Spectral Properties of Non-Unitary Band Matrices*

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Quantum Walk: Particle with spin hopping on $\mathbb{Z}^2$

Setup: $\mathcal{K} = \mathbb{C}^4 \otimes l^2(\mathbb{Z}^2)$

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$\{|x\rangle\}_{x \in \mathbb{Z}^2}$ for $l^2(\mathbb{Z}^2)$

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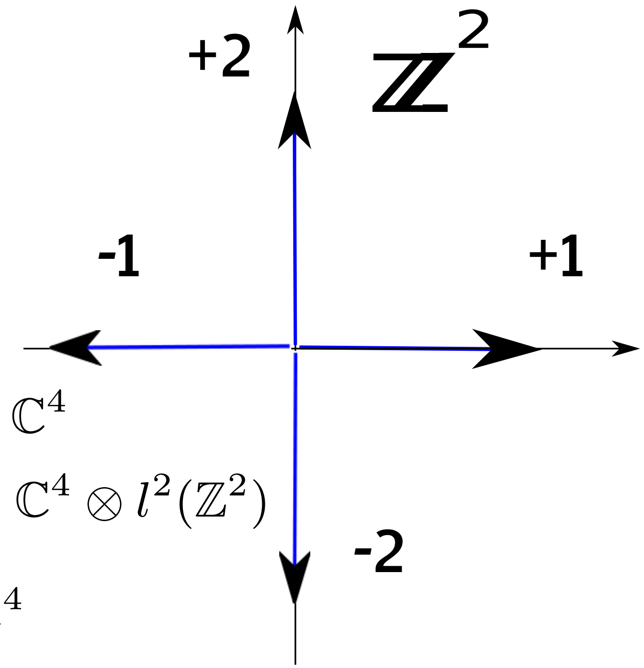

Ingredients:

- Spin dep. shift: Let P_τ the proj. "on" $|\tau\rangle \in \mathbb{C}^4$

$S := \sum_{x \in \mathbb{Z}^2} \sum_{\tau \in I_\pm} P_\tau \otimes |x + \text{sign}(\tau) e_{|\tau|}\rangle\langle x|$ on $\mathbb{C}^4 \otimes l^2(\mathbb{Z}^2)$

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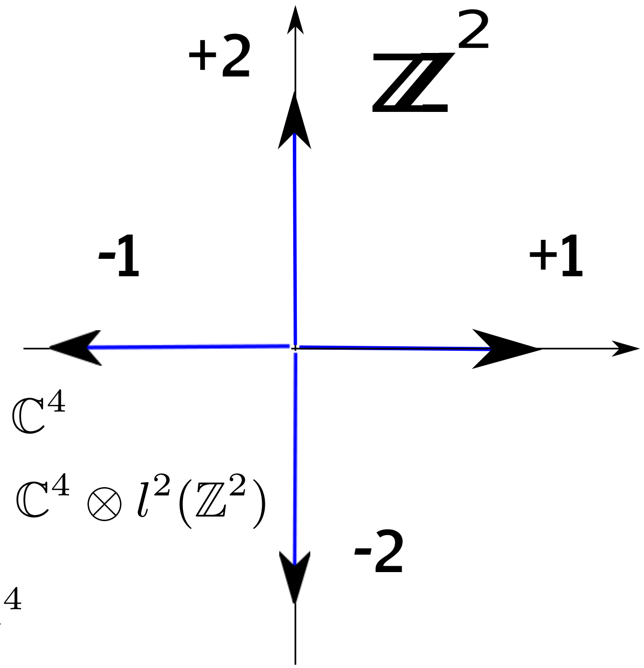

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$$U(C) := S(C \otimes \mathbb{I}) = \sum_{x \in \mathbb{Z}^2} \sum_{\tau \in I_\pm} (P_\tau C) \otimes |x + \text{sign}(\tau) e_{|\tau|}\rangle\langle x|$$

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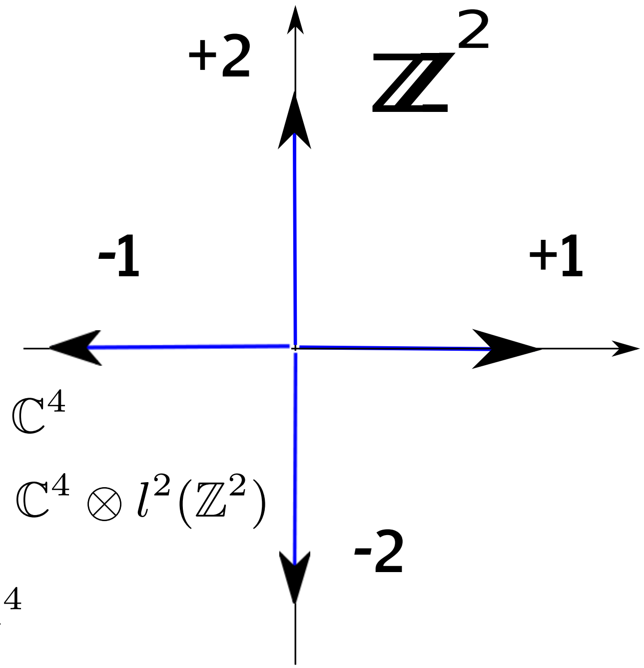

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Time one dynamics of a random QW:

$$\boxed{U_\omega(C) := \sum_{x \in \mathbb{Z}^2} \sum_{\tau \in I_\pm} (P_\tau C_\omega(x)) \otimes |x + \text{sign}(\tau) e_{|\tau|}\rangle\langle x|}$$

Our choice: $C_\omega(x)_{\tau,\sigma} = \exp(i\omega^\tau_{x+\text{sign}(\tau)e_{|\tau|}}) C_{\tau,\sigma}$. JM '10, J '12, HJ '14

Set $\mathbb{D}_\omega = \text{diag}(\exp(i\omega^\tau_x))$, then $\boxed{U_\omega(C) = \mathbb{D}_\omega U(C)}$

Dynamics $\leftrightarrow$ Spectral Properties of RQW

Special case: in the ordered spin basis $\{|+1\rangle, |+2\rangle, |-1\rangle, |-2\rangle\}$

Let $C = \begin{pmatrix} \alpha & r & \beta & 0 \\ q & g & s & 0 \\ \gamma & t & \delta & 0 \\ 0 & 0 & 0 & e^{i\theta} \end{pmatrix} \in U(4)$, where $\tilde{C} = \begin{pmatrix} \alpha & r & \beta \\ q & g & s \\ \gamma & t & \delta \end{pmatrix} \in U(3)$.

Dynamics:

- Spin $|-2\rangle$ decoupled and travels Southwards $\rightsquigarrow \mathcal{H}^{a.c.}(U_\omega(C))$
- Spin $|+1\rangle, |+2\rangle, |-1\rangle$ travel East-, North-, West-wards.
- Vector $|+2\rangle \otimes x_0$ cannot come back to $|+2\rangle \otimes x_0 \rightsquigarrow \mathcal{H}^{a.c.}(U_\omega(C))$

Using: $\sum_{n\in\mathbb{N}} |\langle \tau \otimes x| U^n \tau \otimes x\rangle|^2 < \infty \Rightarrow \tau \otimes x \in \mathcal{H}^{a.c.}(U)$.

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Proposition: Let $\mathcal{H} = \overline{\text{span}\{\tau \otimes x \,|\, \tau \in \{+1, -1\}, x = (x_1, 0), x_1 \in \mathbb{Z}\}}$,

P_0 projector on $\mathcal{H}$ and $T_\omega := P_0 U_\omega(C) P_0|_{\mathcal{H}}$, a contraction op.

Then $\boxed{P_0 U_\omega^n(C) P_0|_{\mathcal{H}} = T_\omega^n}$ $\forall \;\; n \in \mathbb{N}$.

Corollary: $\text{spr}\,(T_\omega) < 1 \;\Rightarrow\; U_\omega(C)$ is purely $a.c.$

Non-Unitary Random Band Matrices

Setup: Hilbert space $\mathcal{H} = l^2(\mathbb{Z})$, canonical basis $\{e_j\}_{j\in\mathbb{Z}}$

Random var. $\{\omega_j\}_{j\in\mathbb{Z}}$ on $\mathbb{T}$, iid, distrib. $d\nu(\theta) = l(\theta)d\theta$, $l \in L^\infty(\mathbb{T})$.

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CMV Type Random Operator: $T_\omega : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ s.t.

$$T_\omega = \begin{pmatrix} \ddots & e^{i\omega_{2j-1}}\gamma & e^{i\omega_{2j-1}}\delta & & & \\ & 0 & 0 & & & \\ & 0 & 0 & e^{i\omega_{2j+1}}\gamma & e^{i\omega_{2j+1}}\delta & \\ & e^{i\omega_{2j+2}}\alpha & e^{i\omega_{2j+2}}\beta & 0 & 0 & \\ & & & 0 & 0 & \\ & & & e^{i\omega_{2j+4}}\alpha & e^{i\omega_{2j+4}}\beta & \ddots \end{pmatrix}$$

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Note: $\boxed{T_\omega = \mathbb{D}_\omega T}$, $\mathbb{D}_\omega = \text{diag}(e^{i\omega_j})$ is unitary, and T is characterized by C_0

Polar Decomposition of $T_\omega = \mathbb{D}_\omega T$

Thm: $\boxed{T_\omega = V_\omega K}$ with

- V_ω unitary and $0 \le K \le \boldsymbol{I}$ deterministic
- $K = P_1 + gP_2, \ \ \sigma(K) = \{1, g\}$, tri-diagonal and dim $P_j = \infty, \ j = 1, 2$.
- $V_\omega = \mathbb{D}_\omega V$ with

$$V = \begin{pmatrix} \ddots & \gamma - \frac{qt}{1+g} & \delta - \frac{st}{1+g} & & & \\ & 0 & 0 & & & \\ & 0 & 0 & \gamma - \frac{qt}{1+g} & \delta - \frac{st}{1+g} & \\ & \alpha - \frac{qr}{1+g} & \beta - \frac{sr}{1+g} & 0 & 0 & \\ & & & 0 & 0 & \\ & & & \alpha - \frac{qr}{1+g} & \beta - \frac{sr}{1+g} & \ddots \end{pmatrix}$$

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V_ω is a Random Quantum Walk JM '10

$$\sigma(V_\omega) = \sigma_p(V_\omega) \quad \text{a.s.} \quad \Leftrightarrow \quad \beta(1+g) \neq sr$$

$$\sigma(V_\omega) = \sigma_{ac}(V_\omega) = \mathbb{S} \;\; \forall \omega \quad \Leftrightarrow \quad \beta(1+g) = sr$$

T_ω s.t. $\|T_\omega\| = 1$, spr$(T_\omega) \leq 1$, T_ω unitary $\Leftrightarrow g = 1, \;\; \forall \omega$.

Infos on $\sigma(VK)$ from $\sigma(V)$ and $\sigma(K)$

Notation: $B_c(r)$ open ball of center c, radius r

Thm: Let V, K bounded, normal, invertible. Then

$$\bigcup_{\tau\in\rho(V)} \bigcap_{k\in\sigma(K)} B_{\tau k}(|k|\text{dist}(\tau,\sigma(V))) \subset \rho(VK)$$ also "$V \leftrightarrow K$".

$\tau = 0$:

$$B_0(\text{dist}(0,\sigma(V))\text{dist}(0,\sigma(K))) \subset \rho(VK)$$

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In our case

$$\sigma(K)=\{1,g\} \text{ and } \sigma(V)\subset\mathbb{S} \text{ so}$$

- $B_0(g)\subset\rho(VK)$
- For any $\tau\in\rho(V)$, intersection on $\sigma(K)$ reduces to

$$B_\tau(\text{dist}(\tau,\sigma(V)))\cap B_{g\tau}(g\ \text{dist}(\tau,\sigma(V))),$$

- $B_{g\tau}(g\ \text{dist}(\tau,\sigma(V)))$ is a dilation by g of $B_\tau(\text{dist}(\tau,\sigma(V)))$

$\bigcup_{\tau\in\rho(V)} B_\tau(\textbf{dist}(\tau,\sigma(V))) \cap B_{g\tau}(g\ \textbf{dist}(\tau,\sigma(V))) \subset \rho(VK)$

Further assume

$\sigma(K) = \{1, g\}$, $1 > g > 0$, and $\sigma(V) = \{e^{iv} \mid \theta \le v \le 2\pi - \theta\}$, $0 \le \theta < \pi/2$

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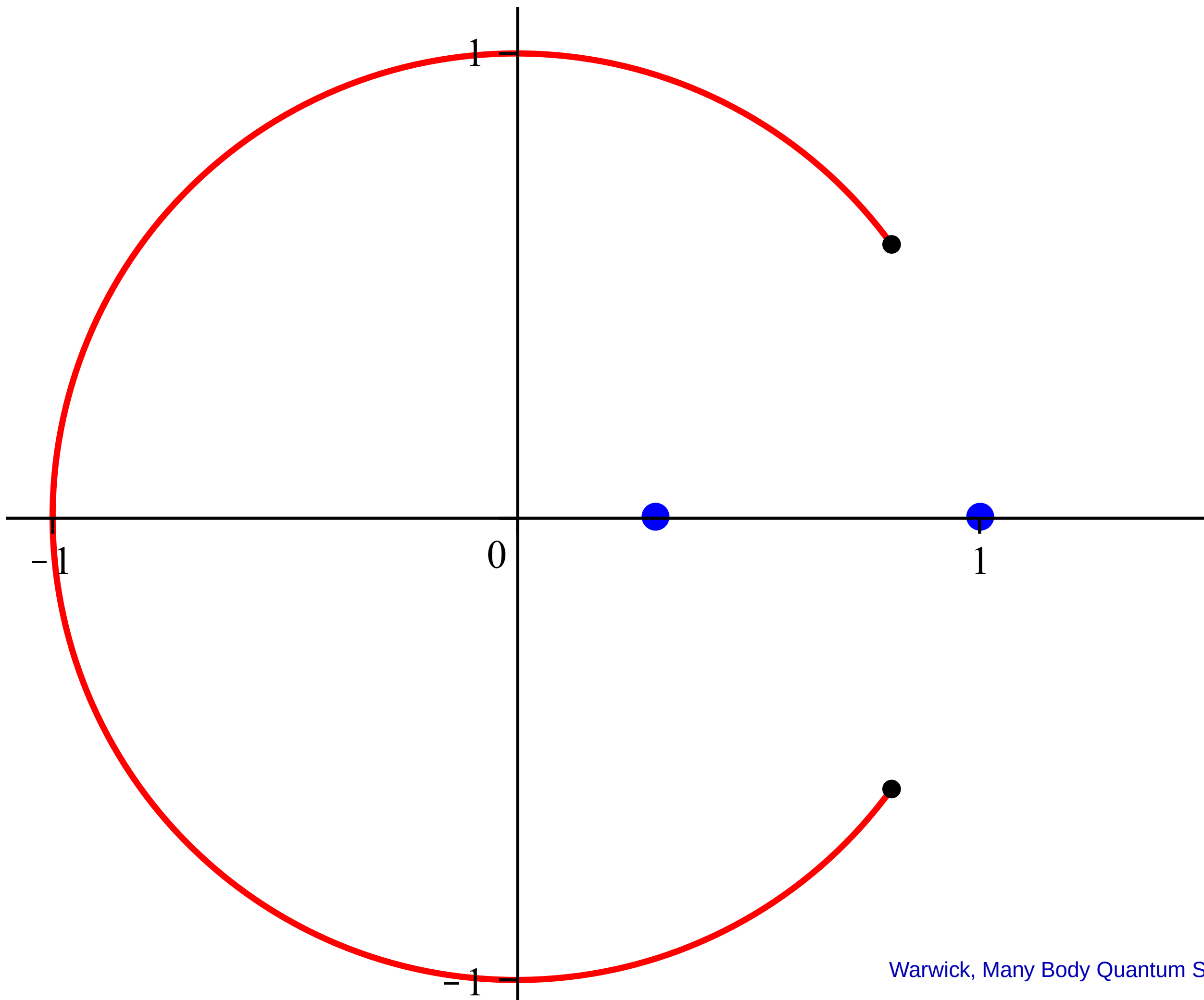

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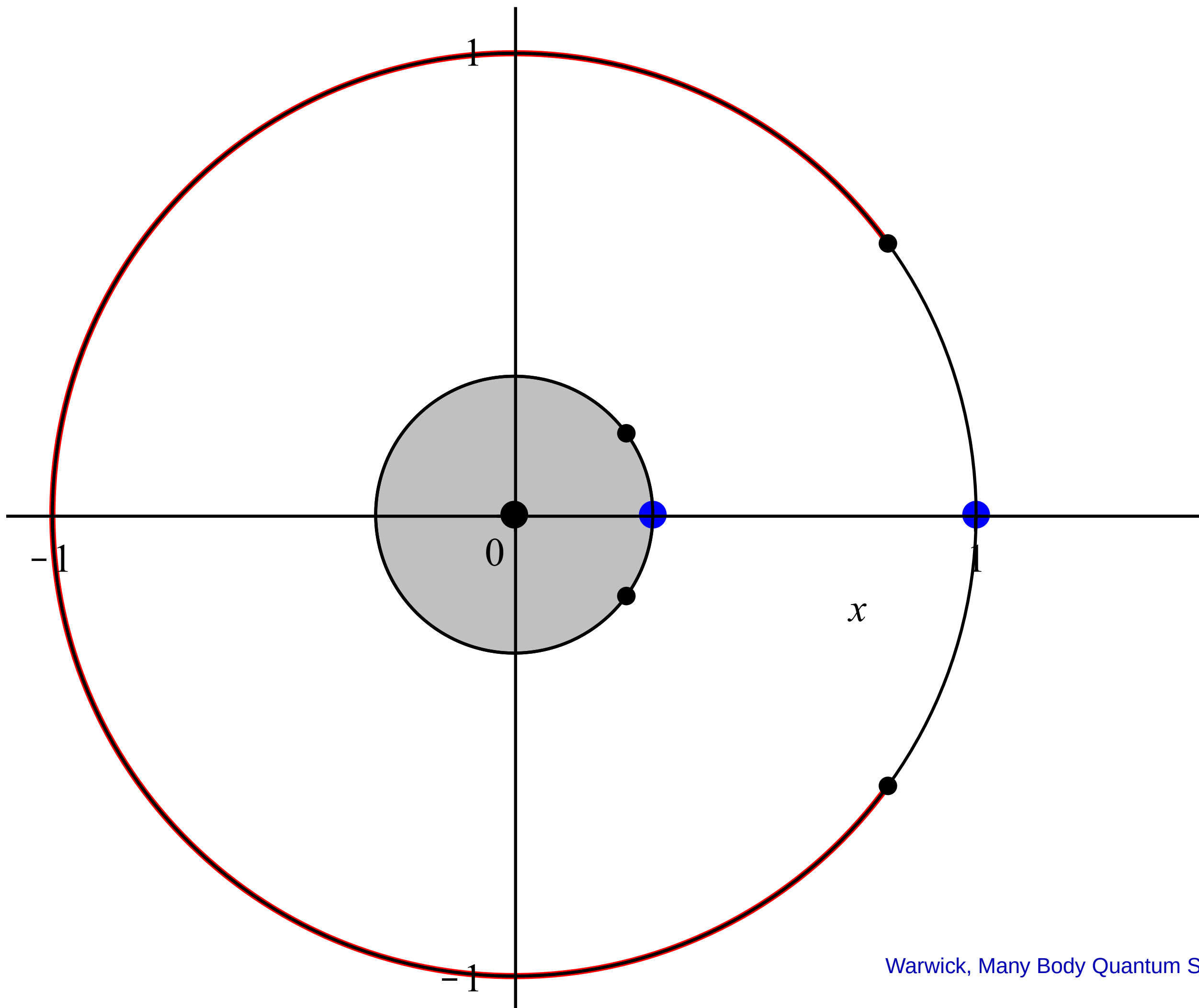

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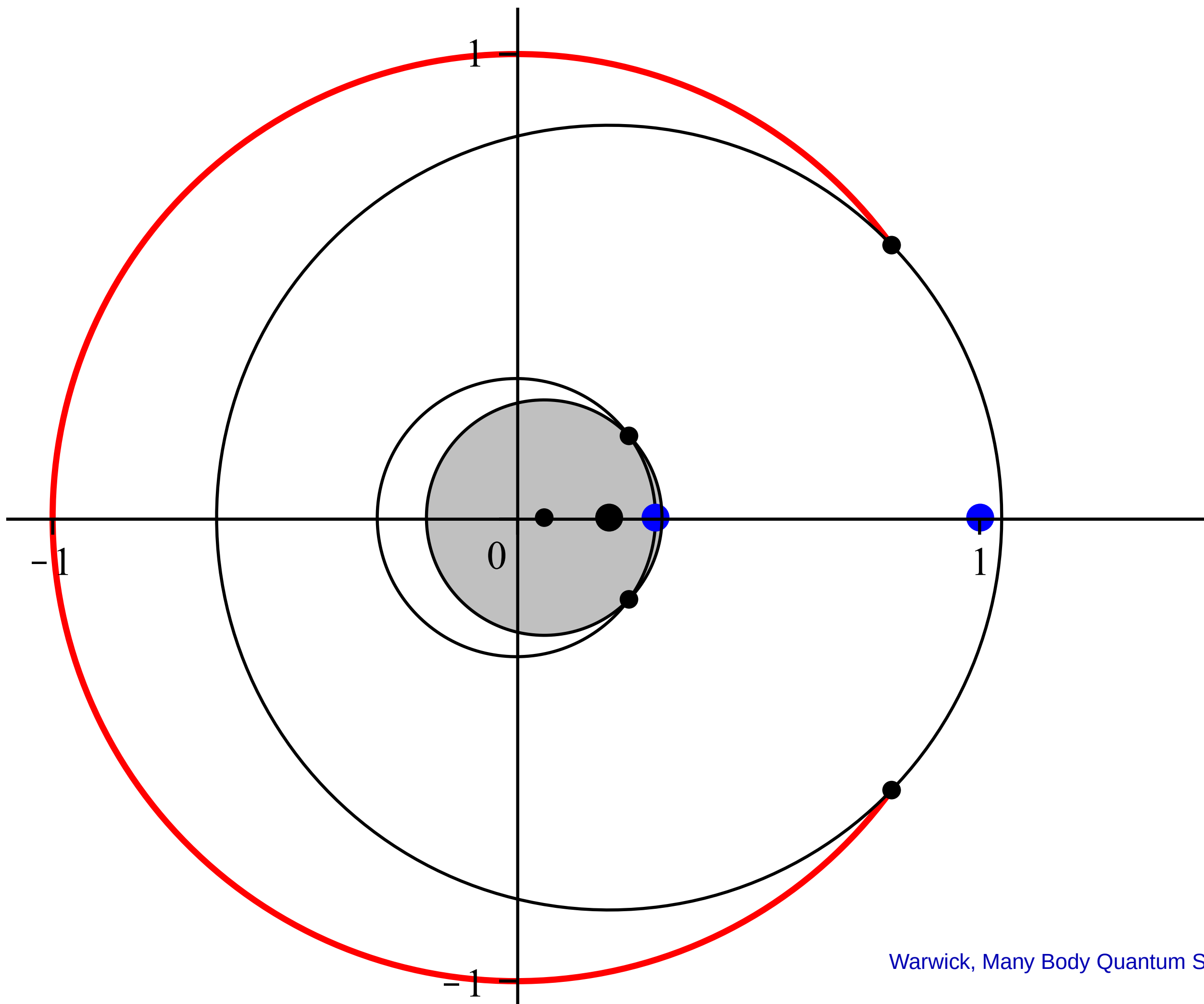

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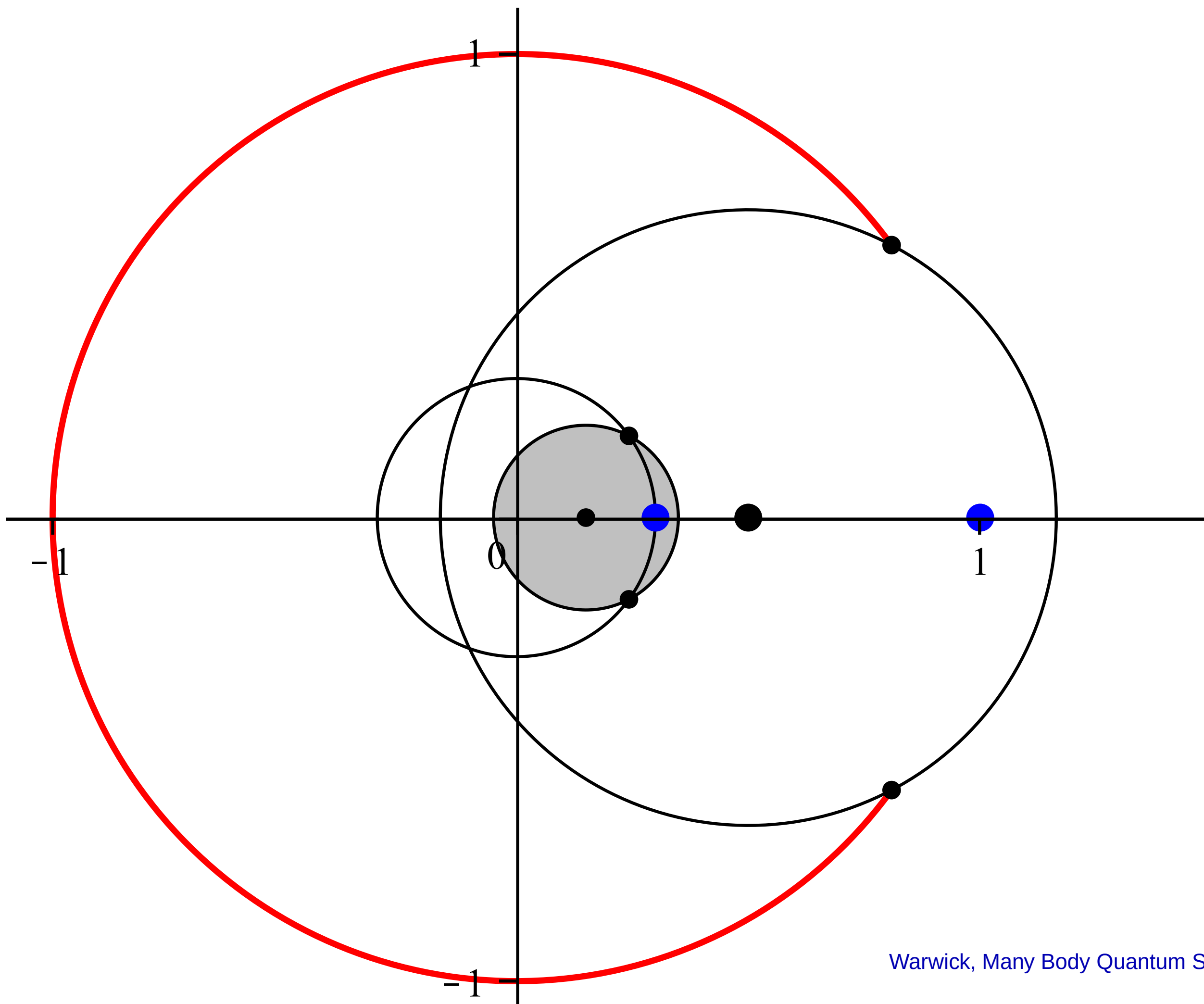

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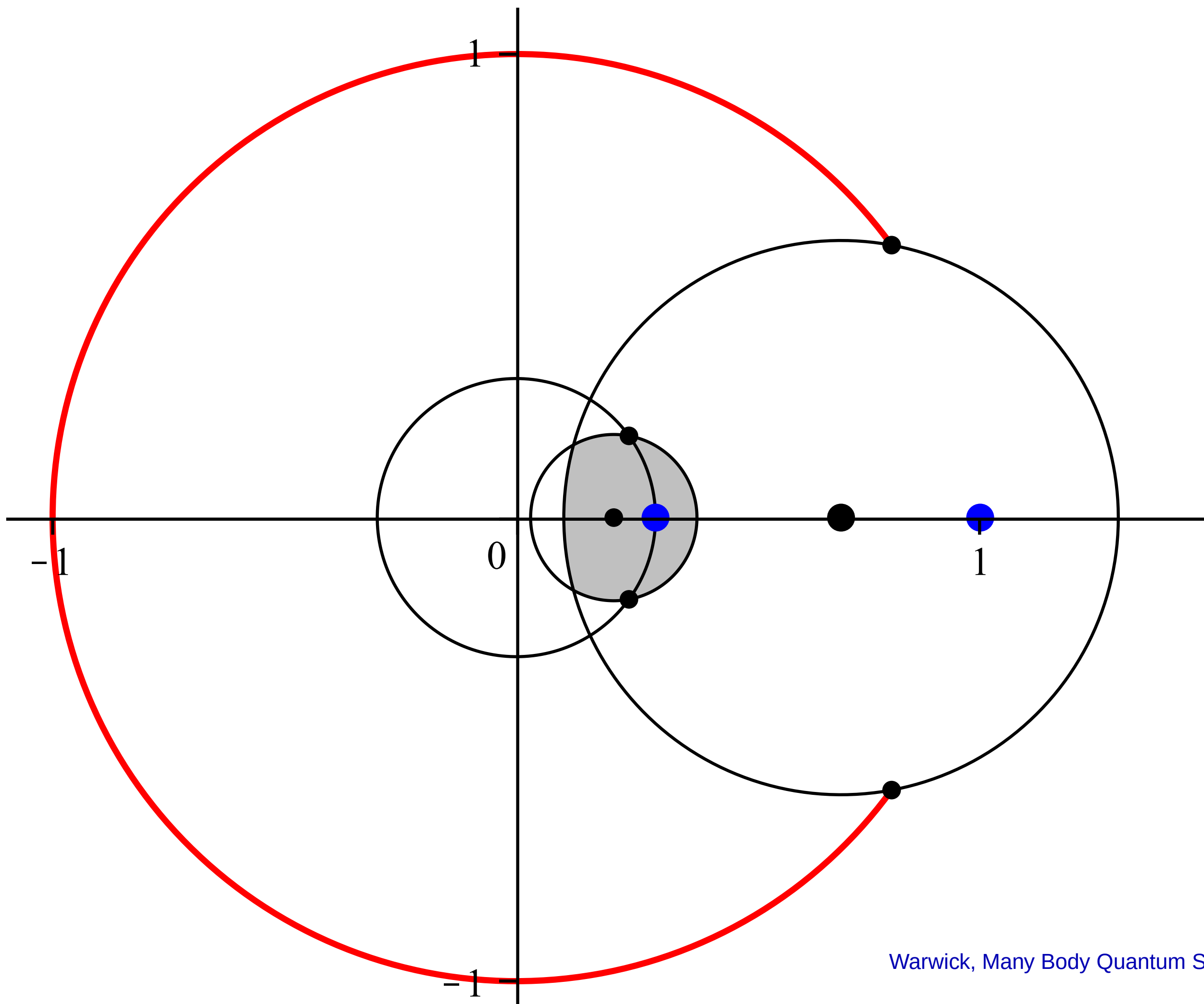

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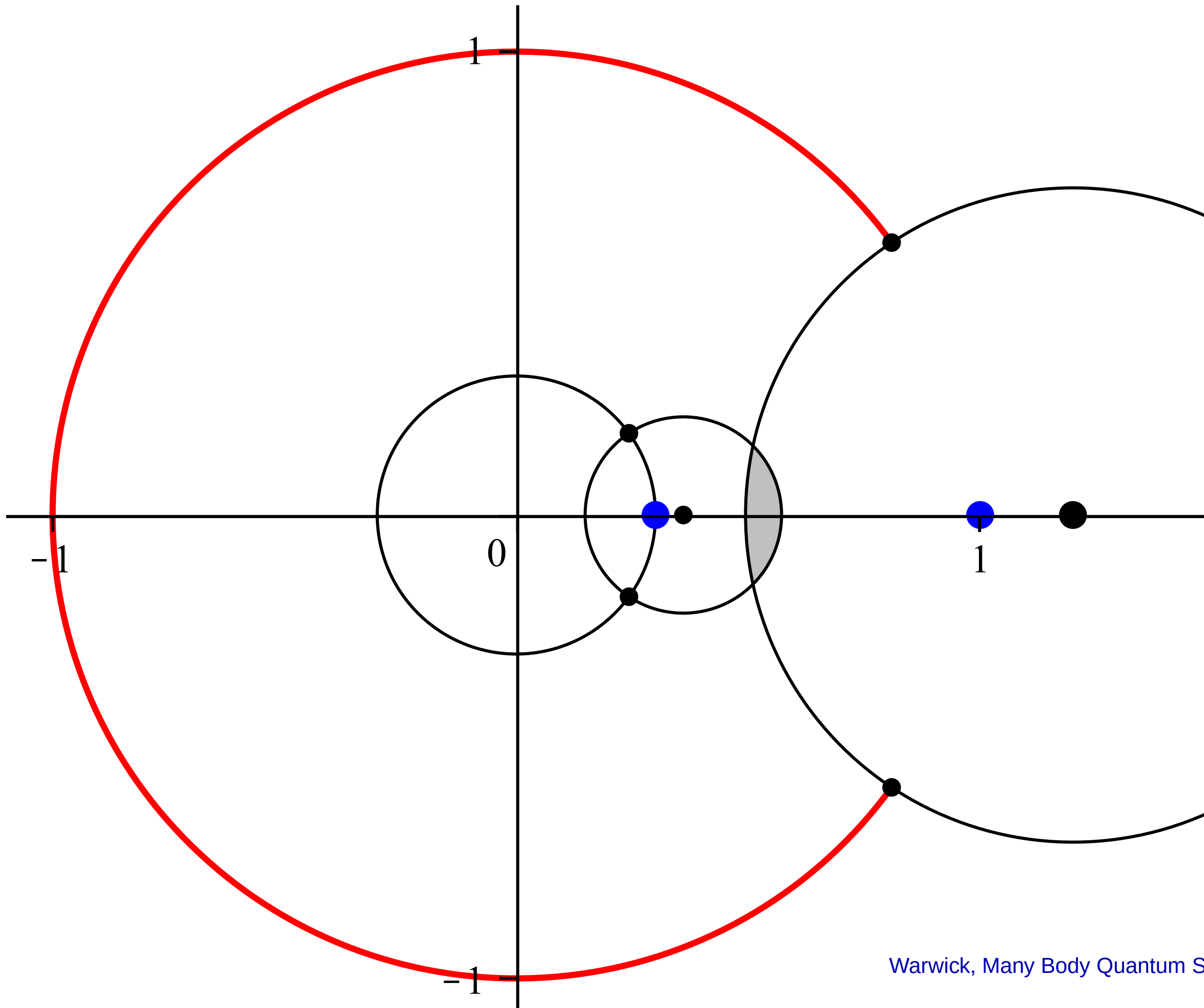

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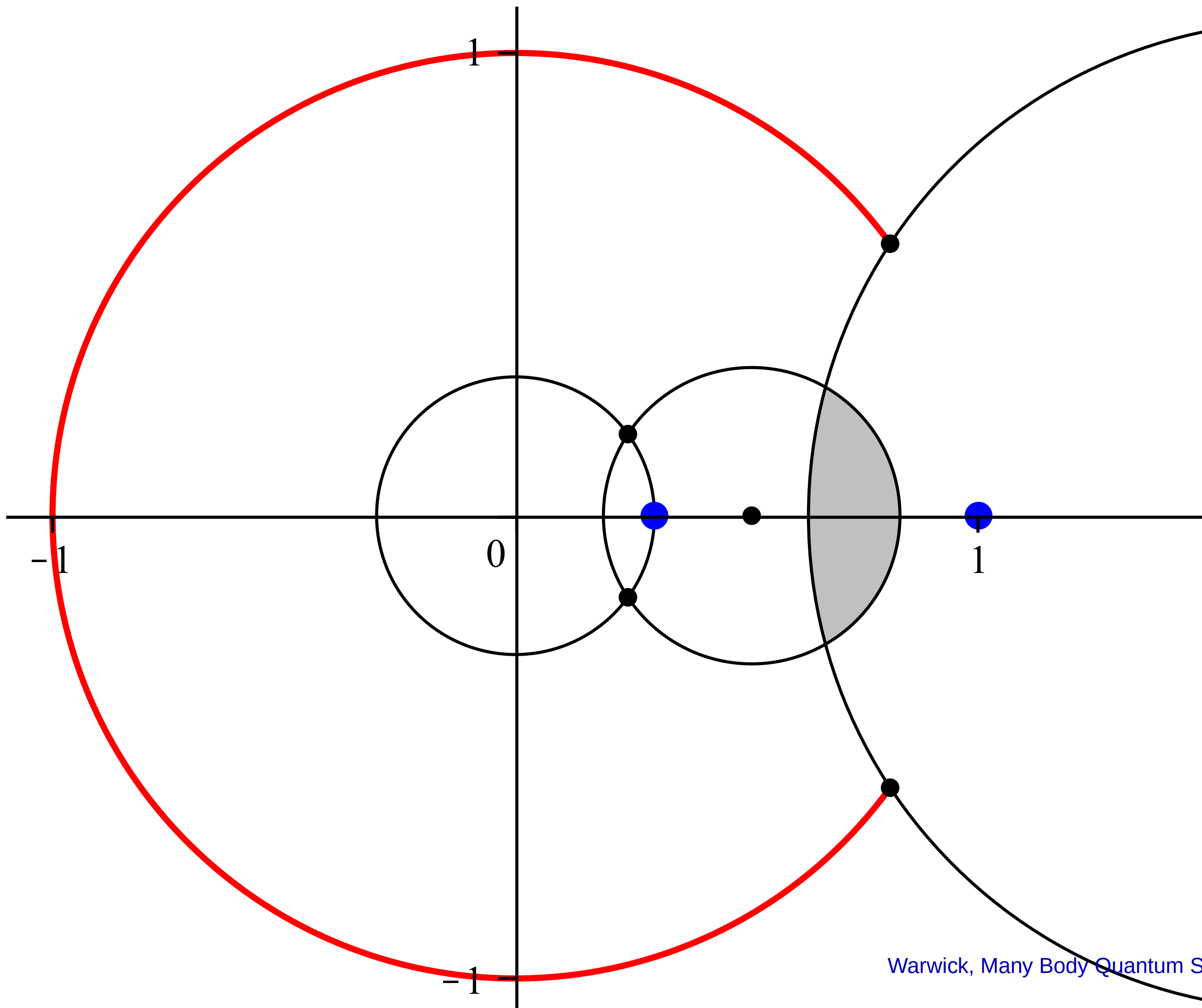

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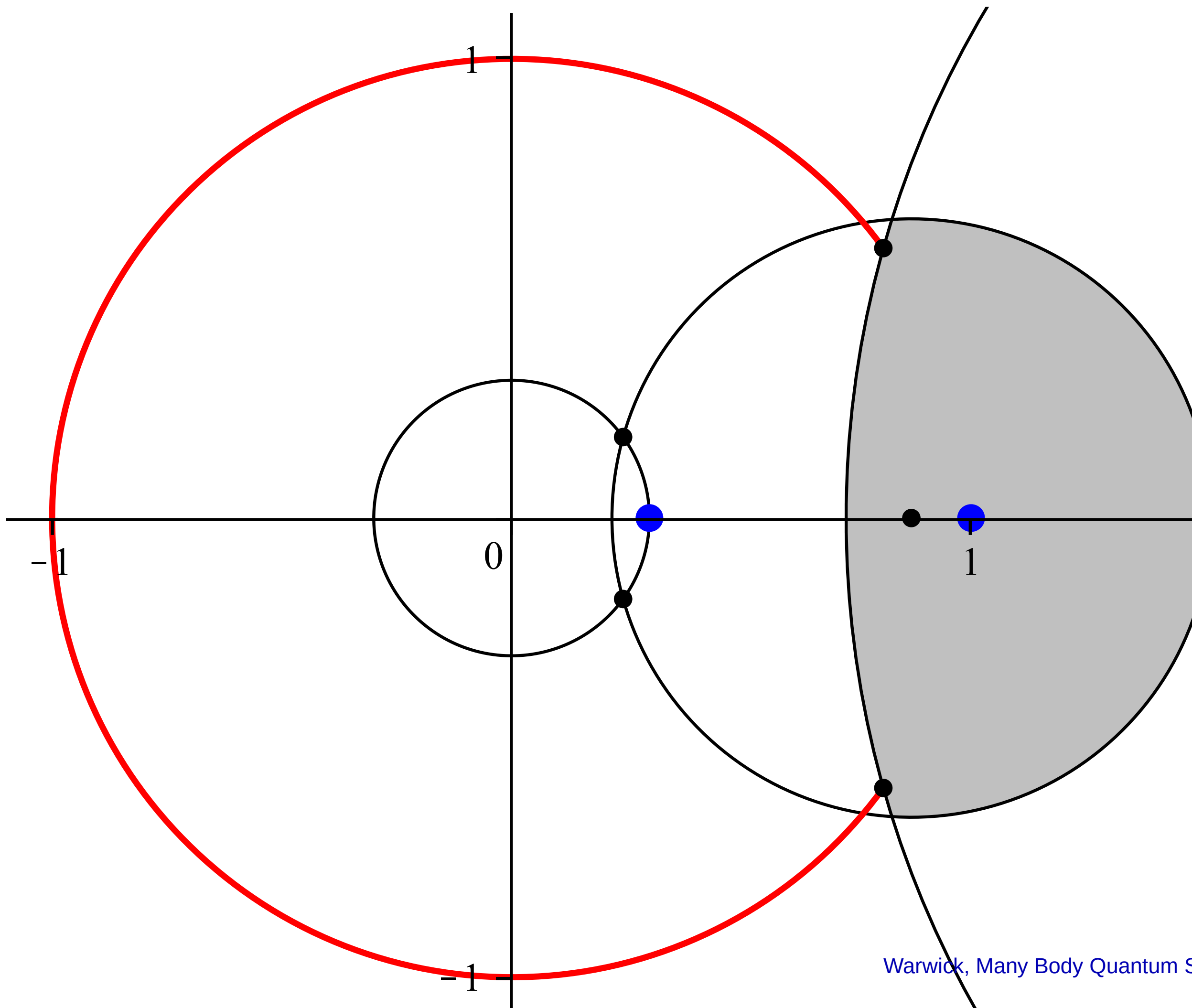

Subset of $\rho(VK)$

Thm: i) $\cup_{\tau \geq 0} \cap_{k \in \{1,g\}} (\cdots) = \cup_{\tau \in \rho(V)} \cap_{k \in \{1,g\}} (\cdots)$

ii) $B_0(g) \cup D \cup \{\text{Re } z \geq \cos(\theta)\} \subset \rho(VK)$, where

$$\partial D = \{y^2 = \frac{x(x^2 - x(1+g)\cos(\theta) + g)}{(1+g)\cos(\theta) - x}\}$$

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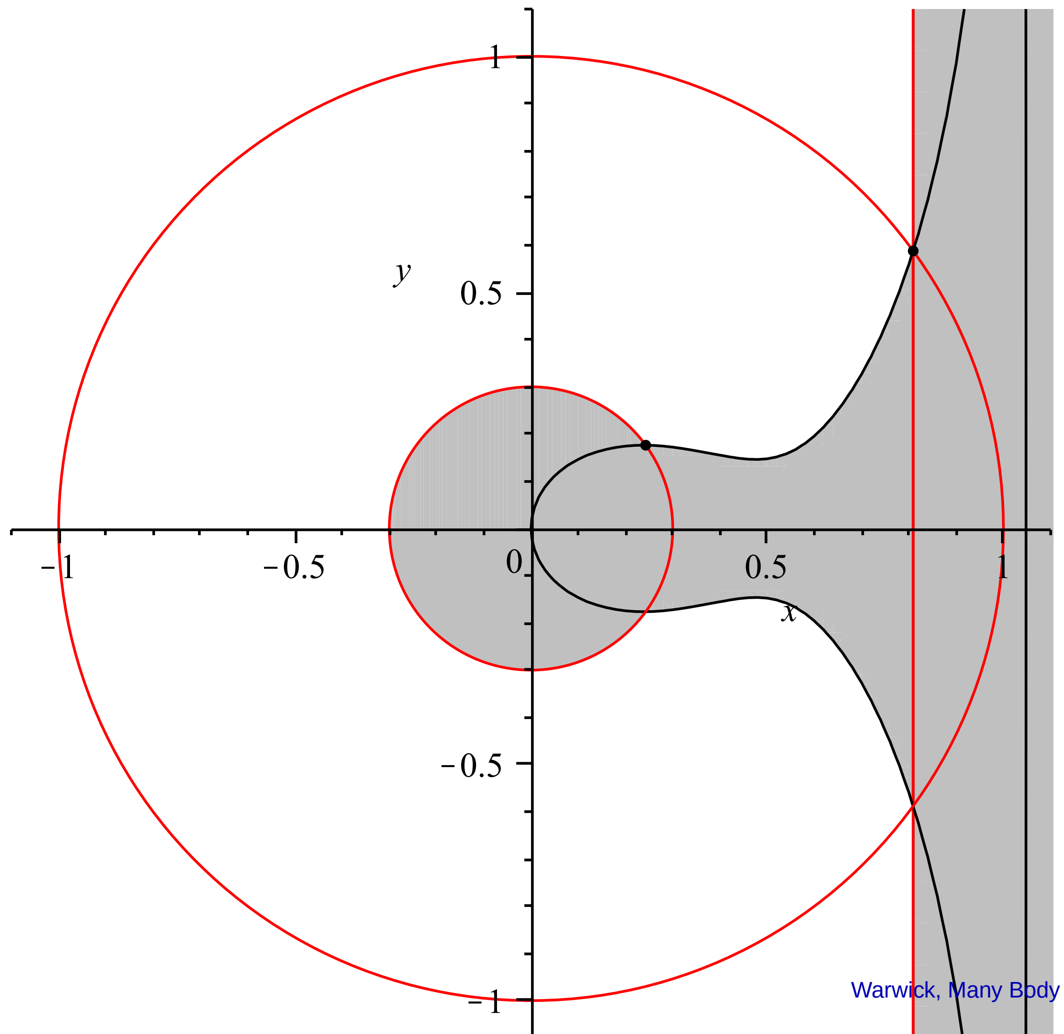

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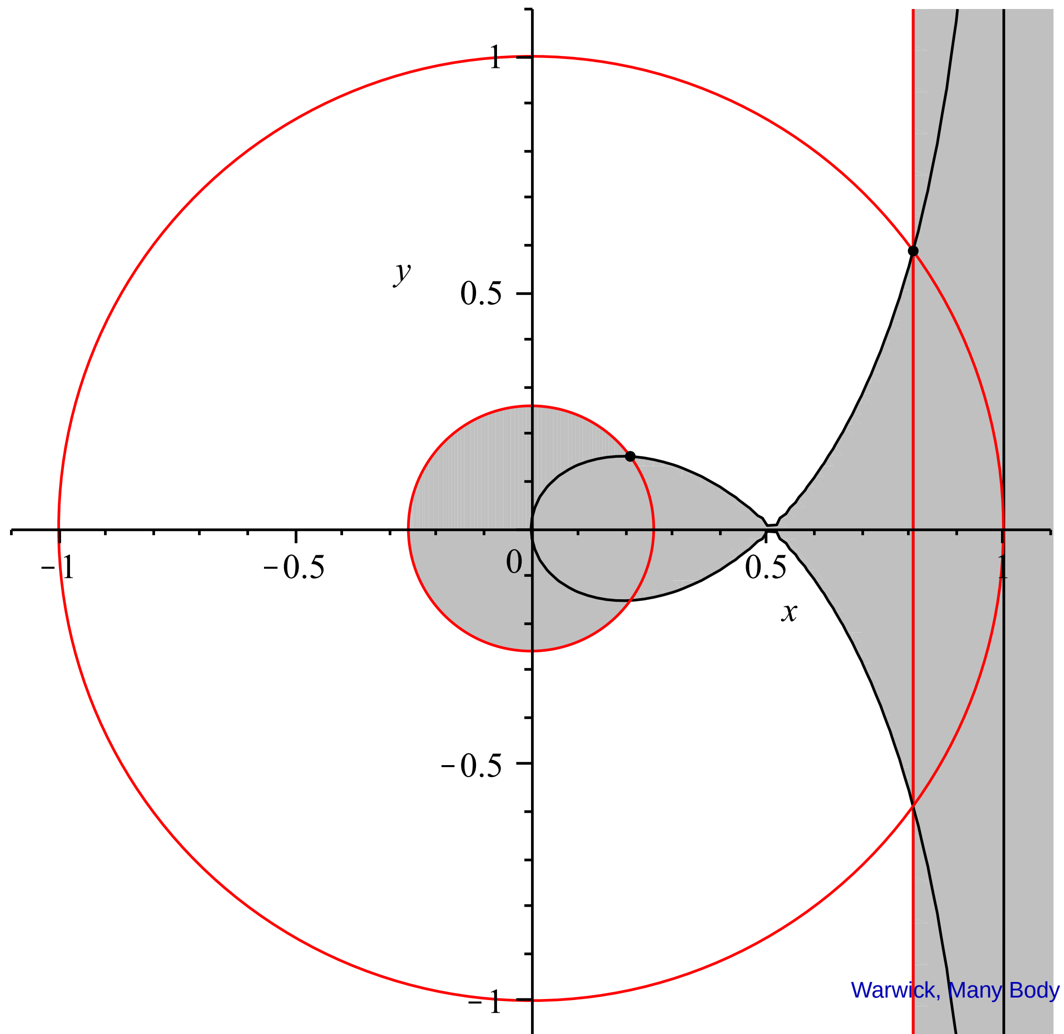

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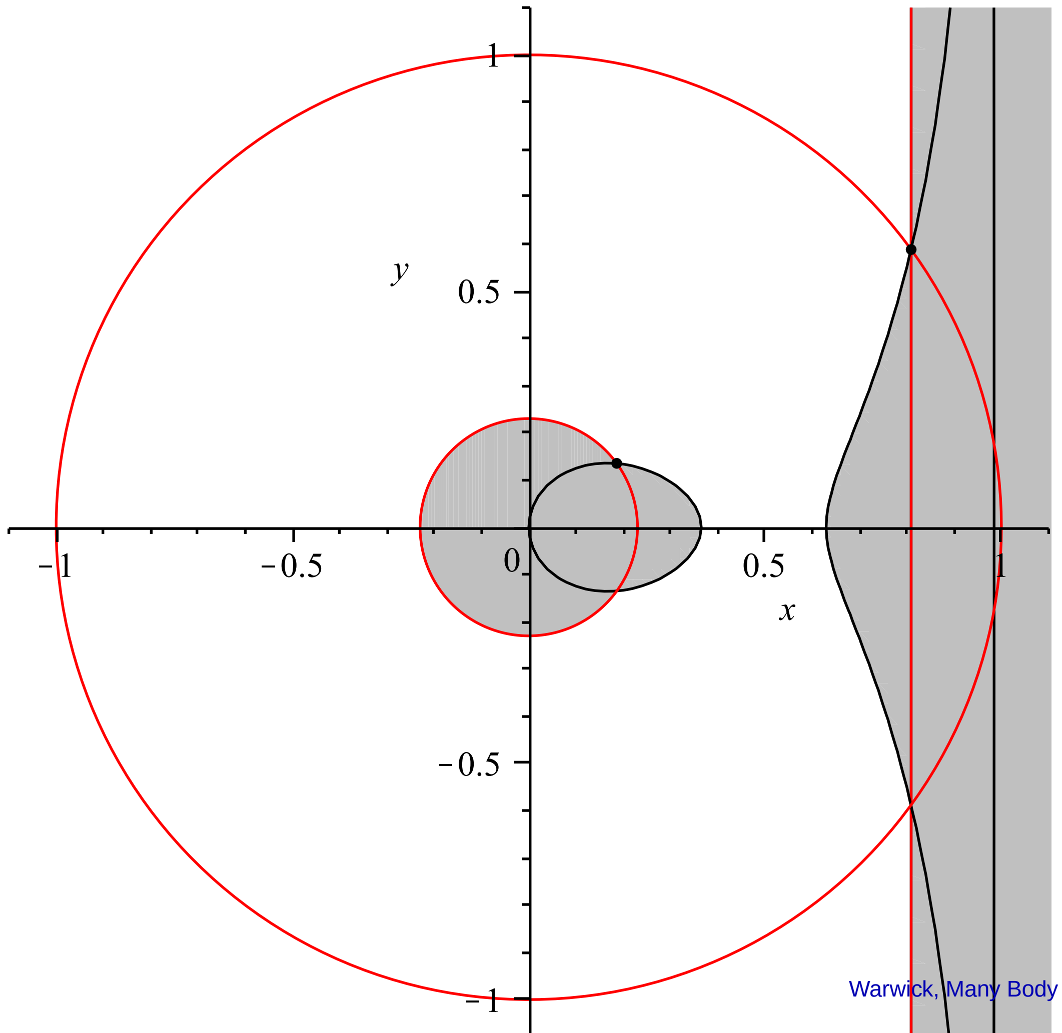

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Remarks:

- If $\sigma(V) \subset \{e^{iv} | \theta \leq v \leq 2\pi - \theta\}$, ii) still holds $\Rightarrow$ Several similar sets if $\sigma(V)$ displays several gaps.
- $[0,1] \subset \rho(VK) \Leftrightarrow \cos^2(\theta) < \frac{4g}{(1+g)^2}$, cf matrix case.
- The set corresponding to "$V \leftrightarrow K$" is

$$\bigcup_{\tau \in \rho(K)} \bigcap_{v \in \sigma(V)} B_{\tau v}(|v| \text{dist}(\tau, \sigma(K))) \subset \rho(VK)$$

and is shown to be contained in ii).

Example: $T_\omega = \mathbb{D}_\omega T$ with $T \leftrightarrow C_0 = \begin{pmatrix} \cos(\eta) & -\sin(\xi)\sin(\eta) \\ \sin(\eta) & \sin(\xi)\cos(\eta) \end{pmatrix}$

Where $\eta, \xi \in]0, \pi/2[$,

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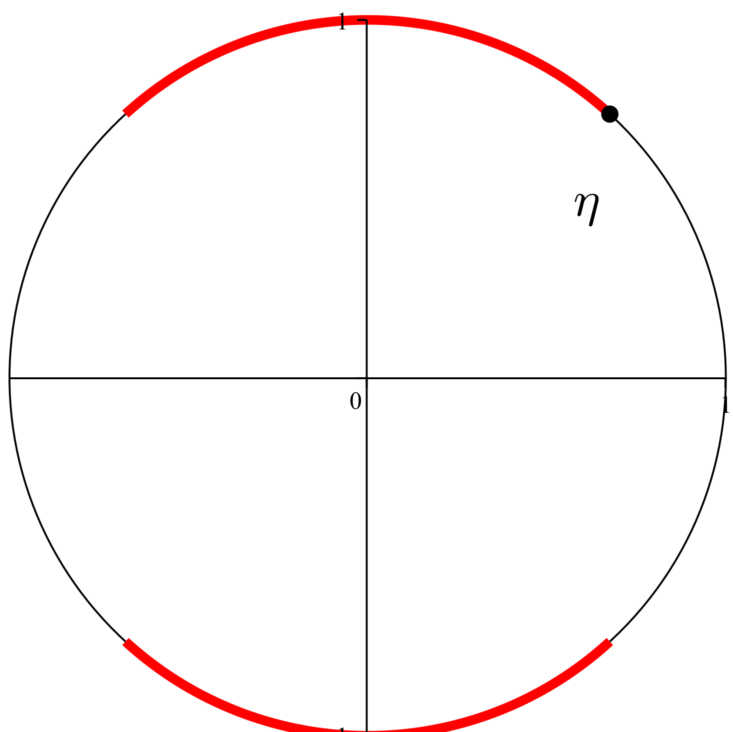

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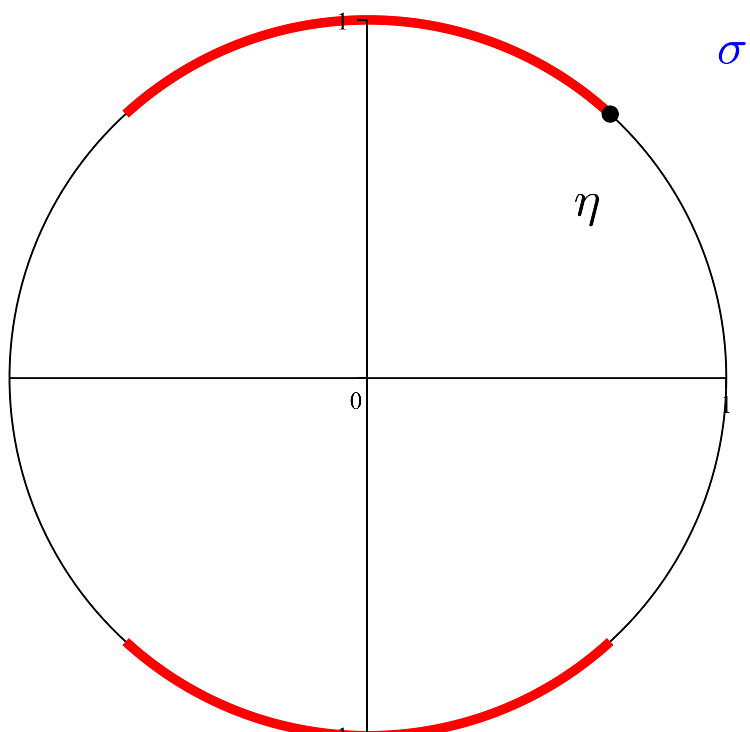

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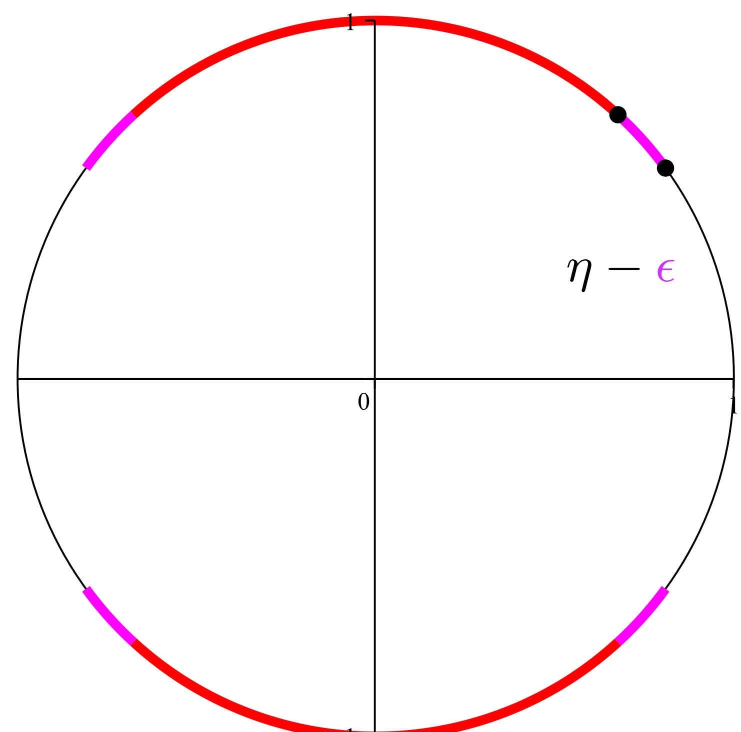

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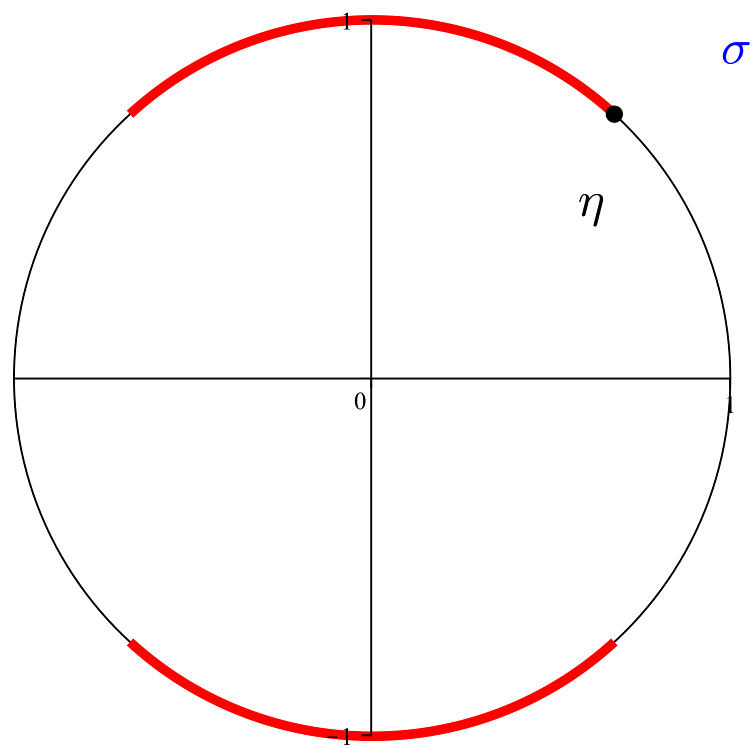

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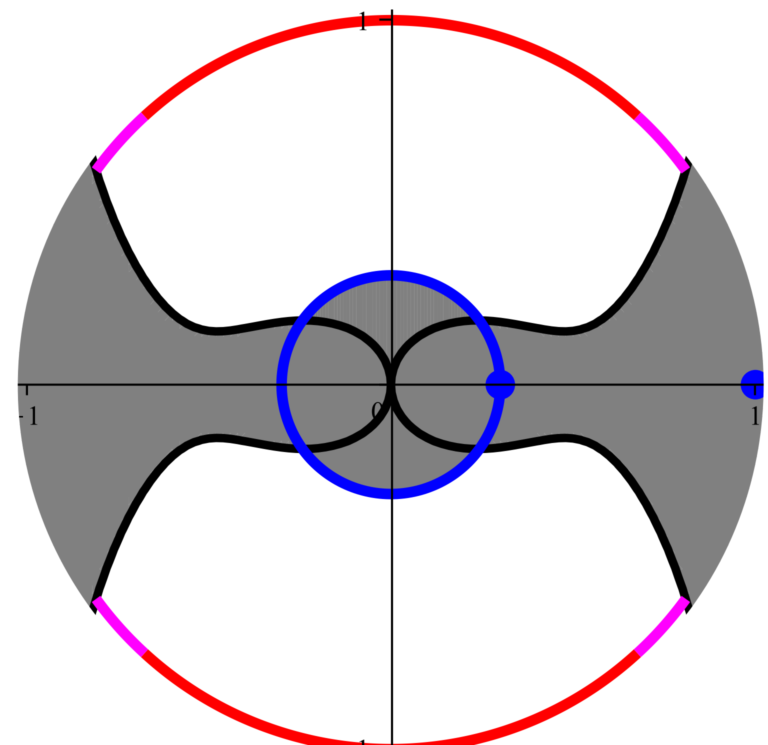

Feschbach-Schur Method $T_\omega = \mathbb{D}_\omega V(P_1 + gP_2)$, $0 < g < 1$

Structure: $T_\omega = \begin{pmatrix} P_1 V_\omega P_1 & g P_1 V_\omega P_2 \\ P_1 V_\omega P_1 & g P_2 V_\omega P_2 \end{pmatrix}$. Let $V_{jk} := P_j V P_k : P_k \mathcal{H} \to P_j \mathcal{H}$

Thm: If $\|V_{11}\| < 1$, then, $\forall\, \omega$

$$\boxed{g < \frac{1 - \|V_{11}\|}{\|V_{21}\|\|V_{12}\| + \|V_{22}\|(1 - \|V_{11}\|)} \quad \Rightarrow \quad \text{spr}\,(T_\omega) < 1}$$

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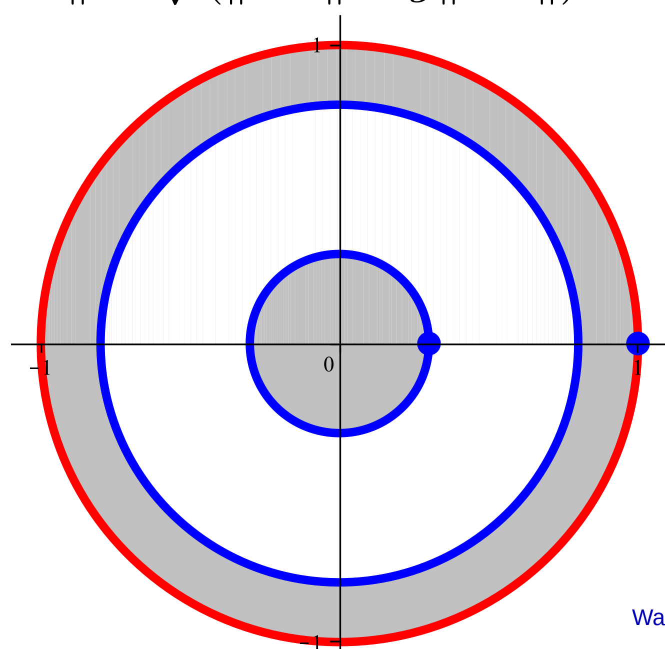
Actually, $\boxed{B_0(g) \cup \{r(V) < |z| \leq 1\} \subset \rho(T_\omega)}$, $\forall\, \omega$, where

$$r(V) = \frac{1}{2}\left(\|V_{11}\| + g\|V_{22}\| + \sqrt{(\|V_{11}\| - g\|V_{22}\|)^2 + 4g\|V_{21}\|\|V_{12}\|}\right) > g.$$



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Remarks:

- $\|V_{11}\| = \frac{|\delta - \bar{\alpha} g e^{i\chi}| + |\alpha - \bar{\delta} g e^{i\chi}|}{1 - g^2}$, where $\det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = g e^{i\chi}$
- $V_{jk} = V_{jk}(g)$
- $F(z) := (V_{11} - z\boldsymbol{I}_1) - gV_{12}(gV_{22} - z\boldsymbol{I}_2)^{-1} V_{21}$ s.t.

  $z \in \rho(T) \cap \rho(gV_{22}) \Leftrightarrow 0 \in \rho(F(z))$ (for $\omega = 0$)

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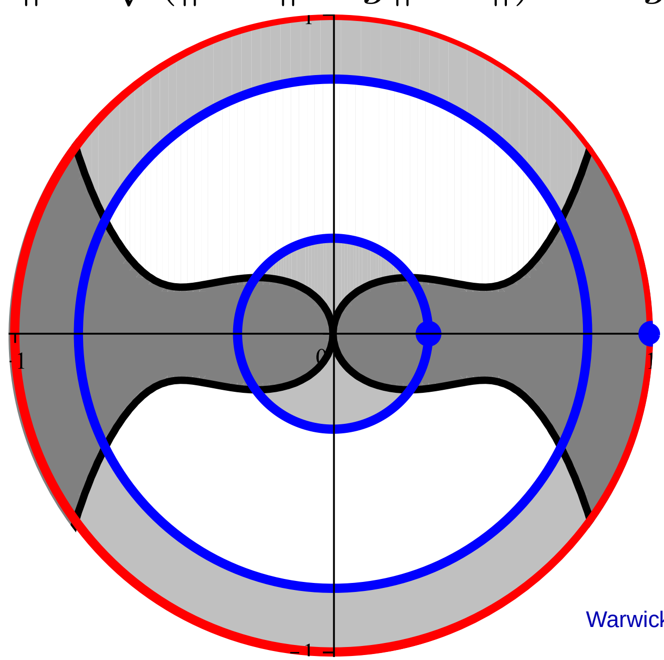
- There are cases where both Theorems hold:



About Eigenvalues of $T_\omega = V_\omega(P_1 + gP_2)$

For $0 < g < 1$: If $T_\omega \varphi = \lambda \varphi$, then

$$|\lambda| = 1 \quad \Rightarrow \quad \varphi = P_1 \varphi \text{ and } V_\omega \varphi = P_1 V_\omega P_1 \varphi = \lambda \varphi,$$

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Consequence,

$$\ker P_2 V_\omega P_1 = \{0\} \quad \Rightarrow \quad \sigma_p(T_\omega) \cap \mathbb{S} = \emptyset,$$

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More on $P_k V_\omega P_j|_{P_j \mathcal{H}}$: $\exists$ O.N.B.'s $\{v_j^{(p)},\}_{p \in \mathbb{Z}}^{j=1,2}$ of $P_j \mathcal{H}$ s.t.

- $P_k V_\omega P_j|_{P_j \mathcal{H}}$ is tri-diagonal and off-diagonal w.r.t. these bases
- $P_k V_\omega P_j|_{P_j \mathcal{H}} \simeq \widetilde{\mathbb{D}}(\omega) P_k V P_j$, where $\widetilde{\mathbb{D}}(\omega)$ is a random diagonal unitary op.
- $\ker P_k V_\omega P_j = \{0\}$, $\forall j, k \in 1, 2$,

$$\forall \ \lambda \text{ eigenval. of } T_\omega \ \Rightarrow \ |\lambda| \notin \{1, g\}.$$

Case $g = 0 \;\Leftrightarrow\; T_\omega = \mathbb{D}_\omega V P_1$

Thm: If $g = 0$, we have for all ω

$$\sigma(T_\omega) = \sigma(P_1 V_\omega P_1|_{P_1\mathcal{H}}) \cup \{0\},$$ Feschbach-Schur

$$\sigma(T_\omega) \setminus \{0\} \subset \Big\{\big||\alpha| - |\delta|\big| \leq |z| \leq |\alpha| + |\delta|\Big\}.$$ Study of $P_1 V_\omega P_1$

Consequence: spr$(T_\omega) < 1$ if $|\alpha| + |\delta| < 1.$

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$|\alpha| + |\delta| = 1 =$ spr(T_ω):

The peripheral spectra coincide

$$\sigma(T_\omega) \cap \mathbb{S} = \sigma(P_1 V_\omega P_1|_{\mathcal{H}_1}) \cap \mathbb{S} = \sigma(V_\omega) = \mathbb{S},$$ a.s.,

their nature differs for $\gamma \neq qt$

$$\sigma_p(T_\omega) \cap \mathbb{S} = \sigma_p(T_\omega^*) \cap \mathbb{S} = \emptyset,$$ whereas $\sigma_c(V_\omega) = \emptyset$ a.s.