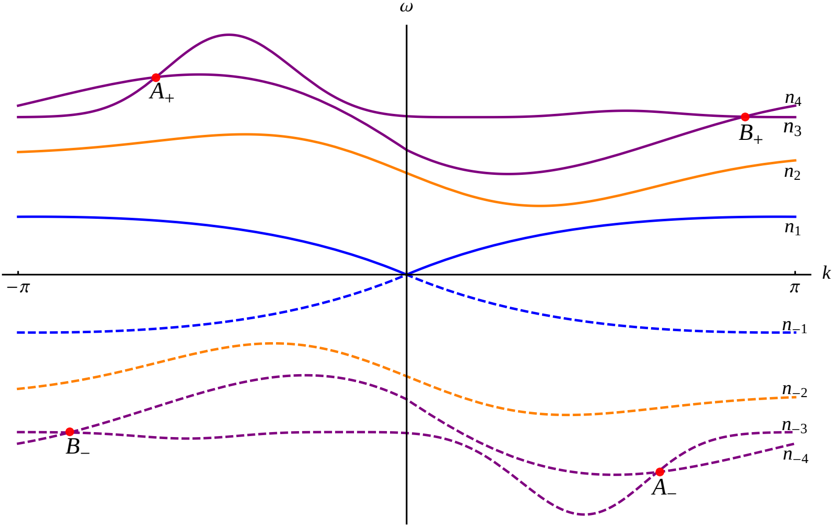
Effective single-band dynamics

Simplest case:

semiclassical dynamics

aka ray optics

Effective single-band dynamics



Effective single-band dynamics

Theorem (De Nittis-L. 2013)

For all suitable observables f, the following Egorov-type theorem holds:

$$\Big\| \Pi_{\pm,\lambda} \Big(\mathrm{e}^{+\mathrm{i}t M_\lambda^{\mathcal{Z}}} \, \mathrm{Op}_\lambda(f) \, \mathrm{e}^{-\mathrm{i}t M_\lambda^{\mathcal{Z}}} - \mathrm{Op}_\lambda\big(f \circ \Phi_t^{\pm}\big) \Big) \Pi_{\pm,\lambda} \Big\| = \mathcal{O}(\lambda^2)$$

$\Phi^{\pm}$ *flow associated to positive/negative frequency band*

$\Phi_t^+(r,k)$ and $\Phi_t^-(r,k) = \Phi_{-t}^+(r,-k)$ related by symmetry

Effective single-band dynamics

Theorem (De Nittis-L. 2013)

$\Phi^{\pm}$ *is the flow associated to the* ray optics equations

$$\begin{pmatrix} 0 & -\mathrm{id} + \lambda\,\Omega^{\pm}_{rk} \\ +\mathrm{id} + \lambda\,\Omega^{\pm}_{kr} & +\lambda\,\Omega^{\pm}_{kk} \end{pmatrix} \begin{pmatrix} \dot{r} \\ \dot{k} \end{pmatrix} = \begin{pmatrix} \nabla_r \mathcal{M}^{\pm}_{\mathrm{sc}} \\ \nabla_k \mathcal{M}^{\pm}_{\mathrm{sc}} \end{pmatrix}$$

$\Omega^{\pm}$ *generalized Berry curvature*

Effective single-band dynamics

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Effective single-band dynamics

Theorem (De Nittis-L. 2013)

Semiclassical Maxwellian

$$\begin{aligned}\mathcal{M}_{\mathrm{sc}}^{\pm} = \tau_{\varepsilon}\,\tau_{\mu}\,\omega_{\pm} + \\ + \lambda\Big(\mathrm{i}\,\tau_{\varepsilon}\,\big\langle \varphi_{\pm}^{E}, \nabla_{r}\tau_{\mu} \times \varphi_{\pm}^{H}\big\rangle - \mathrm{i}\,\tau_{\mu}\,\big\langle \varphi_{\pm}^{H}, \nabla_{r}\tau_{\varepsilon} \times \varphi_{\pm}^{E}\big\rangle + \\ - \tfrac{\mathrm{i}}{2}\Big\langle \varphi_{\pm}, S\,\big\{S^{-2}\,M(\cdot) - \tau_{\varepsilon}\,\tau_{\mu}\,\omega_{\pm}\big\}\,S^{-1}\varphi_{\pm}\Big\rangle\Big)\end{aligned}$$

Symbol M_{λ} and analog of Rammal-Wilkinson term

Effective single-band dynamics

But: single bands **cannot** support real states

Semiclassical twin-band dynamics

Consider **twin-band case**

$$\sigma_{\mathrm{rel}}(k) = \{\omega(k), -\omega(-k)\}$$

Semiclassical twin-band dynamics

Writing $\Pi_\lambda = \Pi_{+,\lambda} + \Pi_{-,\lambda} + \mathcal{O}(\lambda^\infty)$ leads to equations for ψ_{Re} :

$$\mathrm{Re}^{\mathcal{Z}} \, \Pi_\lambda \, \mathrm{e}^{+\mathrm{i}\frac{t}{\lambda} M_\lambda^{\mathcal{Z}}} \, \mathrm{Op}(f) \, \mathrm{e}^{-\mathrm{i}\frac{t}{\lambda} M_\lambda^{\mathcal{Z}}} \, \Pi_\lambda \, \mathrm{Re}^{\mathcal{Z}} = \\ = F_+(t) + F_-(t) + F_{\mathrm{int}}(t)$$

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In-band contribution:

$$F_\pm(t) = \mathrm{Re}^{\,\mathcal{Z}}\,\Pi_{\pm,\lambda}\,\mathrm{e}^{+\mathrm{i}\frac{t}{\lambda}M^{\mathcal{Z}}_\lambda}\,\mathrm{Op}(f)\,\mathrm{e}^{-\mathrm{i}\frac{t}{\lambda}M^{\mathcal{Z}}_\lambda}\,\Pi_{\pm,\lambda}\,\mathrm{Re}^{\,\mathcal{Z}}$$
$$= \Pi_{\pm,\lambda}\,\mathrm{Op}_\lambda\big(f\circ\Phi^\pm_t\big)\,\Pi_{\pm,\lambda} + \mathcal{O}(\lambda^2)$$

$\Rightarrow$ Can be approximated with **semiclassical flow** $\Phi^\pm_t$

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Band transitions:

$$F_{\mathrm{int}}(t) = \mathrm{Re}^{\mathcal{Z}} \, \Pi_{+,\lambda} \, \mathrm{e}^{+\mathrm{i}\frac{t}{\lambda} M^{\mathcal{Z}}_\lambda} \, \mathrm{Op}(f) \, \mathrm{e}^{-\mathrm{i}\frac{t}{\lambda} M^{\mathcal{Z}}_\lambda} \, \Pi_{-,\lambda} \, \mathrm{Re}^{\mathcal{Z}} \\ + \mathrm{Re}^{\mathcal{Z}} \, \Pi_{-,\lambda} \, \mathrm{e}^{+\mathrm{i}\frac{t}{\lambda} M^{\mathcal{Z}}_\lambda} \, \mathrm{Op}(f) \, \mathrm{e}^{-\mathrm{i}\frac{t}{\lambda} M^{\mathcal{Z}}_\lambda} \, \Pi_{+,\lambda} \, \mathrm{Re}^{\mathcal{Z}} \\ = \mathcal{O}(\lambda) \qquad \text{(easy)}$$

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(relative phases remain locked)

Semiclassical twin-band dynamics

Conjecture

$F_{\mathrm{int}}(t) = \mathcal{O}(\lambda^2)$, and thus ray optics dynamics approximate light dynamics up to $\mathcal{O}(\lambda^2)$.

$\rightsquigarrow$ Work in progress

Semiclassical twin-band dynamics

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Ray optics: Notable previous results

*All **non-rigorous**, **single-band** results:*

1. *Haldane & Raghu* (Phys. Rev. A 78, 033834 (2008))
   - »derivation by analogy«
   - necessity of slow variation recognized, but small parameter λ not used
   - $\rightsquigarrow$ eom are missing $\mathcal{O}(\lambda)$ terms, **only leading-order correct**
2. *Esposito & Gerace* (Phys. Rev. A 88, 013853 (2013))
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   - semiclassical states $\Psi(r, k, z)$ parametrized by $(r, k) \in T^*\mathbb{R}^3$, $z \in S^2 \rightsquigarrow$ find extremals of functional
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Recap

1. Maxwell $\approx$ quantum? Question of dynamics!
2. $\rightsquigarrow$ Rewrite Maxwell equations as Schrödinger equation
3. Effective light dynamics: *bona fide* multiband problem
4. Complex conjugation: particle hole-type symmetry, **not** time-reversal symmetry
5. In non-gyroscopic PhCs: frequency bands and Bloch functions come in *conjugate pairs*
6. For ray optics: control of band transition term

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Thank You for your attention!

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