

# Spectral Properties of the Quantum Random Energy Model

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## 1 The Quantum Random Energy Model (QREM)

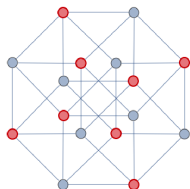
## 2 Low energy regime of the QREM:

A phase transition of the ground state

## 3 Resonant delocalization: Outlook and some conjectures

*(partially based on joint work with M. Aizenman & M. Shamis)*

# 1. The Quantum Random Energy Model



**Hamming cube:**  $\mathcal{Q}_N := \{-1, 1\}^N$

- configuration space of  $N$  spins

**Laplacian on  $\mathcal{Q}_N$ :**  $(-\Delta\psi)(\sigma) := N\psi(\sigma) - \sum_{j=1}^N \psi(F_j\sigma)$

- Spin flip:  $F_j\sigma = (\sigma_1, \dots, -\sigma_j, \dots, \sigma_N)$

Hence the Laplacian acts as a **transversal magnetic field**:  $-\Delta = N - \sum_{j=1}^N \sigma_j^x$

- Eigenvalues:  $2|A|$ ,  $A \subset \{1, \dots, N\}$     Degeneracies:  $\binom{N}{|A|}$

Normalized Eigenvectors:  $f_A(\sigma) = \frac{1}{\sqrt{2^{|A|}}} \prod_{j \in A} \sigma_j$

**Perturbation by a multiplication operator  $U$ :**

$$H = -\Delta + \kappa U$$

- $U = U(\sigma_1^z, \dots, \sigma_j^z)$ ;    Coupling constant  $\kappa \geq 0$ ;     $\|U\|_\infty \approx \mathcal{O}(N)$

- In this talk:

$$U(\sigma) = \sqrt{N} g(\sigma)$$

with  $\{g(\sigma)\}_{\sigma \in \mathcal{Q}_N}$  i.i.d. standard Gaussian r.v.    **REM**

## 1. Adiabatic Quantum Optimization:

Farhi/Goldstone/Gutmann/Snipser '01, ...

**Question:** Find minimum in a complex energy landscape  $U(\sigma)$

e.g. REM, Exact Cover 3, ...

**Idea:** Evolve the ground state through adiabatic quantum evolution, i.e.

$i \partial_t \psi_t = H(t/\tau) \psi_t$  generated by

$$H(s) := (1 - s)(-\Delta) + s U, \quad s \in [0, 1]$$

Required time :  $\tau \approx c \Delta_{min}^{-2}$

## 2. Mean field model for localization transition in disordered $N$ particle systems

Altshuler '06

## 3. Evolutionary Genetics: Rugged fitness landscape for quasispecies ...

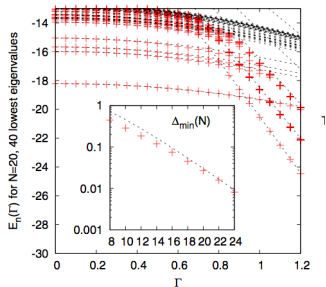
Schuster/Eigner '77, Baake/Wagner '01, ...

## Predicted low-energy spectrum:

$$\hat{H} = \Gamma(-\Delta - N) + U/\sqrt{2}, \quad \text{i.e. } \kappa = (\sqrt{2}\Gamma)^{-1}$$

Jörg/Krzakala/Kurchan/Maggs '08

Presilla/Ostili '10, ...



**First order phase transition of the ground state at**  $\kappa_C = \frac{1}{\sqrt{2 \ln 2}}$ :

$\kappa < \kappa_C$ : Extended ground state with non-random ground-state energy

$$E_0 = -\kappa^2 + o(1)$$

$\kappa > \kappa_C$ : Low lying eigenstates are concentrated on lowest values of  $U$ .

In particular:  $E_0 = N + \kappa \min U + \mathcal{O}(1)$

$\kappa = \kappa_C$ : Energy gap  $\Delta_{min} = E_1 - E_0$  vanishes exponentially in  $N$

One aim in this talk: explain some of the above features!

**Known properties** of the **REM**:

$$U(\sigma) = \sqrt{N} g(\sigma)$$

- Location of the minimum:

$$U_0 := \min U = -\kappa_c^{-1} N + \mathcal{O}(\ln N)$$

- The extreme values  $U_0 \leq U_1 \leq \dots$  form a **Poisson process** about  $-\kappa_c^{-1} N + \mathcal{O}(\ln N)$  of exponentially increasing intensity.



**Perturbation theory:**

- Fate of localized states:  $\langle \delta_\sigma, H \delta_\sigma \rangle = N + \kappa U(\sigma)$ .
- Fate of delocalized states:  $\langle f_A, U f_A \rangle = \frac{1}{2^N} \sum_\sigma U(\sigma) = \mathcal{O}(\sqrt{N} 2^{-N/2})$ .

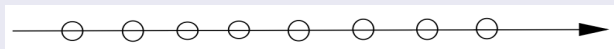
## 2. Low-energy regime of the QREM in case $\kappa < \kappa_C$

### Theorem (Case $\kappa < \kappa_C$ )

For any  $\varepsilon > 0$  and except for events of exponentially small probability, the eigenvalues of  $H$  below  $\left(1 - \frac{\kappa}{\kappa_C} - \varepsilon\right) N$  are within balls centred at

$$2n - \frac{\kappa^2}{1 - \frac{2n}{N}}, \quad n \in \{0, 1, \dots\},$$

of radius  $\mathcal{O}\left(N^{-\frac{1}{2} + \delta}\right)$  with  $\delta > 0$  arbitrary.



There are exactly  $\binom{N}{n}$  eigenvalues in each ball and their eigenfunctions are delocalized:

$$\|\psi_E\|_\infty^2 \leq 2^{-N} e^{\Gamma\left(\frac{x_E}{2}\right)N}$$

where  $\Gamma(x) := -x \ln x - (1-x) \ln(1-x)$  and  $x_E := \frac{E}{N} + \frac{\kappa}{\kappa_C} + \varepsilon$ .

**Step 1:***Hypercontractivity of the Laplacian*

$$\begin{aligned}
 |\psi_E(\sigma)|^2 &\leq \langle \delta_\sigma, P_{(-\infty, E]}(H) \delta_\sigma \rangle = \inf_{t>0} e^{tE} \langle \delta_\sigma, e^{-tH} \delta_\sigma \rangle \\
 &= \inf_{t>0} e^{t(E - \kappa U_0)} \langle \delta_\sigma, e^{t\Delta} \delta_\sigma \rangle = 2^{-N} e^{\Gamma(\frac{x_E}{2})N}.
 \end{aligned}$$

**Step 2:***Reduction of fluctuations*

Projection on centre of band and its complement:

$$Q_\varepsilon := 1 - P_\varepsilon := 1_{[N(1-\varepsilon), N(1+\varepsilon)]}(-\Delta).$$

Note:  $\dim P_\varepsilon \leq 2^N e^{-\varepsilon^2 N/2}$  – take  $\varepsilon = \mathcal{O}\left(N^{-\frac{1}{2} + \delta}\right)$ .



## Lemma

There exist constants  $C, c < \infty$  such that for any  $\varepsilon > 0$  and any  $\lambda > 0$ :

$$\mathbb{P} \left( \left| \|P_\varepsilon U P_\varepsilon\| - \mathbb{E} [\|P_\varepsilon U P_\varepsilon\|] \right| > \lambda \sqrt{\frac{\dim P_\varepsilon}{2^N}} \right) \leq C e^{-c\lambda^2}$$

$$\mathbb{E} [\|P_\varepsilon U P_\varepsilon\|] \leq CN \sqrt{\frac{\dim P_\varepsilon}{2^N}} = CN e^{-\varepsilon^2 N/4}.$$

- 1** Concentration of measure using [Talagrand inequality](#):

Lipschitz continuity of  $F : \mathbb{R}^{Q_N} \rightarrow \mathbb{R}$ ,  $F(U) := \|P_\varepsilon U P_\varepsilon\|$ :

$$\begin{aligned} F(U) - F(U') &\leq \langle \psi, U\psi \rangle - \langle \psi, U'\psi \rangle \\ &\leq \|U - U'\|_2 \|\psi\|_\infty \leq \|U - U'\|_2 \sqrt{\frac{\dim P_\varepsilon}{2^N}}. \end{aligned}$$

- 2** Moment method to estimate  $\mathbb{E} [\|P_\varepsilon U P_\varepsilon\|] \leq (\mathbb{E} [\text{Tr}(P_\varepsilon U P_\varepsilon)^{2N}])^{1/2N} \dots$

**Step 3:***Schur complement formula*

$$P_\varepsilon (H - z)^{-1} P_\varepsilon = \left( P_\varepsilon H P_\varepsilon - z - \kappa^2 P_\varepsilon U Q_\varepsilon (Q_\varepsilon H Q_\varepsilon - z)^{-1} Q_\varepsilon U P_\varepsilon \right)^{-1}$$

... and using Step 2:

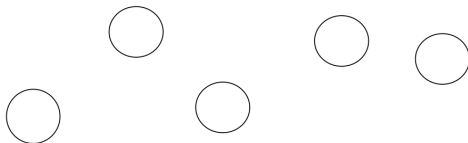
$$P_\varepsilon U Q_\varepsilon (Q_\varepsilon H Q_\varepsilon - z)^{-1} Q_\varepsilon U P_\varepsilon \approx \frac{N}{N - z} P_\varepsilon + \mathcal{O}\left(N^{-\frac{1}{2} + \delta}\right).$$

## Main idea:

## *Geometric decomposition*

For energies below  $E_\delta := \left(1 - \frac{\kappa}{\kappa_c} + \delta\right) N$  the localized eigenstates originate in **large negative deviation sites**:

$$X_\delta := \left\{ \sigma \mid \kappa U(\sigma) < -\frac{\kappa}{\kappa_c} N + \delta N \right\}$$



For  $\delta > 0$  small enough and except for events of exponentially small probability (e.e.p.):

- $X_\delta$  consists of isolated points which are separated by a distance greater than  $2\gamma N$  with some  $\gamma > 0$ .
- On balls  $B_{\gamma, \sigma} := \{\sigma' \mid \text{dist}(\sigma, \sigma') < \gamma N\}$  the potential is larger than  $-\epsilon N$  aside from at  $\sigma$ .

## Theorem

*E.e.p. and for  $\delta > 0$  sufficiently small, there is some  $\gamma > 0$  such that all eigenvalues of  $H$  below  $E_\delta = \left(1 - \frac{\kappa}{\kappa_c} + \delta\right) N$  coincide up to an exponentially small error with those of*

$$\hat{H}_\delta := H_R \oplus \bigoplus_{\sigma \in X_\delta} H_{B_{\gamma, \sigma}}.$$

where  $R := Q_N \setminus \bigcup_{\sigma \in X_\delta} B_{\gamma, \sigma}$ .

- Low energy spectrum of  $H_R$  looks like  $H$  in the delocalisation regime
- Low energy spectrum of  $H_{B_{\gamma, \sigma}}$  is explicit consisting of exactly one eigenstate below  $E_\delta \dots$

**Known properties of Laplacian** on  $B_{\gamma,\sigma}$ :

$$E_0(-\Delta_{B_{\gamma,\sigma}}) = N(1 - 2\sqrt{\gamma(1-\gamma)}) + o(N)$$

Adding a large negative potential  $\kappa U$  at  $\sigma$  and some more moderate background elsewhere, rank-one analysis yields:

- $E_0(H_{B_{\gamma,\sigma}}) = N + \kappa U(\sigma) - s_\gamma(N + \kappa U(\sigma)) + \mathcal{O}(N^{-1/2})$

where  $s_\gamma$  is the self-energy of the Laplacian on a ball of radius  $\gamma N$ .

- for the corresponding normalised ground state:

$$\sum_{\sigma' \in \partial B_{\gamma,\sigma}} |\psi_0(\sigma')|^2 \leq e^{-L_\gamma N} \quad \text{for some } L_\gamma > 0.$$

$$|\psi_0(\sigma)|^2 \geq 1 - \mathcal{O}(N^{-1})$$

- $H_{B_{\gamma,\sigma}}$  has a spectral gap of  $\mathcal{O}(N)$  above the ground state.

### 3. Resonant delocalization: an outlook and some conjectures

Random operators may exhibit regimes in which **extended states** are formed from resonating local quasi-modes. The corresponding eigenstates may satisfy the **eigenfunction thermalization hypothesis** (ETH, i.e., a heuristic version of the equidistribution principle) or not. Our aim is to:

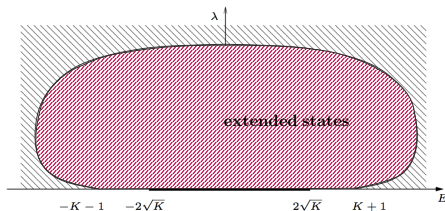
- clarify conditions for resonance of local quasi-modes.
- pinpoint scenarios which **fail** ETH.

#### Example 1:

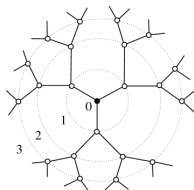
Regular tree graph  $\mathbb{B}$  with coordination number  $K + 1 \geq 3$ :

$$H = -\Delta + \lambda g \quad \text{on } \ell^2(\mathbb{B}).$$

Spectral phase diagram:



M.Aizenman, S.W., *JEMS* 15: 1167-1222 (2013)



- Appearance of extended states within the  $\ell^1$ -spectrum of  $-\Delta$ .
- Transition from Poisson level statistics to GOE within extended state regime?

Biroli/Ribeiro-Teixeira/Tarzia '13.

Given **quasi-modes**  $(e_j, \psi_j)$  of  $H$ , their **tunnelling amplitude**  $\sigma_{jk}$  at energy  $E$  is defined through:

$$P_{jk}(H - E)^{-1}P_{jk} = \begin{bmatrix} e_j + \sigma_{jj}(E) & \sigma_{jk}(E) \\ \sigma_{kj}(E) & e_k + \sigma_{kk}(E) \end{bmatrix}^{-1}$$

If the typical **gap size** for quasi-modes is  $\Delta(E)$ , then the **condition for resonant delocalisation** at  $E$  is:

$$\Delta(E) \leq |\sigma_{jk}(E)|.$$

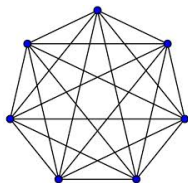
**Example 2:**

M.Aizenman, M. Shamis, S.W. (2013)

**Complete graph** over  $M$  sites:

$$H = -|\varphi_0\rangle\langle\varphi_0| + \kappa_M g$$

with  $|\varphi_0\rangle = (1, 1, \dots, 1)/\sqrt{M}$  and  $\kappa_M := \frac{\lambda}{\sqrt{2 \log M}}$ .



- Toy model for QREM ( $M = 2^N$ )
- Quasi-modes:  $(\kappa_M g(j), \delta_j)$  Gap size:  $\Delta(E)^{-1} = \kappa_M M \varrho(E/\kappa_M)$
- Conditions for resonant delocalisation:

$$\frac{\Delta(E)}{|\sigma_{jk}(E)|} = M \Delta(E) \left( 1 - \int \frac{\varrho(v) dv}{\kappa_M v - E} \right) + \mathcal{O}(1) \leq \mathcal{O}(1)$$

is satisfied for energies near  $E = 0$  and  $E = -1$  in case  $\lambda > \sqrt{2}$ .



## Theorem (Aizenman, Shamir, W. '13)

Consider a sequence of energies  $E_M$  with

$$\lim_{M \rightarrow \infty} M \Delta(E_M) \left( 1 - \int \frac{\varrho(v) dv}{\kappa_M v - E_M} \right) = \alpha \in \mathbb{R}.$$

Then:

- 1 the rescaled eigenvalue point process in a window around in  $E_M$  converges in distribution as  $M \rightarrow \infty$  to the Šeba point process at level  $\alpha$ .
- 2 the corresponding eigenfunctions are almost surely all  $\ell^1$ -delocalized:

$$\lim_{M \rightarrow \infty} \frac{\|\psi_{n,M}\|_1}{\|\psi_{n,M}\|_\infty} = \infty \quad \text{for all } |n| < \infty.$$

Yet, the eigenfunctions are localized in the  $\ell^2$ -sense satisfying the distributional limit

$$\lim_{M \rightarrow \infty} \frac{\|\psi_{n,M}\|_2}{\|\psi_{n,M}\|_\infty} \in (1, \infty).$$

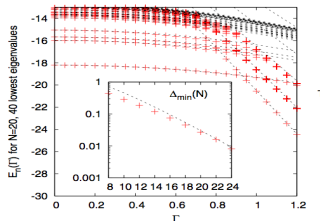
## 4. Conclusion:

### 1 Complete description of the **low-energy** spectrum of the QREM

Ground-state phase transition  
at  $\kappa = \kappa_C$  with  
an exponentially closing gap.

Jörg/Krzakala/Kurchan/Maggs '08

... and generalisations to non-gaussian r.v.'s



### 2 **Resonant delocalisation conjecture** in QREM with eigenfunctions violating quantum ergodicity are expected to occur closer to centre of band within renormalised gaps of Laplacian.