Coulomb scattering in QFT. A status report

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Outline

Relativistic QFT

Definition

A relativistic QFT is given by:

(1) A net of local algebras $\mathbb{R}^4 \supset \mathcal{O} \mapsto \mathcal{A}(\mathcal{O}) \subset B(\mathcal{H})$ s.t.

(a) If $\mathcal{O}_1 \subset \mathcal{O}_2$ then $\mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)$.

(b) If $\mathcal{O}_1 \times \mathcal{O}_2$ then $[\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)] = 0$.

(2) A Hamiltonian H and momentum operators P s.t.

(a) Joint spectrum of H and P is in the closed future lightcone.

(b) If $A \in \mathcal{A}(\mathcal{O})$ then $A(t,x) := e^{i(Ht-Px)} A e^{-i(Ht-Px)} \in \mathcal{A}(\mathcal{O} + (t,x))$.

Relativistic QED

Definition (Fredenhagen-Hertel 81, Bostelmann 04)

A quadratic form ϕ is a pointlike field of a relativistic QFT, if there exist:

(a) $A_r \in \mathcal{A}(\mathcal{O}_r)$, *where $\mathcal{O}_r$ is the ball of radius r centered at zero,*

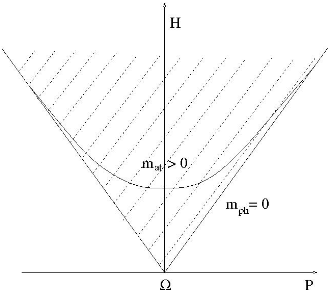
(b) $k > 0$,

s.t. $\|(1+H)^{-k}(\phi - A_r)(1+H)^{-k}\| \underset{r\to 0}{\to} 0.$

Definition

- Relativistic QED is a QFT whose pointlike fields include the Faraday tensor F and a conserved current j which satisfy the Maxwell equations: $dF = 0$, $d * F = j$.
- The electric charge exists and is given (formally) by $Q = \int d^3x\, j^0(x)$.

Vacuum sector of QED



We assume:

(a) Existence of the vacuum vector Ω. (We set $\mathcal{H}_0 = \mathbb{C}\Omega$).

(b) Non-triviality of $\mathcal{H}_{\mathrm{sp}} = \mathbf{1}_{\{m^2_{\mathrm{ph}}\}}(H^2 - P^2)\mathcal{H}_0^{\perp} \oplus \mathbf{1}_{\{m^2_{\mathrm{at}}\}}(H^2 - P^2)\mathcal{H}$.

(c) Hölder continuity of the spectrum of $(H^2 - P^2)$ near $\{m^2_{\mathrm{ph}}, m^2_{\mathrm{at}}\}$.

Asymptotic creation operators

Definition

(a) Free dynamics: $\hat{h}_t(x) := \int \frac{d^3k}{(2\pi)^3} e^{-i\omega(k)t+ikx}\hat{h}(k)$, $\omega(k) = \sqrt{k^2+m^2}$.

(b) Interacting dynamics: $A^*(t,x) := e^{i(Ht-Px)}A^*e^{-i(Ht-Px)}$, $A^* \in \mathcal{A}(\mathcal{O})$.

(c) LSZ creation operator: $A^*_t(\hat{h}) := \int d^3x\, \hat{h}_t(x)A^*(t,x)$.

(d) HR creation operator: $A^*_T(\hat{h}) := \frac{1}{\ln|T|}\int_T^{T+\ln|T|} dt\, A^*_t(\hat{h})$.

Remark: $h := \text{s-}\lim_{T\to\infty} A^*_T(\hat{h})\Omega$ exists and is a single-particle state.

Scattering states

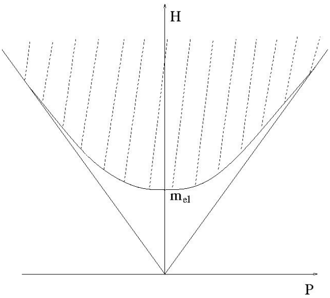
Theorem (Haag 58, Ruelle 62, Hepp 65, Buchholz 77... W.D. 05)

Suppose the particles $h_i = \lim_{T\to\infty} A^*_{i,T}(\hat{h}_i)\Omega$ *have disjoint velocity supports, separated from zero. Then there exists the scattering state*

$$\Psi^+ = \lim_{T\to\infty} A^*_{1,T}(\hat{h}_1)\dots A^*_{n,T}(\hat{h}_n)\Omega$$

which depends only on h_i*. Such states span a space* $\mathcal{H}^+ \subset \mathcal{H}$ *naturally isomorphic to the Fock space* $\Gamma(\mathcal{H}_{\mathrm{sp}})$*.*

Charged sectors



Theorem (Buchholz 86)

$\mathcal{H}_{\mathrm{sp}} := \mathbf{1}_{\{m_{\mathrm{el}}^2\}}(H^2 - P^2)\mathcal{H} = \{0\}$ *in charged representations of QED.*

Remark 1: Electron is an infraparticle.

Remark 2: The proof uses Maxwell equations.

Charged sectors

Theorem (Buchholz 86)

$\mathcal{H}_{\mathrm{sp}} := \mathbf{1}_{\{m_{\mathrm{el}}^2\}}(H^2 - P^2)\mathcal{H} = \{0\}$ *in charged sectors of QED.*

1. Applies to sectors in which one can measure the spacelike asymptotic flux of the electric field: $f(\vec{n}) := \lim_{r\to\infty} r^2 \vec{n}\vec{E}(\vec{n}r)$.
2. $f(\vec{n})$ is a superselection rule (commutes with all elements of $\mathcal{A}$). So $\mathcal{A}$ has uncountably many sectors with the same electric charge.
3. In particular plane-wave configurations of an electron with different velocities live in different sectors.
4. [Buchholz-Roberts 13] constructed charged representations of $\mathcal{A}$ in which $f(\vec{n})$ does not exist. In such representations one can hope for $\mathcal{H}_{\mathrm{sp}} \neq \{0\}$.

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Remarks on models

1. Relativistic QED: $H_{I,QED} = \overline{g} \int d^3x\, \overline{\Psi}(x)\gamma_\mu A^\mu(x)\Psi(x)$.
   1. Bad UV behaviour (in physical spacetime).
   2. Local gauge invariance.

   No complete construction available, not even in lower dimension.

2. Yukawa model: $H_{I,Y} = \overline{g} \int d^3x\, \overline{\Psi}(x)\phi(x)\Psi(x)$.
   1. Bad UV behaviour (in physical spacetime).

   Constructed in lower dimension (for massive ϕ) [Magnen-Sénéor 80].

3. Nelson model: Can be obtained from the Yukawa model by
   1. Fixing a UV cut-off,
   2. Removing terms responsible for electron/positron production,
   3. Changing dispersion relation of electrons $\sqrt{p^2+m^2} \to p^2/2m$.

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Nelson model with many electrons/atoms

Definition

The Nelson model with many atoms/electrons is given by:

(1) Hilbert space $\mathcal{H} = \Gamma(L^2(\mathbb{R}^3)_{\mathrm{at/el}}) \otimes \Gamma(L^2(\mathbb{R}^3)_{\mathrm{ph}})$.

(2) Hamiltonian $H = H_{\mathrm{at/el}} + H_{\mathrm{ph}} + H_I$, where

(a) $H_{\mathrm{at/el}} = \int d^3p \, \frac{p^2}{2m} \, \eta_p^* \eta_p$,

(b) $H_{\mathrm{ph}} = \int d^3k \, \omega(k) a_k^* a_k$, $\omega(k) = |k|$,

(c) $H_I = \int d^3p \, d^3k \, \overline{g} \frac{\tilde{\rho}(k)}{\sqrt{2\omega(k)}} (\eta_{p+k}^* a_k \eta_p + \mathrm{h.c.})$.

(3) Momentum operator: $P = \int d^3p \, p \, \eta_p^* \eta_p + \int d^3k \, k \, a_k^* a_k$.

Nelson model with one electron/atom

Definition

The Nelson model with one electron is given by:

(1) Hilbert space $\mathcal{H}^{(1)} = L^2(\mathbb{R}^3)_{\mathrm{at/el}} \otimes \Gamma(L^2(\mathbb{R}^3)_{\mathrm{ph}})$.

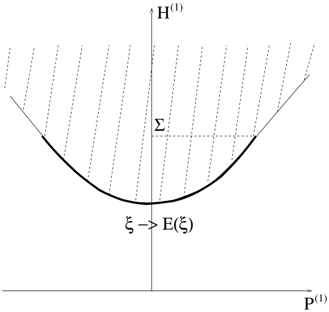
(2) Hamiltonian $H^{(1)} = \frac{p^2}{2m} + H_{\mathrm{ph}} + \phi(G_x)$, where

(a) $H_{\mathrm{ph}} = \int d^3k\, \omega(k) a_k^* a_k$, $\omega(k) = |k|$,

(b) $\phi(G_x) = \int d^3k\, \overline{g} \frac{\tilde{\rho}(k)}{\sqrt{2\omega(k)}} (e^{-ikx} a_k^* + e^{ikx} a_k)$.

(3) Momentum operator: $P^{(1)} = p + \int d^3k\, k\, a_k^* a_k$.

Spectral properties



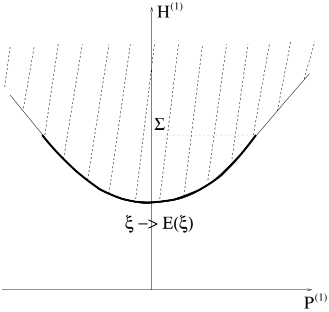
Theorem (Abdesselam-Hasler 10)

There exist $\Sigma > \inf \sigma(H^{(1)})$ *and* $\overline{g} > 0$ *s.t. for* $E(\xi) \leq \Sigma$.

(a) $|\nabla E(\xi)| < 1$,

(b) $\xi \to \nabla E(\xi)$ *is invertible.*

Neutral particle ('atom')



Suppose that the 'charge' of the massive particle is zero, i.e. $\tilde{\rho}(0) = 0$. Then (generically):

$$\mathcal{H}_{\rm sp} := \mathbf{1}_{(-\infty,\Sigma)}(H^{(1)})\mathbf{1}_{\{0\}}(H^{(1)} - E(P^{(1)}))\mathcal{H}^{(1)} \neq \{0\}.$$

Asymptotic creation operators of 'photons'

Definition

(a) Free dynamics: $h_t(y) := \int \frac{d^3k}{(2\pi)^3} e^{-i\omega(k)t+iky} h(k)$, $\omega(k) = |k|$.

(b) Int. dynamics: $a^*(1)(t,y) := e^{i(H^{(1)}t-P^{(1)}y)} a^*(1) e^{-i(H^{(1)}t-P^{(1)}y)}$.

(c) LSZ creation operator: $a_t^*(h) := \int d^3y\, h_t(y) a^*(1)(t,y)$

$$= e^{iH^{(1)}t} a^*(e^{-i\omega t}h) e^{-iH^{(1)}t}.$$

Scattering states of one 'atom' and 'photons'

Theorem (Hoegh-Krohn 69...Griesemer-Zenk 09)

For any $h_i \in L^2(\mathbb{R}^3, \sqrt{|k|^2 + |k|^{-1}}d^3k)$ *and* $\Psi \in \mathcal{H}_{\mathrm{sp}}$ *there exists*

$$\Psi^+ = \lim_{t\to\infty} a_t^*(h_1)\dots a_t^*(h_n)\Psi = a_+^*(h_1)\dots a_+^*(h_n)\Psi.$$

Scattering states of two 'atoms': General considerations

1. Let $\Psi_1, \Psi_2 \in \mathcal{H}_{\mathrm{sp}}$. We want to construct a state

$$\Psi_1 \overset{\mathrm{out}}{\times} \Psi_2 \in \mathcal{H},$$

describing two independent atoms at asymptotic times.

2. We need asymptotic creation operators of atoms $\hat{\eta}^*_{+,1}, \hat{\eta}^*_{+,2}$ s.t.

$$\hat{\eta}^*_{+,1}\Omega = \Psi_1, \quad \hat{\eta}^*_{+,2}\Omega = \Psi_2,$$

where Ω is the vacuum in $\mathcal{H} = \Gamma(L^2(\mathbb{R}^3)_{\mathrm{at}}) \otimes \Gamma(L^2(\mathbb{R}^3)_{\mathrm{ph}})$.

3. Then we can set:

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Fiber Hamiltonians

1. Since $H^{(1)}$ commutes with $P^{(1)}$, we can write

$$H^{(1)} = \Pi^* \int^{\oplus} d^3\xi \, H^{(1)}(\xi) \, \Pi,$$

where $H^{(1)}(\xi)$ are operators on $\Gamma(L^2(\mathbb{R}^3))$.

2. Let $\psi_\xi \in \Gamma(L^2(\mathbb{R}^3))$ be ground-states of $H^{(1)}(\xi)$ i.e.

$$H^{(1)}(\xi)\psi_\xi = E(\xi)\psi_\xi.$$

3. For $h \in C_0^\infty(\mathbb{R}^3)$ consider $\Psi_h \in \mathcal{H}_{\rm sp}$ given by

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Renormalized creation operators of 'atoms'

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$$\Psi_h := \Pi^* \int^{\oplus} d^3\xi\, h(\xi)\psi_\xi.$$

2. Let us define the renormalized creation operator of Ψ_h:

$$\hat{\eta}^*(h) := \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \int d^3\xi d^{3n}k\, h(\xi) f_\xi^n(k_1, \dots, k_n) a_{k_1}^* \dots a_{k_n}^* \eta_{\xi-\underline{k}}^*,$$

where $\{f_\xi^n\}_{n\in\mathbb{N}}$ are the Fock space components of ψ_ξ.

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Asymptotic creation operators of 'atoms'

Definition

For $h \in C_0^\infty(\mathbb{R}^3)$ let us define

$$\hat{\eta}_t^*(h) := \mathrm{e}^{iHt} \hat{\eta}^*(e^{-iEt} h) \mathrm{e}^{-iHt}.$$

$\hat{\eta}_+^*(h) := \lim_{t \to \infty} \hat{\eta}_t^*(h)$ is called the asymptotic creation operator of Ψ_h (if it exists).

Scattering states of two atoms

Theorem (Fröhlich 73, Albeverio 73...W.D.-Pizzo 13)

For $h_1, h_2 \in C_0^\infty(\mathbb{R}^3)$ with disjoint supports the limits

$$\Psi^+_{h_1,h_2} := \lim_{t\to\infty} \hat{\eta}_t^*(h_1)\hat{\eta}_t^*(h_2)\Omega$$

exist and span a subspace naturally isomorphic to $\mathcal{H}_{\mathrm{sp}} \otimes_{\mathrm{s/a}} \mathcal{H}_{\mathrm{sp}}$.

1. This theorem was proven before at fixed infrared cut-off (Fröhlich) and assuming non-zero photon mass (Albeverio).
2. We treat the case of massless photons and $\tilde{\rho}(k) = \chi(k)|k|^\alpha$, $\alpha > 0$, $\chi(k) > 0$ near zero (no infrared cut-off in H).
3. However, we have to replace f_ξ^n with f^n_{ξ,σ_t}, in $\hat{\eta}_t^*(h_i)$. Also, our result holds only in the weak coupling regime.

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Idea of the proof

1. Let $\Psi_t := \mathrm{e}^{itH}\hat{\eta}^*(h_{1,t})\hat{\eta}^*(h_{2,t})\Omega$, with $h_{i,t}(\xi) = e^{-itE(\xi)}h_i(\xi)$.

$$\partial_t \Psi_t = \mathrm{e}^{itH} i[[H_I, \hat{\eta}^*(h_{1,t})], \hat{\eta}^*(h_{2,t})]\Omega.$$

2. This can be expressed by integrals of the form

$$\int d^3\tilde{r}\, \overline{g}\frac{\tilde{\rho}(\tilde{r})}{\sqrt{2|\tilde{r}|}} e^{-i(E(p-\tilde{r})+E(q+\tilde{r}))t} h_1(p-\tilde{r})h_2(q+\tilde{r}) f^{n+1}_{q+\tilde{r}}(r,\tilde{r}) f^m_{p-\tilde{r}}(k).$$

3. (Non-) stationary phase gives integrable decay of $\partial_t\Psi_t$, provided we can control derivatives of $\xi \mapsto f^n_\xi(k)$ up to second order.
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Idea of the proof

1. Let $\Psi_t := \mathrm{e}^{itH}\hat{\eta}^*(h_{1,t})\hat{\eta}^*(h_{2,t})\Omega$, with $h_{i,t}(\xi) = e^{-itE(\xi)}h_i(\xi)$.
$$\partial_t\Psi_t = \mathrm{e}^{itH} i[[H_I, \hat{\eta}^*(h_{1,t})], \hat{\eta}^*(h_{2,t})]\Omega.$$
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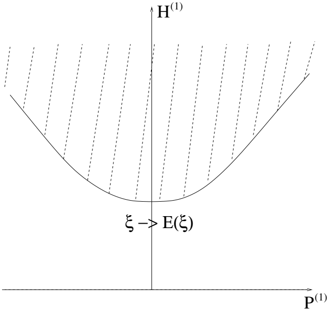
Charged particle ('electron')



Theorem (Fröhlich 74...Hasler-Herbst 07)

$\mathcal{H}_{\mathrm{sp}} = \mathbf{1}_{\{0\}}(H^{(1)} - E(P^{(1)}))\mathcal{H}^{(1)} = \{0\}$ *for* $\tilde{\rho}(0) \neq 0$, $\overline{g} \neq 0$.

Fröhlich-Pizzo-Chen approach

- Time-dependent IR cut-off: $H^{(1)} \to H^{(1)}_{\sigma_t}$. Then $\mathcal{H}_{\mathrm{sp},\sigma_t} \neq \{0\}$.
- "Undressed" electron states: $\Psi^{(t)}_{j,\sigma_t} \in \mathbf{1}_{\Gamma^{(t)}_j}(P^{(1)})\mathcal{H}_{\mathrm{sp},\sigma_t}$, where $\Gamma^{(t)}_j$ is a cube in $P^{(1)}$-space.
- Photon cloud: $W(v_j) = \exp\left[-\overline{g} \int_{\sigma_t}^{\kappa} \frac{a_k - a_k^*}{|k|(1-\hat{k}\cdot v_j)} \frac{d^3k}{\sqrt{2|k|}} \right]$, where $v_j \in \{ \nabla E^{\sigma_t}(\xi) \,|\, \xi \in \Gamma^{(t)}_j \}$.
- "Dressed" electron state:

$$\Psi_\kappa = \lim_{t\to\infty} \sum_j e^{i\gamma_{j,t}} (e^{iH^{(1)}t} e^{-iH_{\mathrm{ph}}t} W(v_j) e^{iH_{\mathrm{ph}}t} e^{-iH^{(1)}_{\sigma_t}t}) \Psi^{(t)}_{j,\sigma_t}$$

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Towards scattering of two 'electrons' in the Nelson model

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$$\Psi_{1,\kappa} \overset{\mathrm{out}}{\times} \Psi_{2,\kappa} = \lim_{t\to\infty} \sum_{j_1,j_2} e^{i\gamma_{j_1,j_2,t}} (e^{iHt} e^{-iH_{\mathrm{ph}}t} W(v_{j_1}) W(v_{j_2}) e^{iH_{\mathrm{ph}}t} e^{-iHt}) \times$$

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3. This limit is not under control (yet). The intermediate case of atom-electron scattering is under investigation.

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🔴 - not understood

🟢 - (partially) understood

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