$\alpha - z$ relative Renyi entropies

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Quantum Relative Entropy

-- a fundamental quantity in Quantum Mechanics & Quantum Information Theory :

The quantum relative entropy of ρ w.r.t σ,

$$\rho \geq 0,\ \mathrm{Tr}\,\rho = 1; \qquad \sigma \geq 0:$$

(density matrix/state)

$$\boxed{D(\rho \,\|\, \sigma) := \mathrm{Tr}\,(\rho \log \rho) - \mathrm{Tr}\,(\rho \log \sigma)}$$

$$\log \equiv \log_2$$

well-defined if $\mathrm{supp}\,\rho \subseteq \mathrm{supp}\,\sigma$

- **Classical counterpart:**

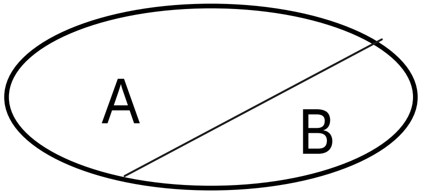
$$\boxed{D(p \,\|\, q) := \sum_{x \in X} p_x \log \frac{p_x}{q_x};} \qquad p = \{p_x\}_{x \in X};\ q = \{q_x\}_{x \in X}$$

- $D(\rho \| \sigma)$ acts as a parent quantity for *von Neumann entropy*:

$$\boxed{S(\rho) := -\mathrm{Tr}\,(\rho \log \rho) \quad = -D(\rho \| I)} \qquad (\sigma = I)$$

- It also acts as a parent quantity for other entropies

ρ_{AB}



  - *Conditional entropy*

$$\boxed{S(A|B)_\rho := S(\rho_{AB}) - S(\rho_B) = -D(\rho_{AB} \| I_A \otimes \rho_B)}$$

$$\rho_B = \mathrm{Tr}_A\, \rho_{AB}$$

  - *Mutual information*

$$\boxed{I(A:B)_\rho := S(\rho_A) + S(\rho_B) - S(\rho_{AB}) = D(\rho_{AB} \| \rho_A \otimes \rho_B)}$$

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Some Properties of $D(\rho \| \sigma)$

“distance”

$$\boxed{\begin{array}{l} D(\rho \| \sigma) \geq 0 \qquad \rho, \sigma \text{ states} \\ \qquad = 0 \text{ if \& only if } \rho = \sigma \end{array}}$$

- Joint convexity:

For two mixtures of states $\rho = \sum_{i=1}^{n} p_i \rho_i$ & $\sigma = \sum_{i=1}^{n} p_i \sigma_i$

$$\boxed{D(\rho \| \sigma) \leq \sum_{i=1}^{n} p_i D(\rho_i \| \sigma_i)}$$

- Invariance under

joint unitaries

$$\boxed{D(U \rho U^* \| U \sigma U^*) = D(\rho \| \sigma)}$$

- Data-processing inequality $\equiv$ Monotonicity under quantum operations

Quantum operation: any allowed physical process on a quantum-mechanical system

Most general description given by

a completely positive trace-preserving **(CPTP)** map (Λ)

- Data-processing inequality **(DPI)**

$$\boxed{D(\Lambda(\rho)\,||\,\Lambda(\sigma)) \leq D(\rho\,||\,\sigma)}$$

This is a fundamental property for any relative entropy

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Significance of the quantum relative entropy in Quantum Information Theory

It acts as a parent quantity for optimal rates of information-processing tasks e.g.

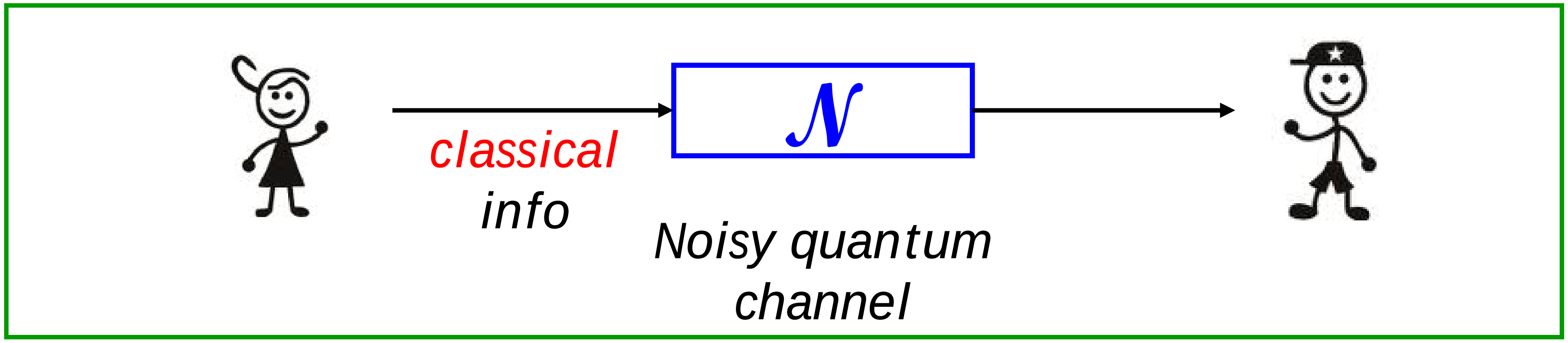
- data compression,
- transmission of information through a channel **etc.**

in the *"asymptotic memoryless setting"*

information sources & channels are assumed to be

- memoryless
- available for infinite number of uses (asymptotic limit) $(n \to \infty)$

E.g. Transmission of classical information



classical info

$\mathcal{N}$

Noisy quantum channel

Optimal rate (of classical information transmission):

classical capacity

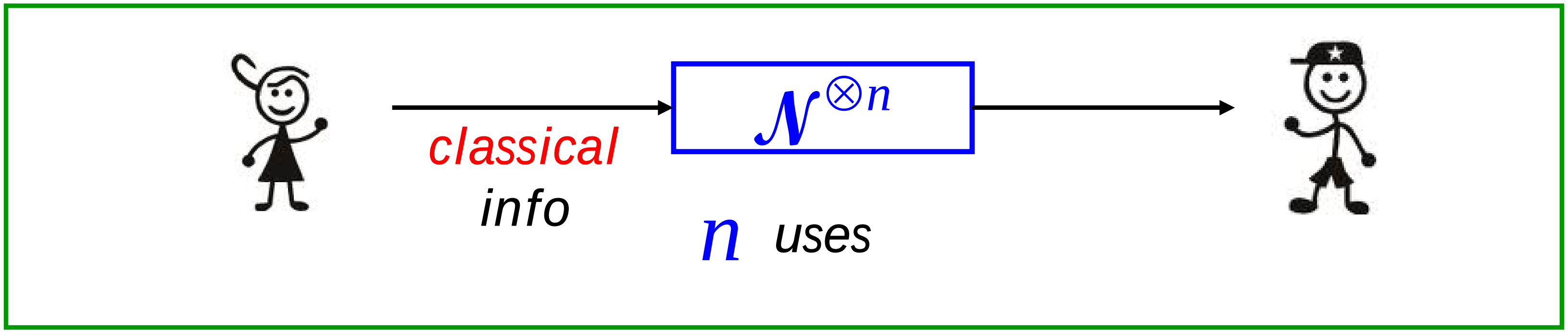
$C(\mathcal{N}) =$ maximum number of bits transmitted per use of $\mathcal{N}$

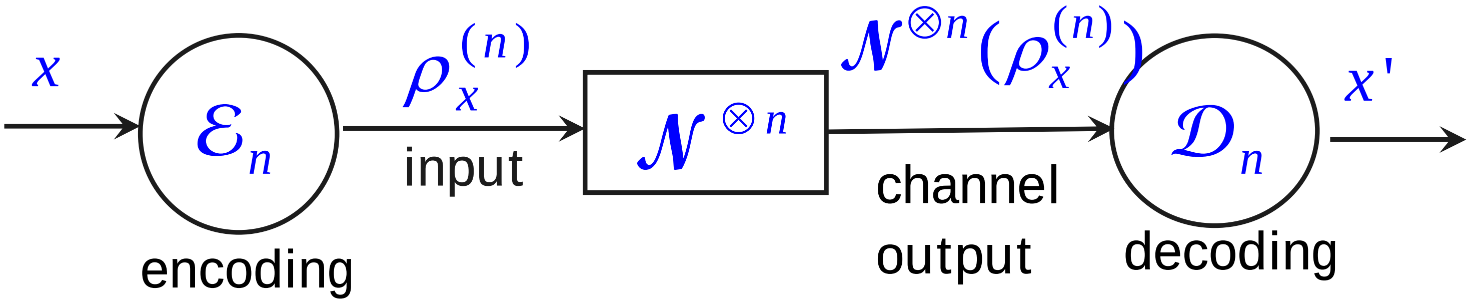
memoryless: there is no correlation in the noise acting on successive inputs

$\mathcal{N}^{\otimes n}$: n successive uses of the channel; independent

“asymptotic, memoryless setting”

- *To evaluate* $C(\mathcal{N})$:





- One requires : **prob. of error** $p_e^{(n)} \to 0$ as $n \to \infty$

$C(\mathcal{N})$: *Optimal rate of reliable information transmission*

-given in terms of a mutual information:

(obtainable from the relative entropy)

Other important relative entropies

(1) $\alpha -$ relative Renyi entropies: $\alpha - \mathrm{RRE}$

(2) Max- and min-relative entropies

(1) $\alpha -$ *relative Renyi entropies:* $\alpha - \mathrm{RRE}$

$$D_\alpha(\rho \| \sigma) := \frac{1}{\alpha - 1} \log\left[\mathrm{Tr}(\rho^\alpha \sigma^{1-\alpha})\right]$$

$$\lim_{\alpha \to 1} D_\alpha(\rho \| \sigma) = D(\rho \| \sigma)$$

*

- *Also is of important operational significance,*

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(2) Max- and Min- relative entropies

- *Max-relative entropy [ND 2008]*

$$D_{\max}(\rho \| \sigma) := \inf \left\{ \gamma : \rho \leq 2^{\gamma} \sigma \right\}$$

- *Min-relative entropy [Renner et al 2012]*

$$D_{\min}(\rho \| \sigma) := -2 \log \| \sqrt{\rho} \sqrt{\sigma} \|_1$$

*

Properties of the min-max relative entropies

$$\boxed{\text{for } * = \max, \min}$$

- Positivity: If $\rho, \sigma \in \mathcal{D}(\mathcal{H})$,

$$\boxed{D_*(\rho \| \sigma) \geq 0}$$

just as $D(\rho \| \sigma)$

- Data-processing inequality:

$$\boxed{D_*(\Lambda(\rho) \| \Lambda(\sigma)) \leq D_*(\rho \| \sigma)}$$

for any CPTP map Λ

- Invariance under joint unitaries:

$$\boxed{D_*(U\rho U^\dagger \| U\sigma U^\dagger) = D_*(\rho \| \sigma)}$$

for any unitary operator U

- *Interestingly,*

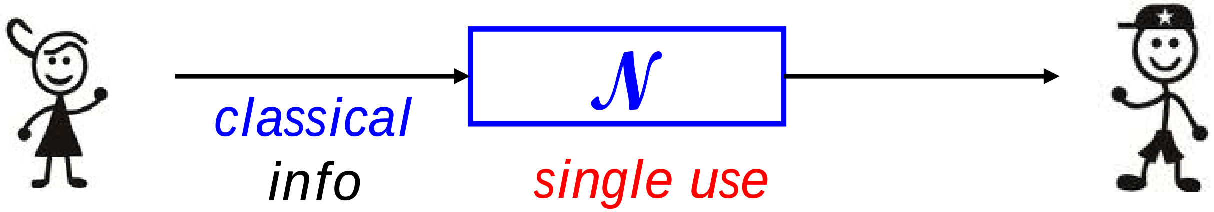
$$\boxed{D_{\min}(\rho \| \sigma) \leq D(\rho \| \sigma) \leq D_{\max}(\rho \| \sigma)} \quad *$$

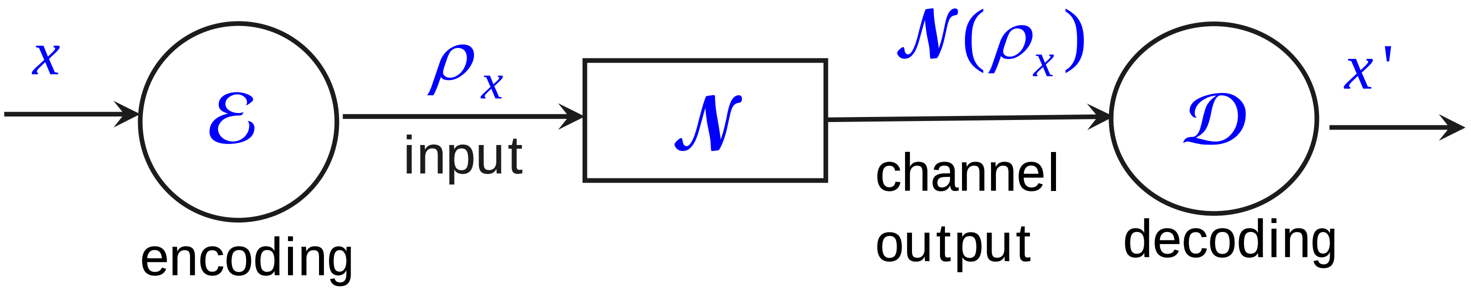
Operational significance of the Max- and Min- relative entropies in:

One-shot Information Theory [Renner; ND]

~~*asymptotic, memoryless*~~

One-shot information theory





One-shot classical capacity := *max. number of bits that can be transmitted on a single use*

$C^{(1)}(\mathcal{N})$:given in terms of a mutual information obtained from the *max-relative entropy*

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In Summary……

- there is a plethora of different entropic quantities which arise in Quantum Information theory

-- which are interesting both from the mathematical and operational points of view;

- hence it is desirable to have a

  unifying mathematical framework

  for the study of these different quantities.

- Recently, such a framework was partially provided:

by a non-commutative generalization of the $\alpha-\mathrm{RRE}$

[Wilde et al; Muller-Lennert et al]

$$\tilde{D}_{\alpha}(\rho\,\|\,\sigma) := \frac{1}{\alpha-1}\log \mathrm{Tr}\left[\left(\sigma^{\frac{(1-\alpha)}{2\alpha}}\rho\sigma^{\frac{(1-\alpha)}{2\alpha}}\right)\right]^{\alpha}$$

$\alpha-$ *Quantum Renyi Divergence*

(sandwiched Renyi entropy)

$\alpha -$ *Quantum Renyi Divergence*

- Recently, such a framework was partially provided:

by a non-commutative generalization $\alpha - \mathrm{RRE}$

[Wilde et al; Muller-Lennert et al]

$\alpha - \mathrm{QRD}$

$$\tilde{D}_{\alpha}(\rho \| \sigma) := \frac{1}{\alpha - 1} \log \mathrm{Tr}\left[\left(\sigma^{\frac{(1-\alpha)}{2\alpha}} \rho \sigma^{\frac{(1-\alpha)}{2\alpha}} \right) \right]^{\alpha}$$

$$\Downarrow$$

IF $[\rho, \sigma] = 0$ **THEN**

$$\mathrm{Tr}\left[\rho \sigma^{\frac{(1-\alpha)}{\alpha}} \right]^{\alpha}$$

$$\Downarrow$$

$$\mathrm{Tr}(\rho^{\alpha} \sigma^{1-\alpha})$$

$\alpha -$ Quantum Renyi Divergence

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$\Downarrow$

$$\mathrm{Tr} \left[\rho \sigma^{\frac{(1-\alpha)}{\alpha}} \right]^{\alpha}$$

$\Downarrow$

IF $[\rho, \sigma] = 0$ $\Downarrow$ **THEN**

$\alpha - \mathrm{RRE}$

$$D_{\alpha}(\rho \| \sigma) = \frac{1}{\alpha - 1} \log \mathrm{Tr}(\rho^{\alpha} \sigma^{1-\alpha})$$

"super-parent":

$$\begin{tikzcd}
D_{\min}(\rho\|\sigma) & D(\rho\|\sigma) & D_{\max}(\rho\|\sigma) \\
 & \tilde{D}_\alpha(\rho\|\sigma) \arrow[ul, "\alpha = \frac{1}{2}"'] \arrow[u, "\alpha \to 1"] \arrow[ur, "\alpha \to \infty"'] \arrow[d, "\textbf{if} \quad [\rho,\sigma]=0"] & \\
 & \alpha - \mathrm{RRE} \quad D_\alpha(\rho\|\sigma) &
\end{tikzcd}$$

- Properties of

$$\tilde{D}_\alpha(\rho\|\sigma) \longrightarrow$$

properties of

$$D_{\min}(\rho\|\sigma), \;\; D(\rho\|\sigma)$$
$$D_{\max}(\rho\|\sigma)$$

“*super-parent*”:

$$\begin{tikzcd}
D_{\min}(\rho\|\sigma) & D(\rho\|\sigma) & D_{\max}(\rho\|\sigma) \\
& \tilde{D}_\alpha(\rho\|\sigma) \arrow[ul, "\alpha = \frac{1}{2}"'] \arrow[u, "\alpha \to 1"] \arrow[ur, "\alpha \to \infty"'] &
\end{tikzcd}$$

- *Joint convexity of* $\tilde{D}_\alpha(\rho\|\sigma)$ *for* $\frac{1}{2} \le \alpha \le 1$

  [Frank & Lieb] $\Rightarrow$ *joint convexity of* $D_{\min}(\rho\|\sigma)$, $D(\rho\|\sigma)$

- $\tilde{D}_\alpha(\rho\|\sigma)$ *monotonically increasing in* α

  $D_\alpha(\rho\|\sigma) \le D_\beta(\rho\|\sigma)$,
  for $\alpha \le \beta$.

  [Muller-Lennert et al] $\Rightarrow$ $D_{\min}(\rho\|\sigma) \le D(\rho\|\sigma) \le D_{\max}(\rho\|\sigma)$

- *Data-processing inequality for* $\tilde{D}_\alpha(\rho\|\sigma)$ *for* $\alpha \ge \frac{1}{2}$

  [Frank & Lieb; Beigi] $\Rightarrow$ DPI for $D_{\min}(\rho\|\sigma), D(\rho\|\sigma), D_{\max}(\rho\|\sigma)$

Limitations of the $\alpha - \mathrm{QRD}$

- The data-processing inequality is not satisfied for $\alpha \in \left(0, \tfrac{1}{2}\right)$

- The important family of $\alpha - \mathrm{RRE}$ can only be obtained from the $\alpha - \mathrm{QRD}$ in the special case of commuting operators

➡

(Q) Can one define a more general family of relative entropies which overcomes these limitations ?

(A) Yes!

The more general family is a……

- Two-parameter family of relative entropies

$$D_{\alpha,z}(\rho \| \sigma); \quad \alpha, z \in \mathbb{R}$$

$\alpha - z$ relative Renyi entropies: $\alpha - z - \mathrm{RRE}$

- They stem from quantum entropic functionals defined by

Jaksic, Ogata, Pautrat and Pillet

for the study of entropic fluctuations in non-equilibrium statistical mechanics

ρ: reference state of a dynamical system

$\sigma \equiv \rho_t$: state resulting from ρ due to time evolution under the action of a Hamiltonian for a time t. *

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$\alpha - z - \mathrm{RRE}$

Definition:

$\forall \rho \in \mathcal{D}(\mathcal{H});\ \sigma \in \mathcal{P}(\mathcal{H}):\ \mathrm{supp}\ \rho \subseteq \mathrm{supp}\ \sigma$

$$D_{\alpha,z}(\rho\|\sigma) = \frac{1}{\alpha-1}\log f_{\alpha,z}(\rho\|\sigma)$$

with the trace functional

$$f_{\alpha,z}(\rho\|\sigma) = \mathrm{Tr}\left(\rho^{\frac{\alpha}{z}}\sigma^{\frac{(1-\alpha)}{z}}\right)^{z}$$

$$= \mathrm{Tr}\left(\sigma^{\frac{(1-\alpha)}{2z}}\rho^{\frac{\alpha}{z}}\sigma^{\frac{(1-\alpha)}{2z}}\right)^{z}$$

$$= \mathrm{Tr}\left(\rho^{\frac{\alpha}{2z}}\sigma^{\frac{(1-\alpha)}{z}}\rho^{\frac{\alpha}{2z}}\right)^{z}$$

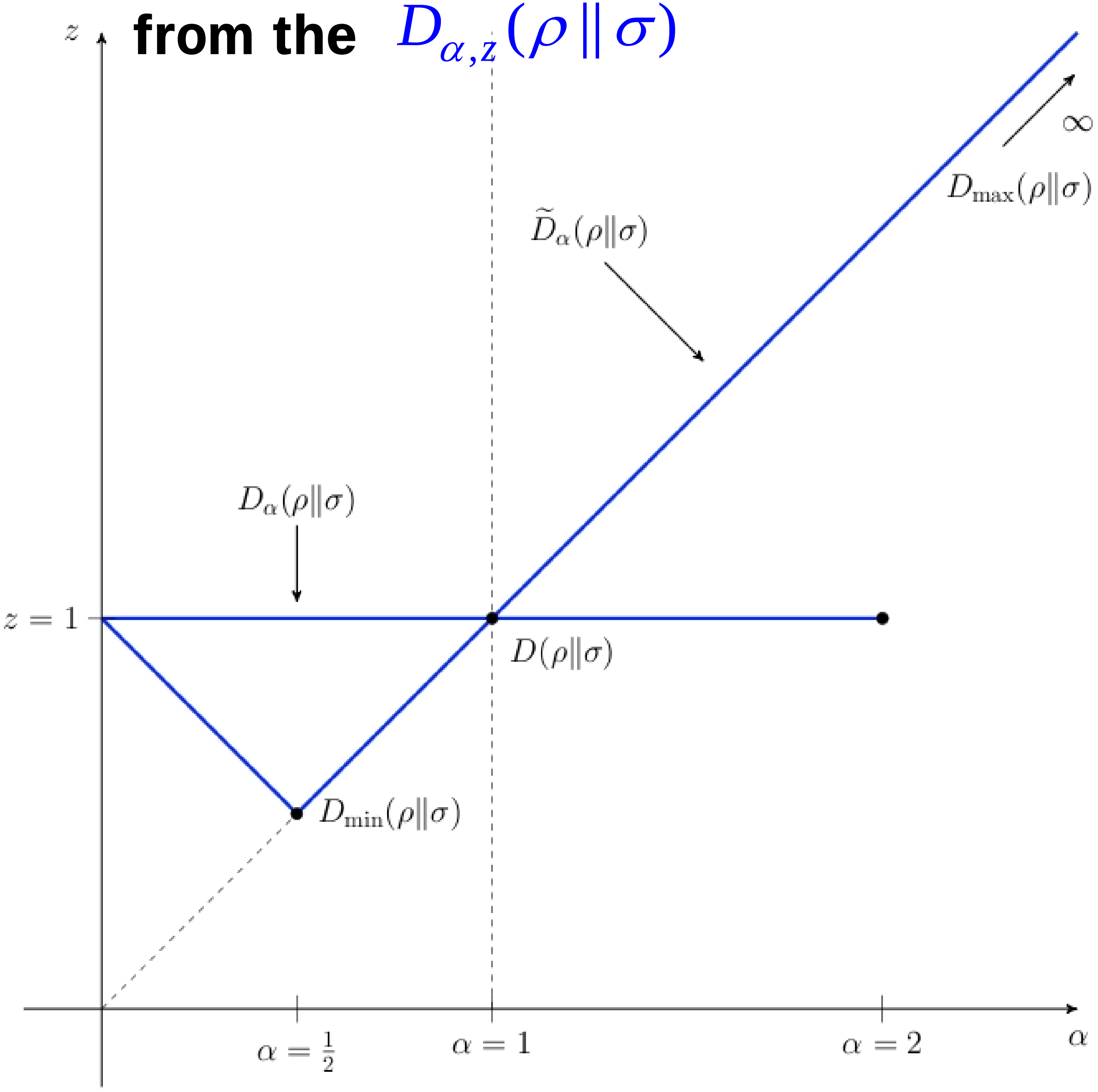
*

$\alpha, z \in \mathbb{R}$

Take limits for

$\alpha \to 1; \quad z \to 0$

Retrieving all other relative entropies from the $D_{\alpha,z}(\rho\|\sigma)$



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Quantum Renyi axioms for a relative entropy

- **Unitary invariance :** $D_{\alpha,z}(\rho\|\sigma)=D_{\alpha,z}(U\rho U^{\dagger}\|U\sigma U^{\dagger})$

- **Tensor property:**

$$D_{\alpha,z}(\rho\otimes\kappa\|\sigma\otimes\omega)=D_{\alpha,z}(\rho\|\sigma)+D_{\alpha,z}(\kappa\|\omega)$$

*

- **Order Axiom:** $\rho\geq\sigma \;\Rightarrow\; D_{\alpha,z}(\rho\|\sigma)\geq 0$

$\forall z\geq|\alpha-1|$ $\rho\leq\sigma\Rightarrow\; D_{\alpha,z}(\rho\|\sigma)\leq 0$

etc.

Order Axiom

$$\forall z \geq |\alpha - 1| \quad \rho \geq \sigma \quad \Rightarrow \quad D_{\alpha,z}(\rho \| \sigma) \geq 0$$

- **Proof:**

Let $0 < \alpha < 1$

$$D_{\alpha,z}(\rho \| \sigma) = \frac{1}{\alpha - 1} \log f_{\alpha,z}(\rho \| \sigma)$$

- **r.t.p.** $\rho \geq \sigma \quad \Rightarrow \quad \log f_{\alpha,z}(\rho \| \sigma) \ \leq 0$

$$\rho \geq \sigma \quad \Rightarrow f_{\alpha,z}(\rho \| \sigma) \ \leq 1$$

r.t.p.

$$\rho \geq \sigma \quad \Rightarrow f_{\alpha,z}(\rho \| \sigma) \ \leq f_{\alpha,z}(\rho \| \rho)$$

$$\because \ f_{\alpha,z}(\rho \| \rho)) = 1$$

$0 < \alpha < 1$ **r.t.p.** $\rho \geq \sigma \Rightarrow f_{\alpha,z}(\rho \| \sigma) \leq f_{\alpha,z}(\rho \| \rho)$

………(a)

$$\mathrm{Tr}\left[\rho^{\frac{\alpha}{z}}\sigma^{\frac{(1-\alpha)}{z}}\right]^{z} \leq \mathrm{Tr}\left[\rho^{\frac{\alpha}{z}}\rho^{\frac{(1-\alpha)}{z}}\right]^{z}$$

$$\boxed{\nu = \frac{(1-\alpha)}{z}}$$

$$\mathrm{Tr}\left[\rho^{\frac{\alpha}{z}}\sigma^{\nu}\right]^{z} \leq \mathrm{Tr}\left[\rho^{\frac{\alpha}{z}}\rho^{\nu}\right]^{z}$$

For $0 < \nu < 1$, , x^{ν} is operator monotone: $\rho \geq \sigma \Rightarrow \rho^{\nu} \geq \sigma^{\nu}$

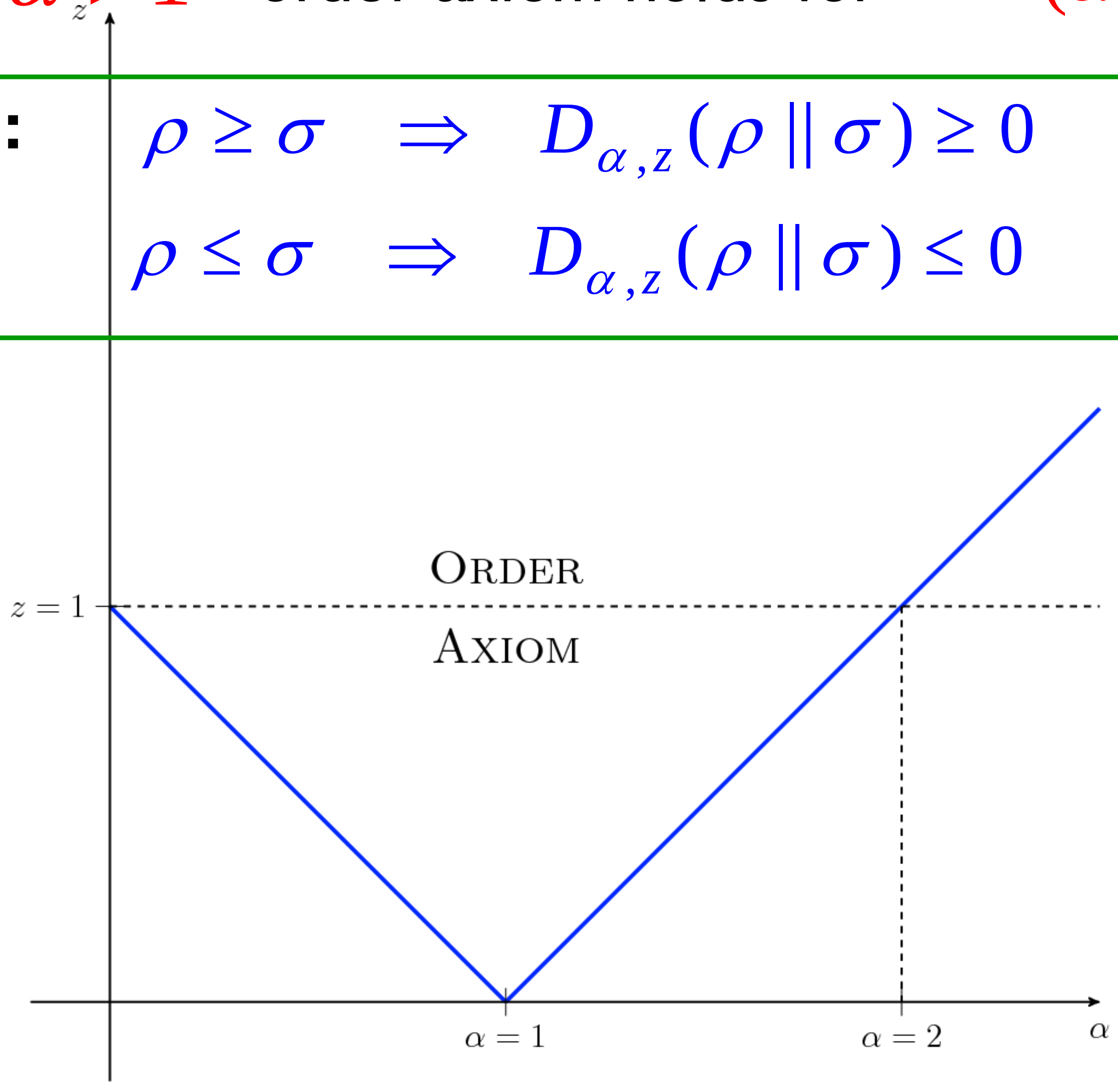
$$f_{\alpha,z}(\rho \| \rho) = \mathrm{Tr}\left[\rho^{\frac{\alpha}{z}}\rho^{\nu}\right]^{z} \geq \mathrm{Tr}\left[\rho^{\frac{\alpha}{z}}\sigma^{\nu}\right]^{z} = f_{\alpha,z}(\rho \| \sigma)$$

$\therefore$ **(a)** holds if $0 < \nu < 1$, i.e. if $\frac{(1-\alpha)}{z} < 1$, i.e. $z > (1-\alpha)$

- For $0 < \alpha < 1$ order axiom holds for $z > (1 - \alpha)$
- Similarly, for $\alpha > 1$ order axiom holds for $z > (\alpha - 1)$
- $\therefore$ **Order Axiom:** $\forall z \geq |\alpha - 1|$

$$\rho \geq \sigma \quad \Rightarrow \quad D_{\alpha,z}(\rho \| \sigma) \geq 0$$

$$\rho \leq \sigma \quad \Rightarrow \quad D_{\alpha,z}(\rho \| \sigma) \leq 0$$



Data-processing inequality (DPI)

$$\forall \Lambda : \mathrm{CPTP} \quad \boxed{D_{\alpha,z}(\Lambda(\rho) \| \Lambda(\sigma)) \leq D_{\alpha,z}(\rho \| \sigma)}$$

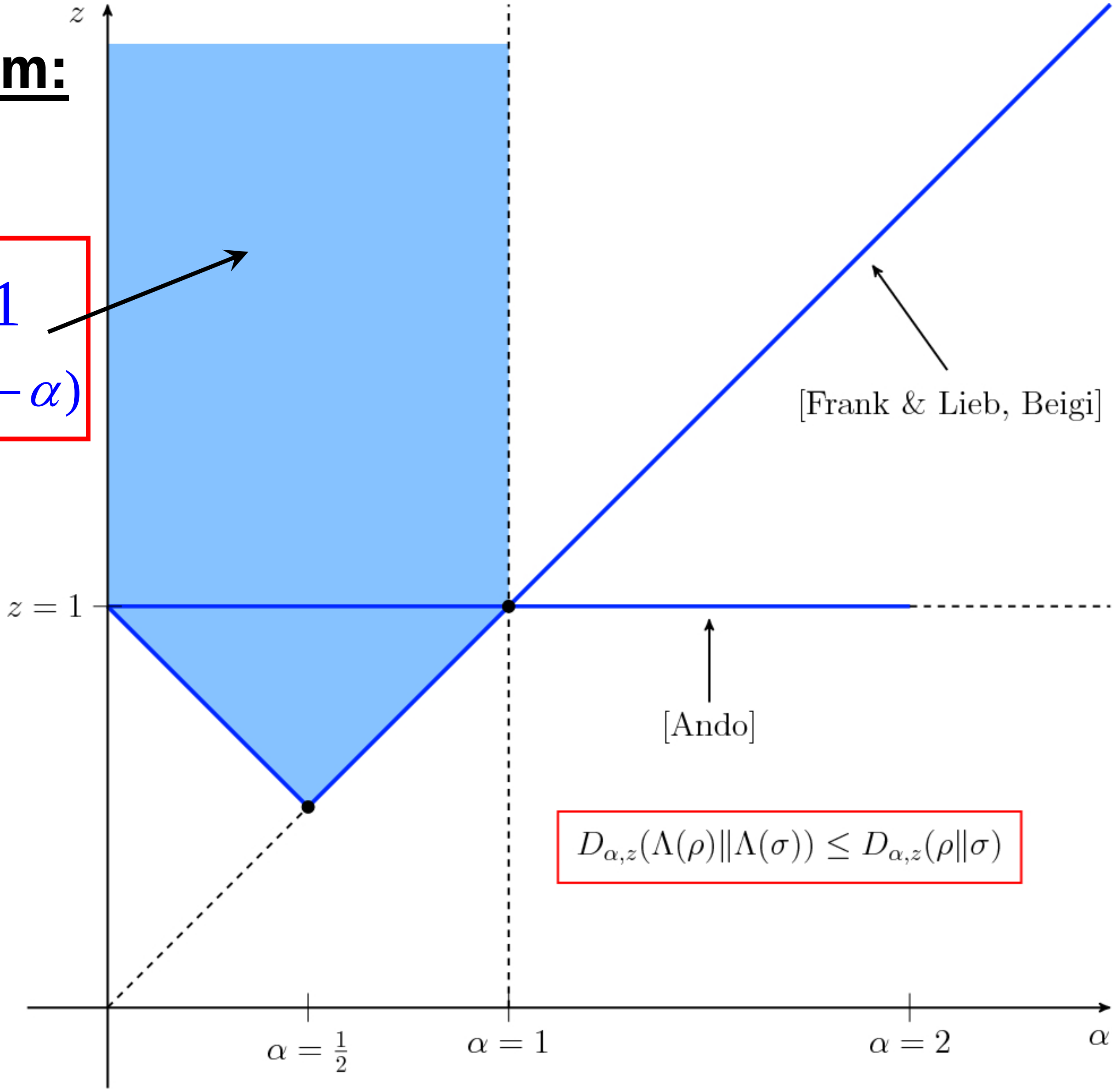
(Q) For which parameter ranges does $D_{\alpha,z}$ satisfy the DPI ?

- **Data-processing inequality for the** $\alpha - z - \mathrm{RRE}$

- **Theorem:**

$$0 \leq \alpha < 1$$
$$z \geq \max(\alpha, 1-\alpha)$$

[KA,ND],
[Hiai]



$$D_{\alpha,z}(\Lambda(\rho)\|\Lambda(\sigma)) \leq D_{\alpha,z}(\rho\|\sigma)$$

Data-processing inequality (DPI)

Proof of DPI in the blue region:

$0 \leq \alpha < 1; \quad z \geq \max(\alpha, 1-\alpha)$

$$D_{\alpha,z}(\Lambda(\rho) \| \Lambda(\sigma)) \leq D_{\alpha,z}(\rho \| \sigma)$$

[Frank & Lieb]:

- To prove DPI it suffices to prove that

$$f_{\alpha,z}(\rho \| \sigma)$$

is jointly concave

for $0 \leq \alpha \leq 1$

(trace functional)

Data-processing inequality (DPI) contd.

Joint concavity of $f_{\alpha,z} \Rightarrow$ *DPI for* $D_{\alpha,z}$ *for* $0 \le \alpha < 1$

- **Proof**

$$\boxed{\Lambda : \text{CPTP} \quad D_{\alpha,z}(\Lambda(\rho) \| \Lambda(\sigma)) \le D_{\alpha,z}(\rho \| \sigma)}$$

Stinespring's Dilation Theorem:

Action of Λ on a state $\rho \in \mathcal{D}(\mathcal{H})$:

$\rho \otimes \tau \in \mathcal{D}(\mathcal{H} \otimes \mathcal{H}_2)$:

τ: some fixed state in $\mathcal{H}_2$

$$\begin{tikzcd}
\rho \otimes \tau \arrow[r, "\text{unitary}"] & U(\rho \otimes \tau)U^* \arrow[d, "\mathrm{Tr}_2"] \\
\rho \arrow[u] \arrow[r, "\Lambda"'] & \mathrm{Tr}_2[U(\rho \otimes \tau)U^*]
\end{tikzcd}$$

$$\boxed{\Lambda(\rho) = \mathrm{Tr}_2[U(\rho \otimes \tau)U^*]}$$

Joint concavity of $f_{\alpha,z} \Rightarrow$ *DPI for* $D_{\alpha,z}$ *for* $0 \le \alpha < 1$

Proof: contd.

$$\boxed{\Lambda(\rho)=\mathrm{Tr}_2[U(\rho\otimes\tau)U^*]}$$

- Let

du: normalized Haar measure on all unitaries on $\mathcal{H}_2$

$$\boxed{\int du\; uAu^* = (\mathrm{Tr}\,A).\kappa}$$

where $\kappa = \dfrac{I}{N}$

$N \longrightarrow \dim \mathcal{H}_2$

Set: $\boxed{X = U(\rho\otimes\tau)U^*}$ $\in \mathcal{D}(\mathcal{H}\otimes\mathcal{H}_2)$

$$\int du\; (I\otimes u)X(I\otimes u^*) = (\mathrm{Tr}_2 X)\otimes\kappa$$

$$=\mathrm{Tr}_2[U(\rho\otimes\tau)U^*]\otimes\kappa$$

Integral representation

$$=\Lambda(\rho)\otimes\kappa$$

Joint concavity of $f_{\alpha,z}$ $\Rightarrow$ DPI for $D_{\alpha,z}$ for $0 \le \alpha < 1$

$$D_{\alpha,z}(\rho\,\|\,\sigma) = \frac{1}{\alpha - 1}\log f_{\alpha,z}(\rho\,\|\,\sigma)$$

$$D_{\alpha,z}(\Lambda(\rho)\,\|\,\Lambda(\sigma)) \le D_{\alpha,z}(\rho\,\|\,\sigma) \quad \Leftrightarrow \quad f_{\alpha,z}(\Lambda(\rho)\,\|\,\Lambda(\sigma)) \ge f_{\alpha,z}(\rho\,\|\,\sigma)$$

$$\Lambda(\rho)\otimes\kappa = \int du\ (I\otimes u)\ U(\rho\otimes\tau)U^*(I\otimes u^*)$$

$$= \int du\ V_u(\rho\otimes\tau)\,V_u^* \qquad V_u = (I\otimes u)\ U$$

Joint concavity of $f_{\alpha,z} \Rightarrow$ *DPI for* $D_{\alpha,z}$ *for* $0 \le \alpha < 1$

$$D_{\alpha,z}(\rho \| \sigma) = \frac{1}{\alpha - 1} \log f_{\alpha,z}(\rho \| \sigma)$$

$$D_{\alpha,z}(\Lambda(\rho) \| \Lambda(\sigma)) \le D_{\alpha,z}(\rho \| \sigma) \quad \Leftrightarrow \quad f_{\alpha,z}(\Lambda(\rho) \| \Lambda(\sigma)) \ge f_{\alpha,z}(\rho \| \sigma)$$

$$f_{\alpha,z}(\Lambda(\rho) \| \Lambda(\sigma)) = f_{\alpha,z}(\Lambda(\rho) \otimes \kappa \| \Lambda(\sigma) \otimes \kappa)$$

← *Tensor property*

$$= f_{\alpha,z}\left(\int du \; V_u (\rho \otimes \tau) V_u^* \, \| \int du \; V_u (\sigma \otimes \tau) V_u^* \right)$$

IF jointly concave

$$\ge \int du \;\; f_{\alpha,z}(V_u (\rho \otimes \tau) V_u^* \, \| V_u (\sigma \otimes \tau) V_u^*)$$

$$= \int du \;\; f_{\alpha,z}(\rho \otimes \tau \| \sigma \otimes \tau)$$

unitary invariance

$$\Lambda(\rho) \otimes \kappa = \int du \; (I \otimes u) \; U(\rho \otimes \tau) U^* (I \otimes u^*)$$

$$= \int du \; V_u (\rho \otimes \tau) V_u^* \qquad\qquad V_u = (I \otimes u) \; U$$

Joint concavity of $f_{\alpha,z} \Rightarrow$ DPI for $D_{\alpha,z}$ for $0 \le \alpha < 1$

$$D_{\alpha,z}(\rho\|\sigma) = \frac{1}{\alpha-1}\log f_{\alpha,z}(\rho\|\sigma)$$

$$D_{\alpha,z}(\Lambda(\rho)\|\Lambda(\sigma)) \le D_{\alpha,z}(\rho\|\sigma) \quad \Leftrightarrow \quad \boxed{f_{\alpha,z}(\Lambda(\rho)\|\Lambda(\sigma)) \ge f_{\alpha,z}(\rho\|\sigma)}$$

$$f_{\alpha,z}(\Lambda(\rho)\|\Lambda(\sigma)) = f_{\alpha,z}(\Lambda(\rho)\otimes\kappa\|\Lambda(\sigma)\otimes\kappa)$$

$$= f_{\alpha,z}\left(\int du\ V_u(\rho\otimes\tau)V_u^*\ \|\int du\ V_u(\sigma\otimes\tau)V_u^*\right)$$

IF jointly concave

$$\ge \int du\ f_{\alpha,z}(V_u(\rho\otimes\tau)V_u^*\|V_u(\sigma\otimes\tau)V_u^*)$$

$$= \int du\ f_{\alpha,z}(\rho\otimes\tau\|\sigma\otimes\tau)$$ unitary invariance

$$= f_{\alpha,z}(\rho\otimes\tau\|\sigma\otimes\tau)$$ normalization of the Haar measure

$$= f_{\alpha,z}(\rho\|\sigma) \longleftarrow$$ Tensor property

$$\therefore f_{\alpha,z}(\Lambda(\rho)\|\Lambda(\sigma)) \ge f_{\alpha,z}(\rho\|\sigma)$$

*

In fact:To prove DPI it suffices to prove that

$f_{\alpha,z}(A) \equiv f_{\alpha,z}(A,K) := \mathrm{Tr}(A^{\frac{\alpha}{z}} K A^{\frac{1-\alpha}{z}} K^*)^z$ is **concave** in A.

$A \geq 0,\ K$ fixed matrix

[Carlen & Lieb]

- **Why?** Because for the choice $K = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}; \quad A = \begin{pmatrix} \rho & 0 \\ 0 & \sigma \end{pmatrix}$

$$\boxed{f_{\alpha,z}(A) = f_{\alpha,z}(\rho \| \sigma)}$$

- Concavity of $f_{\alpha,z}(A,K)$ in A $\Rightarrow$ Joint concavity of $f_{\alpha,z}(\rho \| \sigma)$

- So focus on proving concavity of $f_{\alpha,z}(A)$ *

- Concavity of $$f_{\alpha,z}(A) := \mathrm{Tr}\left[A^{\frac{\alpha}{z}} K A^{\frac{1-\alpha}{z}} K^* \right]^{z}$$

for $\boxed{\alpha \in (0,1);\ z \geq \max\{\alpha, 1-\alpha\}}$

- Set $p = \dfrac{\alpha}{z};\ \ q = \dfrac{1-\alpha}{z};\ \ \ z = \dfrac{1}{p+q}$

$$f_{p,q}(A) = \mathrm{Tr}\left[A^p K A^q K^* \right]^{\frac{1}{p+q}}$$

- Concavity of $f_{p,q}(A)$ for $p,\ q \in (0,1)$ $*$

Key ingredients: Pick functions

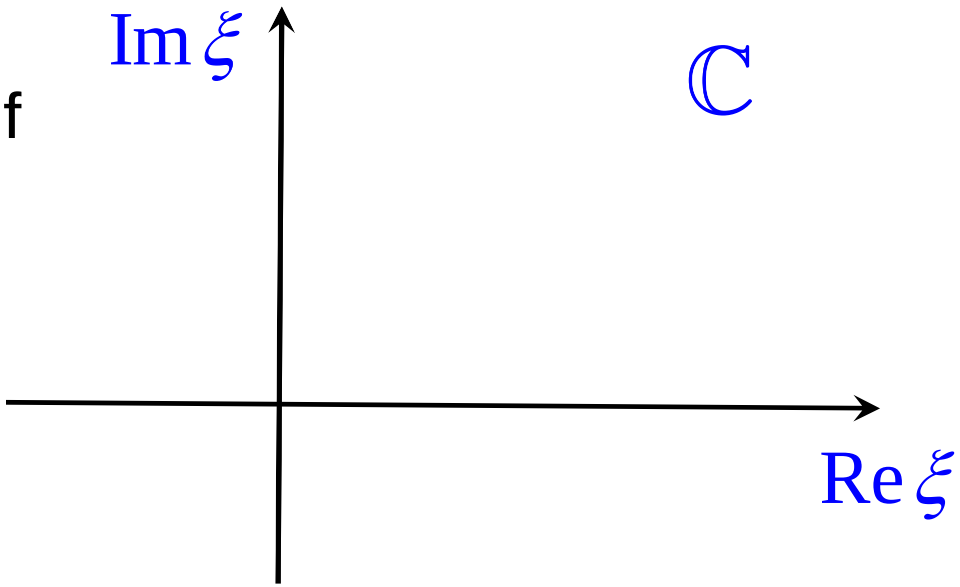
(holomorphic functions that map the upper half plane into itself)

Pick Functions: Holomorphic functions defined on the upper-half plane: $I^+(\mathbb{C}) := \{\xi \in \mathbb{C} : \operatorname{Im}\xi > 0\}$

with their ranges in the closed upper half plane $\{\xi \in \mathbb{C} : \operatorname{Im}\xi \geq 0\}$

Then $f(\xi)$ is a Pick function if

$$\operatorname{Im}\xi > 0 \quad \Rightarrow \quad \operatorname{Im} f(\xi) \geq 0.$$



Also known as: Herglotz functions or Nevanlinna functions

Example of a Pick function

- Let $\boxed{f(\xi)=\xi^p;}$ $0< p< 1$ defined on the cut plane

$$\xi^p = e^{p\log|\xi|}e^{ip\arg\xi} \qquad \arg\xi\in(-\pi,+\pi)$$

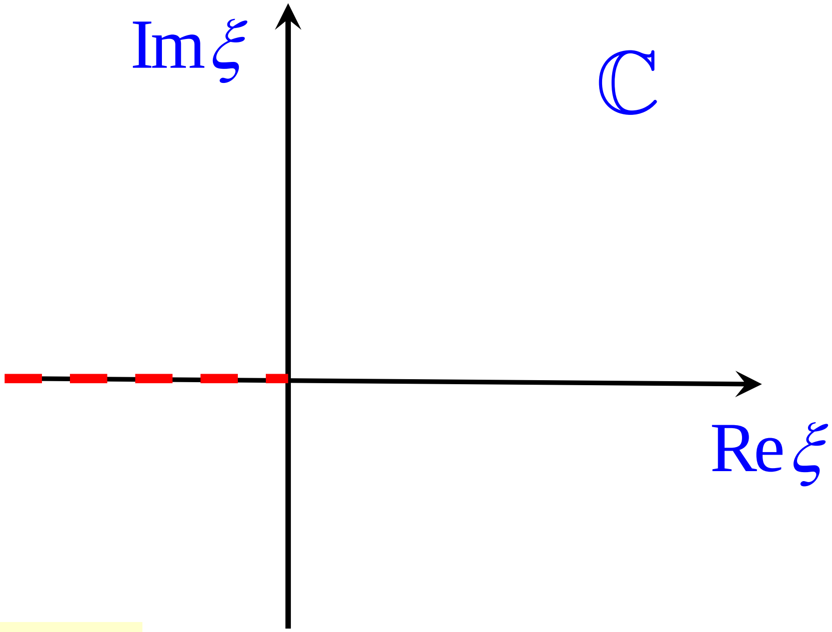
$$\xi =|\xi|e^{i\arg\xi}$$

$$\operatorname{Im}\xi>0 \quad \Leftrightarrow \quad \arg\xi\in(0,\pi)$$

$$\Rightarrow \arg\xi^p = p\arg\xi\in(0,\pi)$$

$$(\because 0< p< 1)$$

$$\therefore \operatorname{Im}\xi^p\geq 0$$

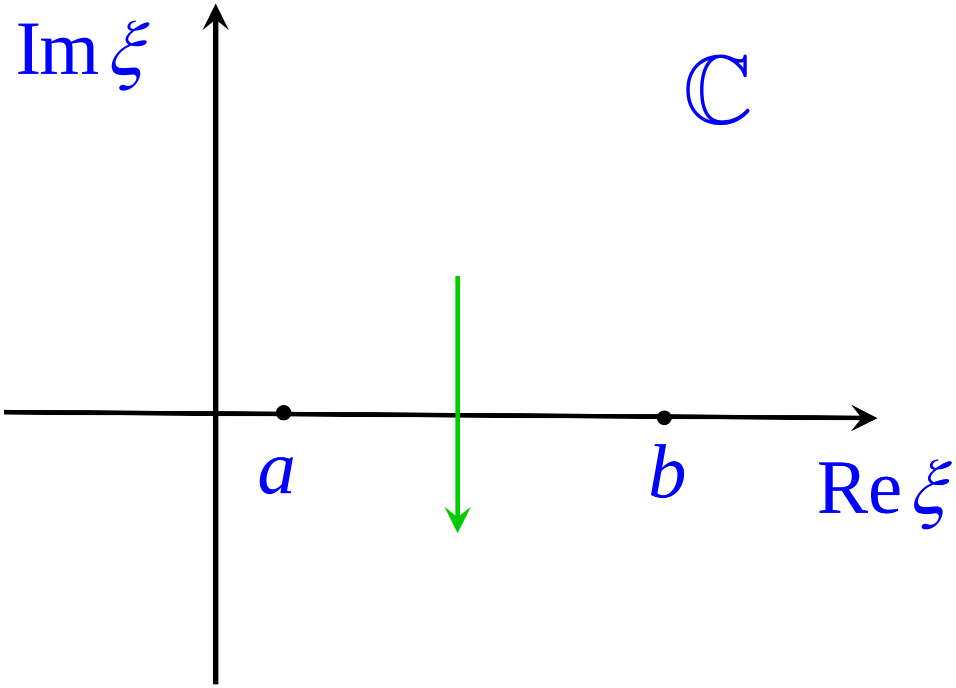


Hence $f(\xi)=\xi^p$ is a Pick function

Subclass of Pick functions $P(a,b)$

- Let (a,b) be an open interval on the real axis
- Then $P(a,b):$ the subclass of Pick functions which can be analytically continued across the interval (a,b) into the lower half plane such that the continuation is by *reflection w.r.t. the real axis.* $f(\bar{\xi}) = \overline{f(\xi)}$
- *Loewner's theorem:*

$P(a,b):$ the class of functions that are operator monotone on (a,b)



Relevance of Pick functions for our proofs

- A Pick function $f(\xi) \in P(a,b)$ has a unique canonical integral representation of the form

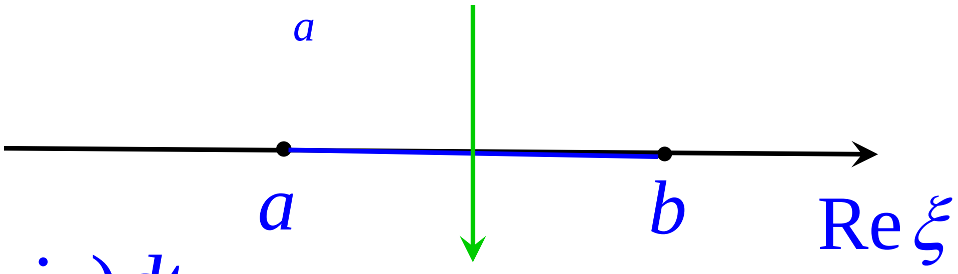
$$f(\xi) = \alpha\xi + \beta + \int_a^b \left(\frac{1}{t-\xi} - \frac{t}{t^2+1}\right) d\mu(t)$$

where $\alpha \geq 0; \quad \beta \in \mathbb{R};$ and

$d\mu(t):$ a positive Borel measure on the real $t-$ axis for which

$$\int_a^b (t^2+1)^{-1} d\mu(t) < \infty$$

$$\alpha = \lim_{y\to\infty} \frac{f(iy)}{iy}; \ \beta = \operatorname{Re} f(i);$$



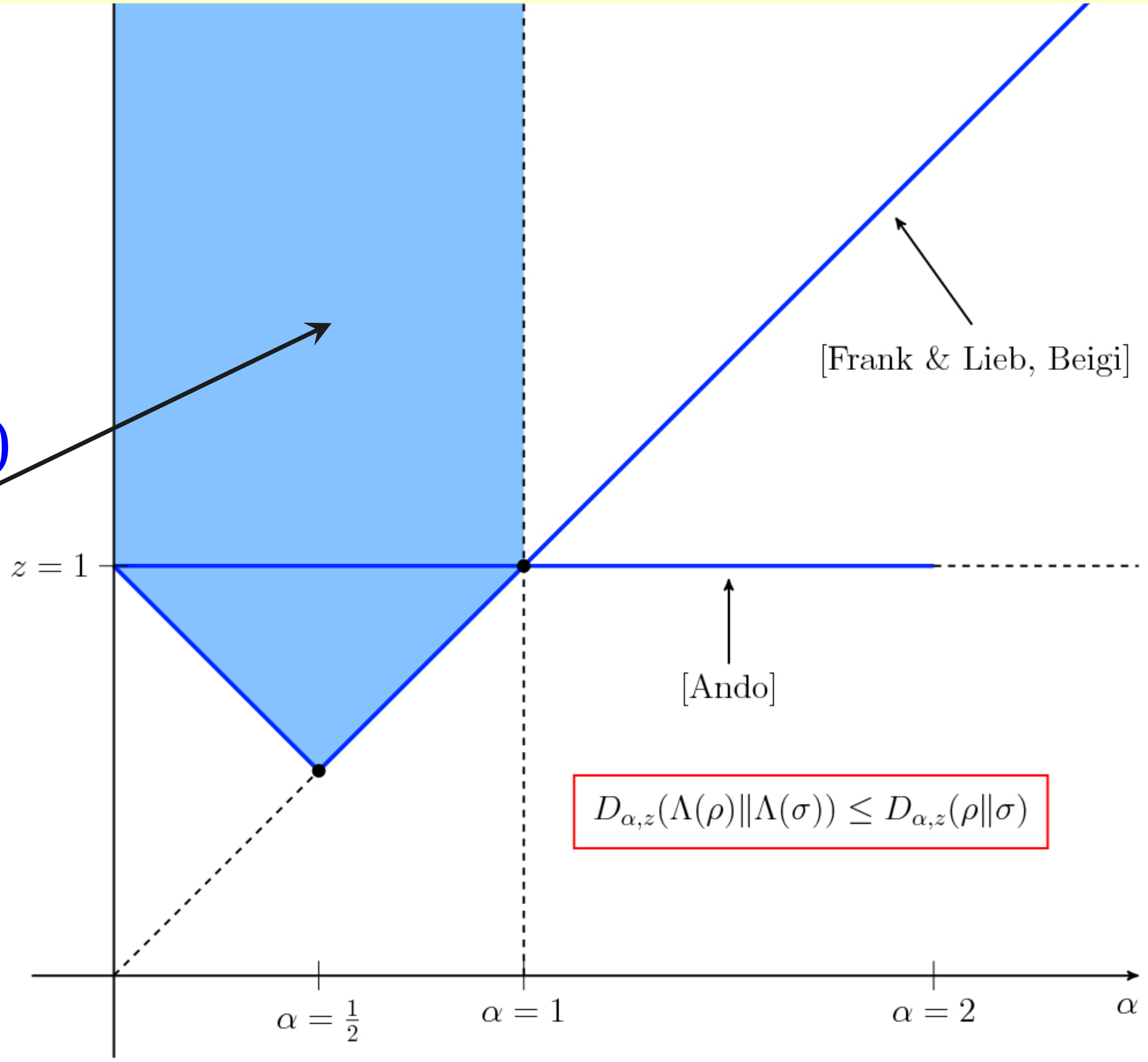
$$\mu(b) - \mu(a) = \lim_{y\to 0^+} \frac{1}{\pi} \int_a^b \operatorname{Im} f(t+iy)dt$$

Stieltjes inversion formula

We want to prove

- Concavity of $f_{p,q}(A)$ for $p,\ q \in (0,1)$

This would imply *DPI* *for* $D_{\alpha,z}(\rho \| \sigma)$ *here*



- Concavity of $f_{p,q}(A)$ for $p,\ q \in (0,1)$

(Q) How do Pick functions enter into the proof?

- Consider 2 related functions:

$$A \geq 0$$

$$F(\xi) = f_{p,q}(Q+\xi R)$$

$$G(\xi) = f_{p,q}(\xi Q + R) \qquad Q > 0,\ R = R^*$$

where: domain of $f_{p,q}$ has been extended to complex matrices

- **The 2 functions are related :**

$$f_{p,q}(A) = \mathrm{Tr}\left[A^p K A^q K^*\right]^{\frac{1}{p+q}}$$

homogeneous of order 1

$$F(\xi) = f_{p,q}(Q+\xi R)$$

$$= \xi f_{p,q}(\frac{1}{\xi}Q + R) = \xi G(\frac{1}{\xi})$$

$$F(\xi) = \xi G(\frac{1}{\xi})$$

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$$f_{p,q}(A) \longrightarrow F(\xi) = f_{p,q}(Q+\xi R); \quad G(\xi) = f_{p,q}(\xi Q + R)$$

- **Claim:** Concavity of $f_{p,q}(A)$ for $A \geq 0$

amounts to proving : Concavity of $F(x)$, for $x \in \mathbb{R}$;

(over the domain of F *)*

Concavity of $f_{p,q}(A)$ for $A \geq 0$ $\Leftrightarrow$ Concavity of $F(x)$, for $x \in \mathbb{R}$;

- Suppose $A_1, A_2 \geq 0$ & for $\forall x \in [0,1]$

$$f_{p,q}(xA_1 + (1-x)A_2) \geq x f_{p,q}(A_1) + (1-x) f_{p,q}(A_2)$$ *concavity*

Set $Q = A_2,\ R = A_1 - A_2;$ $\quad \xi = x$

$$\boxed{F(\xi) = f_{p,q}(Q + \xi R)}$$

$$\begin{aligned} F(x) &= f_{p,q}(Q + xR); \\ &= f_{p,q}(A_2 + x(A_1 - A_2)) \\ &= f_{p,q}(xA_1 + (1-x)A_2) \geq x f_{p,q}(A_1) + (1-x) f_{p,q}(A_2) \end{aligned}$$

$$[\, f_{p,q}(A_1) = F(1);\ \ f_{p,q}(A_2) = F(0)]$$

$$F(x) \geq xF(1) + (1-x)F(0)$$

$$f_{p,q}(A) \longrightarrow F(\xi) = f_{p,q}(Q+\xi R); \quad G(\xi) = f_{p,q}(\xi Q + R)$$

Concavity of $F(x)$, for $x \in \mathbb{R}$; **implies DPI**

Proof of concavity outline in 4 lines

- *Prove that* $G(\xi)$ *is a Pick function*
- *It hence has an integral representation*
- *This carries over to* $F(x)$; $\quad \because F(\xi) = \xi G(\frac{1}{\xi})$
- *Then proving concavity of* $F(x)$ *amounts to proving concavity of the integral's kernel*

  – which is straightforward!

To prove DPI for $0 \le \alpha < 1; \quad z \ge \max(\alpha, 1-\alpha)$

Concavity of $F(x)$, for $x \in \mathbb{R}$; *(over the domain of* F *)*

domain of holomorphy of $F(\xi) = f_{p,q}(Q + \xi R);$

$$\xi = x + iy, \quad x < 1/|c|; \quad c = \|Q\| / \lambda_{\min}(R)$$

$$F(x) = \alpha x + \beta + \int_{-c}^{c} \frac{x^2}{tx-1} \, d\mu(t)$$

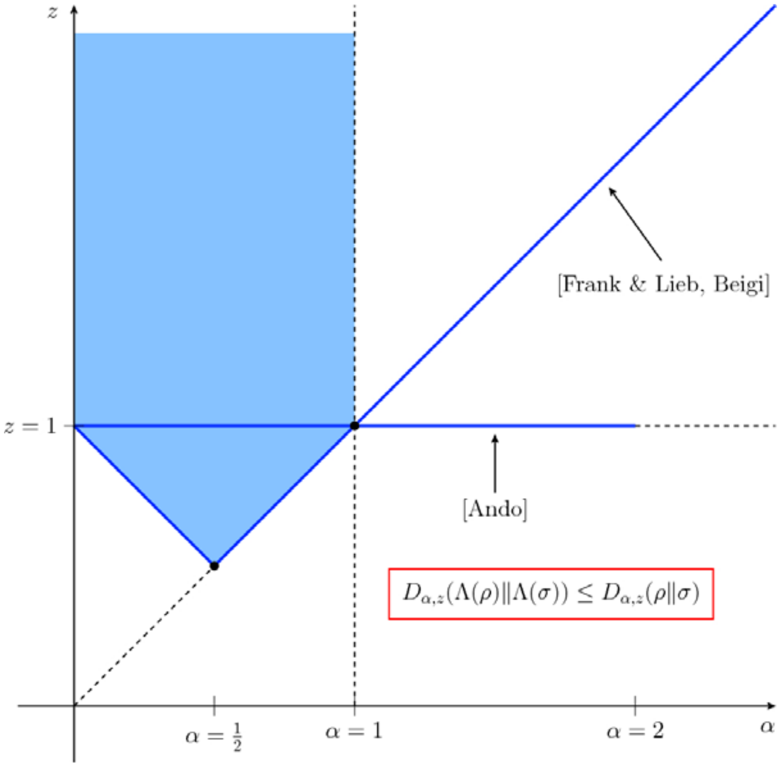
- kernel

$$g(x) = \frac{x^2}{tx-1}$$

$$g''(x) = \frac{2}{(tx-1)^3} < 0$$

for $x < 1/|c| < 1/t$

$$\Rightarrow F''(x) \le 0 \Rightarrow F(x)$$ concave ∎



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SUMMARY

- Introduced a 2-parameter family of relative entropies,

-- that unifies the study of all known relative entropies

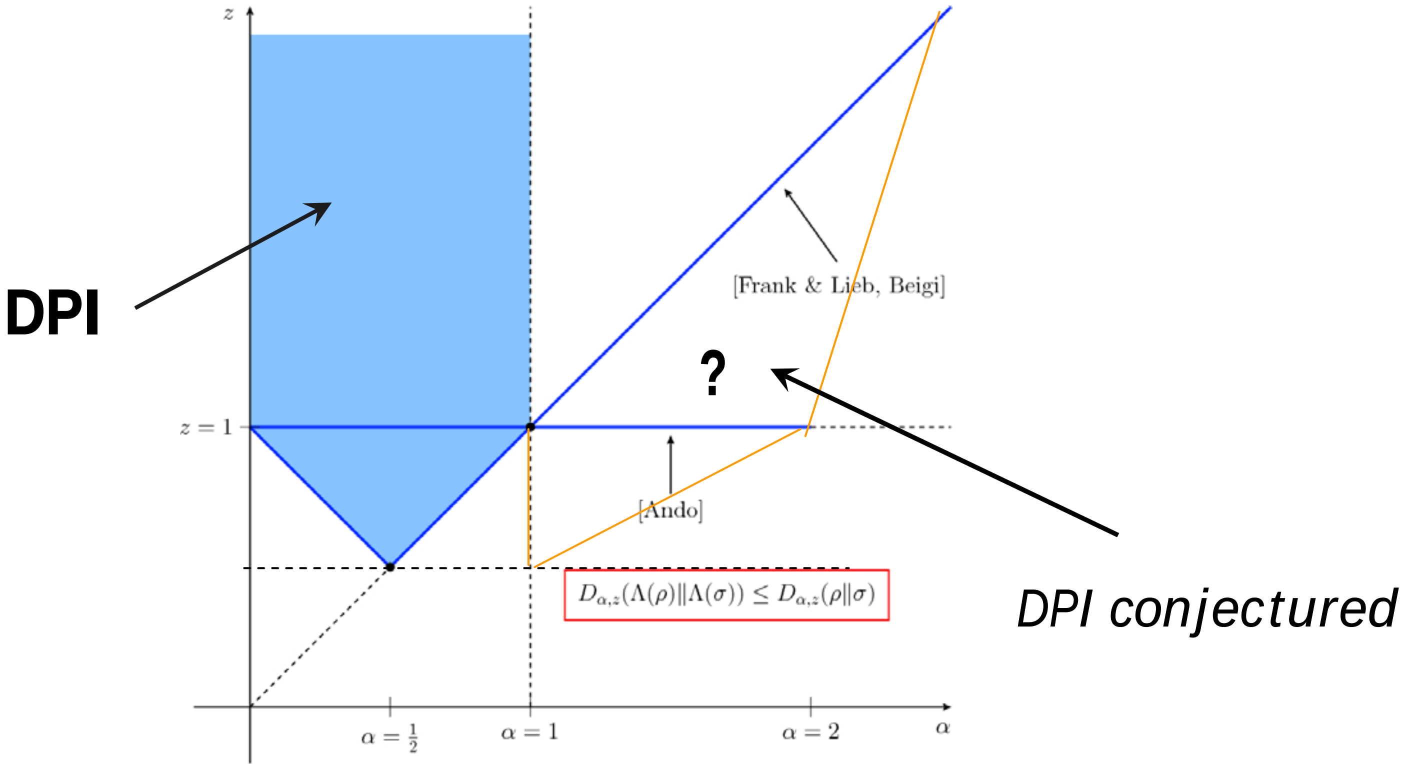
$\alpha - z -$ relative Renyi entropies: $\alpha - z - \mathrm{RRE}$

$$D_{\alpha,z}(\rho \| \sigma)$$

- These satisfies the quantum generalizations of Renyi's axioms for a relative entropy.

- Focus: For which parameter ranges does it satisfy the DPI ?

- Proved the DPI for $\boxed{0 \leq \alpha < 1; \quad z \geq \max(\alpha, 1-\alpha)}$ using the theory of Pick functions.



Thank You!

- Thanks to **Koenraad Audenaert;**
- & to **Felix Leditzky** for preparing the figures.

Summary and conjectures regarding DPI

- *Regions of concavity (blue) and conjectured convexity (orange) of the (reparametrizede) trace functional*

$$f_{p,q};\ p=\alpha/z\ ;\ q=(1\text{-}\alpha)/z$$