Invariants of ground state phases in one dimension

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What is a quantum phase transition?

A simple answer: A phase transition at zero temperature
A slightly more precise answer: Consider:

- A smooth family of Hamiltonians $H(s), s \in [0, 1]$
- The associated family of ground states $\Omega_i(s)$
- A quantum phase transition occurs at singularities of $s \mapsto \Omega_i(s_c)$

In this talk:

- Quantum spin systems
- Hamiltonians $H_\Lambda(s)$ are continuously differentiable
- Spectral gap above the ground state energy $\gamma_\Lambda(s)$ such that

$$\gamma_\Lambda(s) \geq \gamma(s) \begin{cases} > 0 & (s \neq s_c) \\ \sim C\,|s - s_c|^\mu & (s \to s_c) \quad \text{QPT} \end{cases}$$

Local vs topological order

Ordered phases $\sim$ non-unique ground state

- The usual picture: Local order parameter distinguishes between possible ground states
  Example: Local magnetization in the quantum Ising model
- 'Topological order': Local disorder, for any local A,

$$\|P_\Lambda A P_\Lambda - C_A \cdot 1\| \leq C|\Lambda|^{-\alpha}, \qquad C_A \in \mathbb{C},$$

  P_Λ: The spectral projection associated to the ground state energy
  The ground state space depends on the topology of the lattice
  Example: Ground state degeneracy in Kitaev's 2d model

Basic question: What is a ground state phase?

Automorphic equivalence

$$H_\Lambda(s) = \sum_{X\subset\Lambda} \Phi_X(s), \qquad s \in [0,1]$$

with $s \mapsto \Phi_X(s)$ of class C^1, and uniform spectral gap:

$$\gamma := \inf_{\Lambda\subset\Gamma, s\in[0,1]} \gamma_\Lambda(s) > 0$$

Define $\mathcal{S}_\Gamma(t)$: ground state space on Γ at $s = t$.
Then there exists an automorphism $\alpha_\Gamma^{t_1,t_2}$ of $\mathcal{A}_\Gamma$ such that

$$\mathcal{S}_\Gamma(t_2) = \mathcal{S}_\Gamma(t_1) \circ \alpha_\Gamma^{t_1,t_2}$$

$\alpha_\Gamma^{t_1,t_2}$ is local: satisfies a Lieb-Robinson bound

Now: Invariants of the equivalence classes? Classification of phases?

Finitely correlated states

A special class of states on a spin chain $\mathcal{A}_{\mathbb{Z}}$ with local algebra $\mathcal{A}$

- A finite dimensional C*-algebra $\mathcal{B}$
- A completely positive map $\mathbb{E} : \mathcal{A} \otimes \mathcal{B} \to \mathcal{B}$
- Two positive elements $e \in \mathcal{B}$ and $\rho \in \mathcal{B}^*$ such that

$$\mathbb{E}(1 \otimes e) = e, \qquad \rho \circ \mathbb{E}(1 \otimes b) = \rho(b)$$

Notation: $\mathbb{E}(A \otimes b) = \mathbb{E}_A(b)$. Finitely correlated state:

$$\omega(A_n \otimes \cdots \otimes A_m) := \rho(e)^{-1} \rho \left(\mathbb{E}_{A_n} \circ \cdots \circ \mathbb{E}_{A_m}(e) \right)$$

Exponential decay of correlations if $\sigma(\mathbb{E}_1) \setminus \{1\} \subset \{z \in \mathbb{C} : |z| < 1\}$

$$\omega(A \otimes 1^{\otimes l} \otimes B) = \rho(e)^{-1} \rho \left(\mathbb{E}_A \circ (\mathbb{E}_1)^l \circ \mathbb{E}_B(e) \right)$$

Finitely correlated states

- 'Finite correlation': The set of functionals on $\mathcal{A}_{\mathbb{N}}$ defined by

$$\omega_X(A) = \omega(X \otimes A),$$

  with $X \in \mathcal{A}_{\mathbb{Z}\setminus\mathbb{N}}$, generates a finite dimensional linear space.
- Purely generated FCS: Consider $\mathcal{B} = \mathcal{M}_k$ and

$$\mathbb{E}(A \otimes b) = V^*(A \otimes b)V$$

  for $V : \mathbb{C}^k \to \mathbb{C}^n \otimes \mathbb{C}^k$.
- In a basis $\{e_\mu\}$ of $\mathbb{C}^n$: $V\chi = \sum_{\mu=1}^n e_\mu \otimes v_\mu^* \chi$ with $v_i \in \mathcal{M}_k$ i.e.

$$\mathbb{E}(A \otimes b) = \sum_{\mu,\nu=1}^{n} \langle e_\mu, Ae_\nu \rangle \, v_\mu b v_\nu^* \quad \text{(MPS)}$$

Example: the AKLT model

- Affleck-Kennedy-Lieb-Tasaki, 1987
- $\mathrm{SU}(2)$-invariant, antiferromagnetic spin-1 chain
- Nearest-neighbor interaction

$$H_{[a,b]} = \sum_{x=a}^{b-1} \left[\frac{1}{2} (S_x \cdot S_{x+1}) + \frac{1}{6} (S_x \cdot S_{x+1})^2 + \frac{1}{3} \right] = \sum_{x=a}^{b-1} P^{(2)}_{x,x+1}$$

  where $P^{(2)}_{x,x+1}$ is the projection on the spin-2 space of $\mathcal{D}_1 \otimes \mathcal{D}_1$
- Uniform spectral gap γ of $H_{[a,b]}$, $\gamma > 0.137194$
- Ground state is finitely correlated: $\mathcal{B} = \mathcal{M}_2$ and

$$(\mathcal{D}_1 \otimes \mathcal{D}_{1/2})V = V\mathcal{D}_{1/2}$$

Hamiltonians

Let $\mathbb{V} = (v_1, \dots, v_n) \in B_{n,k}(p,q)$ and $\omega^{\mathbb{V}}$ be such that

- $v_i \in \mathcal{M}_k$
- spectral radius of $\mathbb{E}_1^{\mathbb{V}}$ is 1, and it is a non-degenerate eigenvalue
- $\sigma(\mathbb{E}_1^{\mathbb{V}}) \setminus \{1\} \subset \{z \in \mathbb{C} : |z| < 1\}$ trivial peripheral spectrum
- there are projections p, q such that $pe^{\mathbb{V}}p$ and $q\rho^{\mathbb{V}}q$ are invertible

Then there is a canonical Hamiltonian $H^{\mathbb{V},p,q}$ such that

- positive, finite range interaction
- uniform spectral gap above the ground state energy
- ground state spaces:

$$\mathcal{S}_{\mathbb{Z}} = \{\omega^{\mathbb{V}}\}, \quad \mathcal{S}_{[1,\infty)} \cong \mathcal{M}^*_{\dim(p)} \quad \mathcal{S}_{(-\infty,0]} \cong \mathcal{M}^*_{\dim(q)}$$

Invariants of gapped phases

Theorem. *Consider* $\mathbb{I} \in B_{n,k_i}(p_i, q_i)$ *and* $\mathbb{F} \in B_{n,k_f}(p_f, q_f)$ *and the canonically associated Hamiltonians* $H^{\mathbb{I},p_i,q_i}, H^{\mathbb{F},p_f,q_f}$*.*
There is a continuous path $H(s), s \in [0,1]$ *such that*

1. $H(0) = H^{\mathbb{I},p_i,q_i}$ *and* $H(1) = H^{\mathbb{F},p_f,q_f}$
2. $H(s)$ *are uniformly gapped*
3. *There is a unique ground state on* $\mathbb{Z}$

if and only if $\dim(p_i) = \dim(p_f)$ *and* $\dim(q_i) = \dim(q_f)$*.*

In words: The pair $\big(\dim(p), \dim(q)\big)$ is the invariant of the gapped phase with a unique state on $\mathbb{Z}$.

Corollary & Comments

Corollary. *Each gapped phase contains a model with a pure product state in the thermodynamic limit*

Remarks:

- The theorem emphasizes the role of edge states in the non-trivial classification of gapped phases in $d = 1$
- No bulk-edge correspondence
- No symmetry requirements
- Conjecture: The theorem extends to arbitrary gapped models with a unique ground state in the thermodynamic limit
- The interaction length is constant and the smallest such l is $l \leq (k^2 - n + 1)k^2$
- The case of the AKLT model: belongs to the phase $(2, 2)$

About the proof

Key:

$$\mathbb{V} = (v_1, \ldots, v_n) \quad \longrightarrow \quad \mathbb{E}^{\mathbb{V}} \quad \longrightarrow \quad \omega^{\mathbb{V}} \quad \longrightarrow \quad H^{\mathbb{V}}$$

and $\quad \text{Gap}(\mathbb{E}_1^{\mathbb{V}}) \quad \longrightarrow \quad \text{Gap}(H^{\mathbb{V}})$

i.e. Construct a gapped path of Hamiltonians by constructing a path $\mathbb{V}(s)$ with the right properties
But: $\mathbb{V} \mapsto H^{\mathbb{V}}$ not always continuous!

The theorem reduces to a statement about the pathwise connectedness of a certain subspace of $(\mathcal{M}_k)^{\times n}$

Note:

$$\mathbb{E}_1^{\mathbb{V}}(b) = \sum_{\mu=1}^{n} v_\mu b v_\mu^*$$

the matrices v_μ are the Kraus operators for the CP map $\mathbb{E}_1^{\mathbb{V}}$.

Primitive maps

One way to enforce the spectral gap condition: Perron-Frobenius theory

- ▷ Irreducible positive map $\implies$
  1. Spectral radius r is a non-degenerate eigenvalue
  2. Corresponding eigenvector $e > 0$
  3. Eigenvalues λ with $|\lambda| = r$ are $r\mathrm{e}^{2\pi\mathrm{i}\alpha/\beta}$, $\alpha \in \mathbb{Z}/\beta\mathbb{Z}$
- ▷ A primitive map is an irreducible map with $\beta = 1$

Lemma. *A CP map with Kraus operators $\{v_1, \dots, v_n\}$ is primitive iff there exists $m \in \mathbb{N}$ such that*

$$\mathrm{span}\left\{v_{\mu_1} \cdots v_{\mu_m} : \mu_i \in \{1, \dots, n\}\right\} = \mathcal{M}_k$$

Note: m fixed!

Primitive maps

How to construct paths of primitive maps? Consider

$$Y_{n,k} := \left\{ \mathbb{V} : v_1 = \sum_{\alpha=1}^{k} \lambda_\alpha \left| e_\alpha \right\rangle \left\langle e_\alpha \right|, \text{and} \quad \langle v_2 e_\alpha, e_\beta \rangle \neq 0 \right\}$$

with the choice

$$(\lambda_1, \ldots, \lambda_k) \in \Omega := \{\lambda_i \neq 0, \lambda_i \neq \lambda_j, \lambda_i/\lambda_j \neq \lambda_k/\lambda_l\}$$

Then,

$$\left| e_\alpha \right\rangle \left\langle e_\beta \right| \in \operatorname{span}\{v_{\mu_1} \cdots v_{\mu_m} : \mu_i \in \{1,2\}\}$$

for $m \geq 2k(k-1)+3$.

Problem reduced to the pathwise connectedness of $\Omega \subset \mathbb{C}^k$

Use transversality theorem

Backbone of proof

1. Embed $\mathbb{I}, \mathbb{F}$ into a common matrix algebra $\mathcal{M}_k$
2. Construct $\mathbb{V}(s), s \in [0,1]$ such that
   - $\mathbb{V}(0) = \mathbb{I}$, $\mathbb{V}(1) = \mathbb{F}$
   - $\mathbb{V}(s) \in Y_{n,k}$ for $s \in (0,1)$

   At the edges $s \in \{0,1\}$: perturb the Jordan blocks of v_1
3. If $\dim(p_i) = \dim(p_f)$, then $p_f = u^* p_i u$ and interpolate in $\mathrm{SU}(k)$
   If $\dim(q_i) = \dim(q_f)$, then $q_f = w^* q_i w$ and interpolate in $\mathrm{SU}(k)$

Result: continuous $\mathbb{V}(s), p(s), q(s)$ generating a continuous $H(s) := H^{\mathbb{V}(s),p(s),q(s)}$ with uniform spectral gap

Note: If $\dim(p_i) \neq \dim(p_f)$ then $\dim(\mathcal{S}_{i,[0,\infty)}) \neq \dim(\mathcal{S}_{f,[0,\infty)})$: There is no automorphism, different phases

Local symmetries

Next question: What if $H(s)$ all share a symmetry?

Automorphic equivalence and local symmetries:

- ▷ Lie group G, and π^g the action of G on $\mathcal{A}_\Gamma$
- ▷ G is a local symmetry of the interaction if

$$\pi^g(\Phi_X(s)) = \Phi_X(s)$$

for all $g \in G$, $X \subset \Gamma$ and $s \in [0, 1]$

Then:

$$\alpha_\Gamma^{t_1,t_2} \circ \pi^g = \pi^g \circ \alpha_\Gamma^{t_1,t_2}$$

i.e. $\alpha_\Gamma^{t_1,t_2}$ is compatible with the symmetries

Edge representations

Let now $\Pi_\Gamma(s)$ be the subrepresentation of G on $\mathcal{S}_\Gamma(s)$

Proposition. *Assume* $H(s), s \in [0,1]$ *is a smooth path of gapped Hamiltonians with* G*-invariant interactions. Then the representations* $\Pi_\Gamma(t_1)$ *and* $\Pi_\Gamma(t_2)$ *are equivalent for all* $t_1, t_2 \in [0,1]$.

Follows from

$$\Pi_\Gamma(t_2)\left((\alpha_\Gamma^{t_1,t_2\,*}(\omega)\right)(A) = \omega\left(\alpha_\Gamma^{t_1,t_2} \circ \pi^g(A)\right)$$
$$= \omega\left(\pi^g \circ \alpha_\Gamma^{t_1,t_2}(A)\right) = \left(\alpha_\Gamma^{t_1,t_2\,*}\left(\Pi_\Gamma(t_1)(\omega)\right)\right)(A)$$

The representations Π_Γ are invariants of symmetric gapped phases

Now: concrete observables?

The case of FCS chains

Unitary representation of G at one site:

$$U^g = \mathrm{e}^{\mathrm{i}gS}$$

i.e. $\pi^g(A) = U^{g*} A U^g$ for $A \in \mathcal{A}$

Consider the FCS ground state ω of a G-invariant interaction

Theorem. *In the GNS representation $(\mathcal{H}_\omega, \rho_\omega, \Omega_\omega)$, the automorphism $\pi^g_{[1,\infty)}$ is unitarily implementable by $\mathcal{U}^g_{[1,\infty)}$, and*

$$\mathcal{U}^g_{[1,\infty)} \in \rho_\omega(\mathcal{A}_{[1,\infty)})'' \cap \rho_\omega(\mathcal{A}_{(-\infty,0]})'.$$

Rigorous version of the formal $\exp\left(\mathrm{i}g \sum_{x=1}^{\infty} S_x\right)$

The excess spin operator

- $\mathcal{U}^g_{[1,\infty)}$ is an observable
- Generator of $\mathcal{U}^g_{[1,\infty)}$: Excess spin operator
- In fact, we prove

$$\mathcal{U}^g_{[1,\infty)} = \mathrm{s-}\lim_{L\to\infty} \mathrm{e}^{\mathrm{i}g\rho_\omega(S(L))}$$

  where $S(L) \in \mathcal{A}_{[1,L^2]}$
- Similar result for models with stochastic-geometric representation
- c.f. non-local string order parameter

$$O_{x,y} = (-1)^{y-x}\omega\left(S_x \mathrm{e}^{\mathrm{i}\pi\sum_{j=x+1}^{y-1} S^j} S_y\right)$$

  used to describe 'dilute Neel order'

Bulk-edge correspondence

Symmetric FCS is generated by V such that

$$(U^g \otimes u^g)V = Vu^g \qquad i.e. \qquad \mathbb{E}_{U^g}(u^g) = u^g$$

where u^g is a representation of G on $\mathbb{C}^k$.

Simple computation:

$$\Pi^g_{[1,\infty)}(\omega)(A) = \mathrm{Tr}\left(Ad_{u_g^*}(\sigma_\omega)\mathbb{E}_A(1)\right)$$

The excess spin is observable in the correlation structure in the bulk

The case of the AKLT model:

- Symmetry: $G = \mathrm{SU}(2)$
- Auxiliary algebra $\mathcal{B} = \mathcal{M}_2$, i.e. $\Pi^g_{[1,\infty)}$ is a spin $1/2$ representation
- All models in that phase must carry a spin $1/2$ at the edges, see Hagiwara et al., *Observation of S = 1/2 degrees of freedom in an S = 1 linear chain Heisenberg antiferromagnet. PRL 65, 1990.*

Conclusion

So far...

- Automorphic equivalence yields a good notion of a gapped ground state phase
- Valid in any dimension
- Invariants in $d = 1$ without symmetry: dimensions of the edge ground state spaces
- Invariants in any dimension with symmetry: G-representation on ground state spaces
- Invariants in $d = 1$ with symmetry: The observable excess spin operators
- There is more to understand; e.g. the role of entanglement entropy?

... More details this afternoon!