The reduced ℓ^p-cohomology in degree 1 and harmonic functions

Antoine Gournay

TECHNISCHE UNIVERSITÄT DRESDEN

University of Warwick, May 21st 2015

Why ℓ^p-cohomology?

A ball packing in a manifold M is a [countable] set of closed balls in M so that any two balls intersect at most in a point. The incidence graph of a ball packing is a graph whose vertices are the balls and there is an edge if the ball touches.

Theorem (Koebe 1936)

A finite graph can be packed in $\mathbb{R}^2$ if and only if it is planar.

Quasi-round packing: replace balls by generic domains, require there is a K so that the ration “outer radius / inner radius” is $\leq K$.

There is no obstruction for quasi-round packings of finite graphs in $\mathbb{R}^3$.

Why ℓ^p-cohomology?

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There is no obstruction for quasi-round packings of finite graphs in $\mathbb{R}^3$.

Theorem (Benjamini & Schramm 2009)

If an infinite graph can be quasi-roundly packed in $\mathbb{R}^d$ either it is d-parabolic or it has non-trivial reduced ℓ^d-cohomology in degree 1.

d-parabolic $\iff \inf\{\|\nabla f\|_{\ell^p} \mid f$ has finite support and $f(x_0) = 1\} = 0$.
"Easy" to understand, *e.g.* 2-parabolicity is recurrence. A Cayley graph is d-parabolic if and only if it has polynomial growth of degree $\leq d$.

In general, $d_{par} = \inf\{d \mid d\text{-parabolic}\}$ belongs to $[d_{isop}, d_{gr}]$ where d_{isop} is the isoperimetric dimension (see later) and d_{gr} the minimal polynomial degree growth of balls.

What is ℓ^p-cohomology?

In degree 1, the ℓ^p-cohomology of a graph $G = (V, E)$ is defined via incidence operators between vertices and edges. Take $E \subset V \times V$ symmetric, and let

$$\begin{array}{rccc} \nabla : & \{V \to \mathbb{R}\} & \to & \{E \to \mathbb{R}\} \\ & f & \mapsto & \nabla f(x,y) = f(y) - f(x) \end{array}$$

In graphs of bounded valency, $\nabla : \ell^p(V) \to \ell^p(E)$ is a bounded operator.

The space of p-Dirichlet functions is $\mathrm{D}^p(G) = \{f : V \to \mathbb{R} \mid \nabla f \in \ell^p(E)\}$.

It is endowed with a semi-norm $\|f\|_{\mathrm{D}^p} = \|\nabla f\|_{\ell^p}$. ("semi-" $\to$ constant functions).

What is ℓ^p-cohomology?

$$\begin{array}{cccc} \nabla : & \{V \to \mathbb{R}\} & \to & \{E \to \mathbb{R}\} \\ & f & \mapsto & \nabla f(x,y) = f(y) - f(x) \end{array} \qquad \|f\|_{\mathrm{D}^p} = \|\nabla f\|_{\ell^p}$$

Definition

The reduced ℓ^p-cohomology in degree 1 of a graph is

$$\underline{\ell^p H}^1(G) = \frac{\mathrm{Im}\,\nabla \cap \ell^p(E)}{\overline{\nabla \ell^p(V)}^{\ell^p}} = \frac{\mathrm{D}^p(G)}{\overline{\ell^p(V) + \mathrm{cst}}^{\mathrm{D}^p}}$$

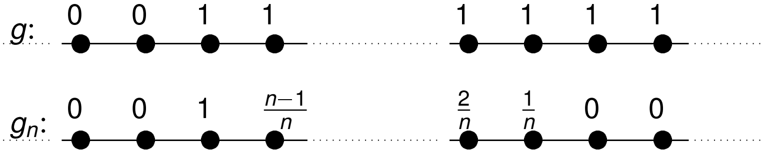
Theorem (Élek 1998, Pansu ∅)

Fix a bound on the geometry (valency, curvature and injectivity radius). Then the [reduced] ℓ^p-cohomology [in degree 1] is an invariant of quasi-isometry.

A simple (yet important) example.

$$\underline{\ell^p H^p}(G) = \mathrm{D}^p(G) / \overline{\ell^p(V) + \mathbb{R}}^{\mathrm{D}^p(G)}$$

Example:



g_n finitely supported so $\in \ell^p(V)$ for any p.

$\nabla(g - g_n)$ takes n times the value $1/n$

$$\implies \|g - g_n\|_{\mathrm{D}^p} = (n/n^p)^{1/p} = n^{-1/p'}.$$

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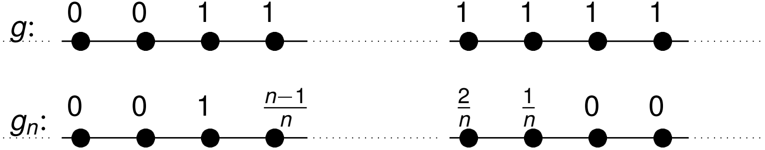


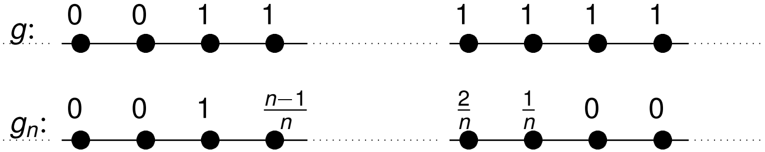
$$\implies \|g - g_n\|_{\mathrm{D}^p} = (n/n^p)^{1/p} = n^{-1/p'}.$$

Then $g_n \overset{\mathrm{D}^p}{\to} g$ if $p > 1$. Thus $\forall 1 < p < \infty, [g] = 0 \in \underline{\ell^p H}^1(G)$.

A simple (yet important) example.

$$\underline{\ell^p H}^p(G) = \mathrm{D}^p(G)/\overline{\ell^p(V)+\mathbb{R}}^{\mathrm{D}^p(G)}$$

Example:



In fact $\underline{\ell^p H}^1(G) = \{0\}$ if $p \in]1,\infty[$ and $\underline{\ell^1 H}^1(G) \simeq \mathbb{R}$.

Remark: If $p < q$, the map $\underline{\ell^p H}^1(G) \xrightarrow{\mathrm{Id}} \underline{\ell^q H}^1(G)$ is not always injective...

Ends

$\underline{\ell^1 H}^1(G)$ is intimately related to the ends of a graph.

Definition (Freudenthal, 193?)

An end of a graph $\Gamma = (V, E)$ is a function from finite subsets of V to infinite ones, such that
· $\xi(F)$ is an infinite connected component of F^c;
· $\forall F, F' \subset V$ (finite), $\xi(F) \cap \xi(F') \neq \varnothing$.

Examples:

- A finite graph has 0 ends.
- The infinite grid (a Cayley graph of $\mathbb{Z}^2$) has 1 end.
- The infinite line (a Cayley graph of $\mathbb{Z}$) has 2 ends.
- Regular trees of even valency ≥ 3 (Cayley graphs of free groups) have ∞ many ends.

Ends

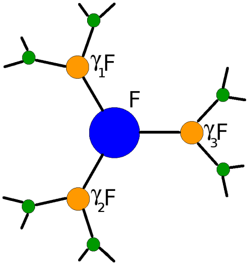
Lemma

The number of ends is a quasi-isometry invariant.

Theorem (Hopf, 1944)

The number of ends of a Cayley graph is 0, 1, 2 or ∞.

Idea: 3 ends $\implies$ ∞ ends



Theorem (Stallings, 1971)

[The Cayley graph of] a group has 2 ends iff it contains $\mathbb{Z}$ as a finite index subgroup. It has ∞ many ends iff it is a “non-trivial” amalgamated product or HNN extension.

Ends and $\underline{\ell^1 H}^1$

$$\underline{\ell^1 H}^1(G) = \mathrm{D}^1(G) / \overline{\ell^1(V) + \mathbb{R}}^{\mathrm{D}^1(G)}$$

Lemma ("well-known")

If G has finitely many ends, $\underline{\ell^1 H}^1(G) \cong \mathbb{R}^{\text{ends}(G)-1}$.

Preliminary claim: $\mathrm{D}^1(G) \subset \ell^\infty(V)$...?

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Hint: $f(y) - f(x) = \sum_{e \in P} \nabla f(e)$ for P a path from x to y.

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$$|f(y) - f(x)| = \Big| \sum_{e \in P} \nabla f(e) \Big| \leq \|\nabla f\|_{\ell^1(E)}.$$

Shows more: the ℓ^1 norm of ∇f tends to 0 outside larger and large balls.

On the [infinite] connected components of B_n^c (B_n = balls centred at some vertex), f becomes uniformly constant as $n \to \infty$.

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On the [infinite] connected components of B_n^c ($B_n =$ balls centred at some vertex), f becomes uniformly constant as $n \to \infty$.

Thence, one defines a value of $f \in \mathrm{D}^1(G)$ on each end:

$$\beta f(\xi) := \lim_{n \to \infty} f(x_n) \quad \text{where } x_n \in \xi(B_n)$$

Ends and $\underline{\ell^1 H}^1$

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If G has finitely many ends, $\underline{\ell^1 H}^1(G) \cong \mathbb{R}^{\mathrm{ends}(G)-1}$.

$$|f(y)-f(x)| = \left|\sum_{e\in P} \nabla f(e)\right| \leq \|\nabla f\|_{\ell^1(E)}.$$

Boundary value of $f \in \mathrm{D}^1(G)$:

$$\beta f(\xi) := \lim_{n\to\infty} f(x_n) \quad \text{where } x_n \in \xi(B_n)$$

β is linear and continuous on $\mathrm{D}^1(G)$.

Also: $\ell^1(V) \subset \operatorname{Ker}\beta$

$\implies$ β sends $\overline{\ell^1(V)+\mathrm{cst}}^{\mathrm{D}^1(G)}$ to constant functions.

Ends and $\underline{\ell^1 H}^1$

$$\underline{\ell^1 H}^1(G) = \mathrm{D}^1(G)/\overline{\ell^1(V)+\mathbb{R}}^{\mathrm{D}^1(G)}$$

Lemma ("well-known")

If G has finitely many ends, $\underline{\ell^1 H}^1(G) \cong \mathbb{R}^{\mathrm{ends}(G)-1}$.

We have a [linear & continuous] map β which associates to $g \in \mathrm{D}^1(G)$ a function on the ends.

$\overline{\ell^1(V)+\mathbb{R}}^{\mathrm{D}^1(G)}$ is sent to constant functions.

Remains to show that if βg is constant, then $g \in \overline{\ell^1(V)+\mathbb{R}}^{\mathrm{D}^1(G)}$.

Truncation Lemma 1

Definition

Say $g : V \to \mathbb{R}$ takes only one value at infinity if there is a $K \in \mathbb{R}$ so that for any $\varepsilon > 0$ one can find a $F_\varepsilon \subset V$ finite so that

$$g(F_\varepsilon^c) \subset]K - \varepsilon, K + \varepsilon[.$$

Lemma ("maximum principle revisited")

If $g \in \mathrm{D}^p(G)$ takes only one value at infinity then $[g] = 0 \in \underline{\ell^p H}^1(G)$.

Proof: WLOG $K = 0$. Define g_ε as

$$g_\varepsilon(x) = \begin{cases} g(x) & \text{if } g(x) < \varepsilon \\ \varepsilon g(x)/|g(x)| & \text{if } g(x) \geq \varepsilon \end{cases}$$

$f_\varepsilon = g - g_\varepsilon$ is finitely supported, so $\in \ell^p(V)$ for any p.

$\|g - f_\varepsilon\|_{\mathrm{D}^p} = \|\nabla g_\varepsilon\|_{\ell^p(E)}$ is pointwise bouded by ∇g and tends pointwise to 0; so tends to 0. ∎

Aim ...

... to describe $\underline{\ell^p H}^1$ as an ideal boundary.

$\underline{\ell^1 H}^1$ is related to the ends ["well-known"].

[Bourdon & Pajot 2003]: In the hyperbolic case, there a strong link between ℓ^p-cohomology in degree 1 and some [Besov] space of functions on the visual boundary; also $p_c = \inf\{p \mid \underline{\ell^p H}^1 \neq 0\}$ is a lower bound on the conformal dimension.

A natural idea, is to try to look at the "values" of this function on the "Poisson boundary".

Questions and answers

Question (Gromov 1992)

If G is the Cayley graph of an amenable group is $\underline{\ell^p H}^1(G) = \{0\}$ for any $p \in]1, \infty[$? [In fact, in all degrees]

Theorem (Gromov 1992, ...)

If G is the Cayley graph of a virtually nilpotent group then $\underline{\ell^p H}^1(G) = \{0\}$ for any $p \in]1, \infty[$.

Holds for any group with [virtually] infinite center. [Tessera 2009] show this also holds for polycyclic groups and some more.

Isoperimetric profiles

Definition

Let $d \in \mathbb{Z}_{\geq 1}$. A graph G satisfies a d-dimensional isoperimetric profiles (noted IS_d) if $\exists K > 0$ such that, $\forall F \subset V$ finite,

$$|F|^{1-\frac{1}{d}} \leq K|\partial F|$$

It has a strong isoperimetric profile (noted IS_ω) if $\exists K > 0$ such that, $\forall F \subset V$ finite, $|F| \leq K|\partial F|$

Examples: Cayley graphs of $\mathbb{Z}^d$ satisfy IS_d.

A group is amenable iff its Cayley graph does not satisfy IS_ω. (Restatement of Følner)

Isoperimetric profiles

Satisfying IS_α (for $\alpha \in \mathbb{Z}_{\geq 1} \cup \{\omega\}$) is invariant under quasi-isometries.

Hyperbolic $\implies \mathrm{IS}_\omega \implies \mathrm{IS}_d, \forall d$.

But $\mathrm{IS}_d, \forall d \not\Longrightarrow \mathrm{IS}_\omega$. For example, Cayley graphs of $\mathbb{Z}^2 \rtimes_\alpha \mathbb{Z}$ where $\alpha(1) = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$.

Theorem (Varopoulos 1985 + Gromov 1981 + Wolf 1968)

Γ has polynomial growth of degree $\leq d \iff \mathcal{C}\mathrm{ay}(\Gamma, S)$ does not have IS_{d+1}.

$\implies$ groups which are not virtually nilpotent have IS_d for all d.

Harmonic functions

Let $P_x^{(n)}$ be the measure defined by $P_x^{(n)}(y) =$ the probability that a simple random walker starting at x ends up in y after n steps.

This gives a kernel: $P^{(n)}g(x) := \int g(y) \mathrm{d}P_x^{(n)}(y)$.

A function $g : V \to \mathbb{R}$ is harmonic if $P^{(1)}g = g$ (mean value property).

$\mathcal{H}(G)$:=space of harmonic functions.
$\mathcal{H}_b(G) := \mathcal{H}(G) \cap \ell^\infty(V)$ =space of bounded harmonic functions.

Boundary values

Theorem (G., 2013)

Assume G satisfies IS_d. Let $1 \le p < d/2$. There is a linear map $\pi : \bigcup_{1 \le p < d/2} \mathrm{D}^p(G) \to \mathcal{H}(G)$ such that

- if $g \in \mathrm{D}^p(G)$, then $\pi(g) \in \mathrm{D}^q(G)$ for all $q > \dfrac{dp}{d-2p}$.
- if $g \in \mathrm{D}^p(G) \cap \ell^\infty(V)$, then $\pi(g) \in \ell^\infty(V)$.
- $[g] = 0 \in \underline{\ell^p H^1}(G) \iff \pi(g)$ is a constant function $\iff g$ takes only one value at ∞.

This extends a result of Lohoué (1990) which requires IS_ω.

Truncation Lemma 2

Lemma (Holopainen & Soardi 1994)

Assume $g \in \mathrm{D}^p(G)$ is such that $[g] \neq 0 \in \underline{\ell^p H}^1(G)$. Let g_K be the function with values truncated in $[-K, K]$. Then for some K_0 and any $K > K_0$, $[g_K] \neq 0 \in \underline{\ell^p H}^1(G)$.

Proof goes along the same lines as Truncation Lemma 1.

Corollary

To show $\underline{\ell^p H}^1(G) = \{0\}$, it suffices to show $[g] = 0$ for any $g \in \mathrm{D}^p(G) \cap \ell^\infty(V)$.

Corollaries

Theorem (G., 2013)

Assume G satisfies IS_d. Let $1 \leq p < d/2$. There is a linear map $\pi : \bigcup_{1 \leq p < d/2} \mathrm{D}^p(G) \to \mathcal{H}(G)$ such that

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- $[g] = 0 \in \underline{\ell^p H}^1(G) \iff \pi(g)$ is a constant function $\iff g$ takes only one value at ∞.

Corollary 1

If G satisfies IS_d and $\mathcal{H}_b(G) = \{0\}$, then $\underline{\ell^p H}^1(G) = \{0\}$ for all $p \in [1, \frac{d}{2}[$.

Indeed, $\pi(g) = 0$, $\forall g \in \mathrm{D}^p(G) \cap \ell^\infty(V)$, so Theorem gives $[g] = 0$. Truncation Lemma 2 allows to conclude.

Corollaries

Theorem (G., 2013)

Assume G satisfies IS_d. Let $1 \leq p < d/2$. There is a linear map $\pi : \bigcup_{1 \leq p < d/2} \mathrm{D}^p(G) \to \mathcal{H}(G)$ such that

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- $[g] = 0 \in \underline{\ell^p H}^1(G) \iff \pi(g)$ is a constant function $\iff g$ takes only one value at ∞.

Corollary 2

If G is the Cayley graph of a group Γ which is not virtually-$\mathbb{Z}$ and $1 \leq p < q < \infty$. Then the identity map $\underline{\ell^p H}^1(G) \to \underline{\ell^q H}^1(G)$ is injective.

[Cheeger-Gromov 1992] show $\underline{\ell^2 H}^1(G) = \{0\}$ for amenable groups $\implies$ for amenable groups, $\underline{\ell^p H}^1(G) = \{0\}$ for all $1 < p \leq 2$.

Corollaries

Theorem (G., 2013)

Assume G satisfies IS_d. Let $1 \leq p < d/2$. There is a linear map $\pi : \bigcup_{1\leq p<d/2} \mathrm{D}^p(G) \to \mathcal{H}(G)$ such that

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- $[g] = 0 \in \underline{\ell^p H}^1(G) \iff \pi(g)$ is a constant function $\iff g$ takes only one value at ∞.

Corollary 3

If G has IS_d and $1 \leq p < d/2$, then

· $\mathcal{H}_b(G) \cap \mathrm{D}^p(G) = \{\text{constants}\} \implies \forall q < \frac{dp}{d+2p}, \underline{\ell^p H}^1(G) = \{0\}$.

· $\underline{\ell^p H}^1(G) = \{0\} \implies \mathcal{H}(G) \cap \mathrm{D}^p(G) = \{\text{constants}\}$

This uses that π is the identity on harmonic functions.

Corollaries

Theorem (G., 2013)

Assume G satisfies IS_d. Let $1 \leq p < d/2$. There is a linear map $\pi : \bigcup_{1 \leq p < d/2} \mathrm{D}^p(G) \to \mathcal{H}(G)$ such that

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- $[g] = 0 \in \underline{\ell^p H}^1(G) \iff \pi(g)$ is a constant function $\iff g$ takes only one value at ∞.

Corollary 4

Assume $G' \subset G$ is a connected spanning subgraph of G so that G' has IS_d and $\underline{\ell^p H}^1(G') = \{0\}$ for $1 \leq p < d/2$. $\underline{\ell^p H}^1(G) = \{0\}$.

If $g \in \mathrm{D}^q(G)$ then $g \in \mathrm{D}^q(G')$. This implies g takes only one value at ∞ as seen on G'. But this is also the case on G.

Corollaries

Corollary 4

Assume $G' \subset G$ is a connected spanning subgraph of G so that G' has IS_d and $\underline{\ell^p H}^1(G') = \{0\}$ for $1 \leq p < d/2$. $\underline{\ell^p H}^1(G) = \{0\}$.

Corollary 5

Let $L \neq \{*\}$ and has at most two ends and H is infinite and has most two ends. Then the lamplighter $L \wr H$ has no harmonic function with gradient in ℓ^p ($p < \infty$).

[Thomassen 1978] shows that up to a quasi-isometry (actually bi-Lipschitz map of constant 6) both L and H contain a connected spanning subgraph which is a line, half-line or cycle. Hence they contain a of $G' = H' \wr L'$ where $H' = C_n, \mathbb{N}$ or $\mathbb{Z}$ and $L' = \mathbb{N}$ or $\mathbb{Z}$. This G' has $\mathcal{H}_b(G') = \{0\}$ and IS_d for all d.

! These graphs have lots of bounded harmonic functions ($\implies$ gradient in ℓ^∞).

Any Cayley graph has harmonic functions with gradient in ℓ^∞ (= Lipschitz).

Corollaries

Corollary 4

Assume $G' \subset G$ is a connected spanning subgraph of G so that G' has IS_d and $\underline{\ell^p H}^1(G') = \{0\}$ for $1 \leq p < d/2$. $\underline{\ell^p H}^1(G) = \{0\}$.

This can also be interpreted as a "forbidden subgraph" result: if $\underline{\ell^p H}^1(G) \neq \{0\}$ then there are no spanning connected subgraphs G' with $\mathcal{H}_b(G') = \{\text{constants}\}$ and IS_d (for some $d > 2p$).

e.g. no spanning $\mathbb{Z}^d$ for $d > 2p$... Recall: [Thomassen 1978] very often there is a spanning line!

Corollary 6

Assume G has IS_d and $\underline{\ell^p H}^1(G) \neq \{0\}$ for $1 \leq p < d/2$. Then some part of the Poisson boundary is quasi-isometry invariant

Both IS_d and $\underline{\ell^p H}^1$ are QI-invariant, so always get a non-trivial bounded harmonic function by looking at $\pi(g)$ where $[g] \neq 0 \in \underline{\ell^p H}^1$ and g bounded.

The proof

Theorem (G., 2013)

Assume G satisfies IS_d. Let $1 \leq p < d/2$. There is a linear map $\pi : \bigcup_{1 \leq p < d/2} \mathrm{D}^p(G) \to \mathcal{H}(G)$ such that

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- if $g \in \mathrm{D}^p(G) \cap \ell^\infty(V)$, then $\pi(g) \in \ell^\infty(V)$.
- $[g] = 0 \in \underline{\ell^p H}^1(G) \iff \pi(g)$ is a constant function $\iff g$ takes only one value at ∞.

π is naively defined: $\pi(g) = \lim_{n \to \infty} P^n g$.

The "difficult" parts are:
· π is [well-]defined.
· $\pi(g)$ is a constant implies g takes only one value at ∞.

π defined, take one

$P^{(n)}g(x)$ Cauchy? $\to P^{(n)}g(x) - P^{(m)}g(x) = \int g \mathrm{d}P_x^{(n)} - \int g \mathrm{d}P_x^{(m)}$.

Problem/Idea ?

$$\begin{aligned} \int g \mathrm{d}\xi - \int g \mathrm{d}\phi &= \langle g \mid \xi - \phi \rangle \\ &= ??? \\ &= \langle \nabla g \mid ?? \rangle \\ &\leq \|\nabla g\|_{\ell^p} \|??\|_{\ell^{p'}} \end{aligned}$$

Duality

∇^* the adjoint of ∇:

$$\begin{array}{llll} \nabla : & \{E \to \mathbb{R}\} & \to & \{V \to \mathbb{R}\} \\ & f & \mapsto & \nabla f(x) = \sum_{y\sim x} f(y,x) - \sum_{y\sim x} f(x,y) \end{array}$$

Note: $\Delta = \nabla^*\nabla$ then $\Delta g = 0 \iff (I-P)g = 0$.

Definition

ξ, ϕ finitely supported probability measures. A transport pattern from ξ to ϕ is a finitely supported function $\tau_{\xi,\phi} : E \to \mathbb{R}$ so that $\nabla^*\tau_{\xi,\phi} = \phi - \xi$.

π defined, take two

$P^{(n)}g(x)$ Cauchy? $\to P^{(n)}g(x) - P^{(m)}g(x) = \int g \mathrm{d}P_x^{(n)} - \int g \mathrm{d}P_x^{(m)}$.

Problem/Idea ?

$$\begin{aligned} \int g \mathrm{d}\xi - \int g \mathrm{d}\phi &= \langle g \mid \xi - \phi \rangle \\ &= \langle g \mid \nabla^* \tau_{\phi,\xi} \rangle \\ &= \langle \nabla g \mid \tau_{\phi,\xi} \rangle \\ |\cdot| &\leq \|\nabla g\|_{\ell^p} \|\tau_{\phi,\xi}\|_{\ell^{p'}} \end{aligned}$$

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$P^{(n)}g(x)$ Cauchy? $\to P^{(n)}g(x) - P^{(m)}g(x) = \int g \mathrm{d}P_x^{(n)} - \int g \mathrm{d}P_x^{(m)}$.

Problem/Idea ?

$$
\begin{aligned}
P^{(m+k)}g(x) - P^{(n)}g(x) &= \langle g \mid P_x^{(m+k)} - P_x^{(m)} \rangle \\
&= \langle g \mid \nabla^* \tau_{P_x^{(m)}, P_x^{(m+k)}} \rangle \\
&= \langle \nabla g \mid \tau_{P_x^{(m)}, P_x^{(m+k)}} \rangle \\
|\cdot| &\leq \|\nabla g\|_{\ell^p} \|\tau_{P_x^{(m)}, P_x^{(m+k)}}\|_{\ell^{p'}}
\end{aligned}
$$

How to define $\tau_{P_x^{(m)}, P_x^{(m+k)}}$?

π defined, take two

$P^{(n)}g(x)$ Cauchy? $\to P^{(n)}g(x) - P^{(m)}g(x) = \int g dP_x^{(n)} - \int g dP_x^{(m)}$.

Problem/Idea ?

$$\begin{aligned} P^{(m+k)}g(x) - P^{(n)}g(x) &= \langle g \mid P_x^{(m+k)} - P_x^{(m)} \rangle \\ &= \langle g \mid \nabla^* \tau_{P_x^{(m)}, P_x^{(m+k)}} \rangle \\ &= \langle \nabla g \mid \tau_{P_x^{(m)}, P_x^{(m+k)}} \rangle \\ |\cdot| \quad &\leq \|\nabla g\|_{\ell^p} \|\tau_{P_x^{(m)}, P_x^{(m+k)}}\|_{\ell^{p'}} \end{aligned}$$

How to define $\tau_{P_x^{(m)}, P_x^{(m+k)}}$?

∇^* is linear so (a possible choice is):

$$\tau_{P_x^{(m)}, P_x^{(m+k)}} = \sum_{i=0}^{k-1} \tau_{P_x^{(m+i)}, P_x^{(m++i+1)}}$$

Simple but inefficient...

How to define $\tau_{P_x^{(m)},P_x^{(m+1)}}$?

There is a very natural transport pattern:

take a random step!

Then $\|\tau_{P_x^{(m)},P_x^{(m+1)}}\|_{\ell^{p'}} \leq K \| P_x^{(m)} \|_{\ell^{p'}}$ (where K depends on the valency).

Transport: to infinity and beyond!

$$|P^{(m+k)}g(x) - P^{(n)}g(x)| \leq \|\nabla g\|_{\ell^p} \|\tau_{P_x^{(m)},P_x^{(m+k)}}\|_{\ell^{p'}} \leq \|\nabla g\|_{\ell^p} \sum_{i=m}^{m+k-1} \|P_x^{(i)}\|_{\ell^{p'}}$$

So it suffices to check that $\sum_{n\geq 0} P_x^{(n)}$ is in $\ell^{p'}(V)$.

Theorem (Varopoulos 1985)

If G has IS_d, then for some $K > 0$, $\|P_x^{(n)}\|_{\ell^\infty(V)} \leq Kn^{-d/2}$.

Thus

$$\|P_x^{(n)}\|_{\ell^q(V)}^q \leq \|P_x^{(n)}\|_{\ell^\infty(V)}^{q-1} \|P_x^{(n)}\|_{\ell^1(V)} \leq K' n^{-d(q-1)/2}$$

and $\sum P^{(n)}$ converges in ℓ^q if $q' < d/2$.

$\pi(g)$ =cst $\implies$ constant at ∞

Want to prove:

$\pi(g)$ is a constant function $\implies$ g is constant at infinity.

WLOG the constant $\pi(g)$ is 0.

Will prove:

$\forall \varepsilon > 0, \exists n_\varepsilon$ such that $x \notin B_{3n_\varepsilon} \implies |g(x)| < K\varepsilon$.

Make a well-chosen splitting of the scalar product:

$$P^{(n)}g(x) - g(x) = \langle \nabla g \mid \tau_{\delta_x, P_x^{(n)}} \rangle$$

$\pi(g) =$ cst $\implies$ constant at ∞

from $\pi(g) \equiv 0$, to prove: for some $K > 0$

$$\forall \varepsilon > 0, \exists n_\varepsilon \text{ such that } x \notin B_{3n_\varepsilon} \implies |g(x)| < K\varepsilon.$$

Given $\varepsilon > 0$, define n_ε to be so that

- $\|\nabla g\|_{B^c_{n_\varepsilon}} < \varepsilon$
- $\sup\limits_{x \in V} \sum_{i \geq n_\varepsilon} \|P_x^{(n)}\|_{\ell^{p'}} < \varepsilon$

$$\begin{aligned} P^{(n)}g(x) - g(x) &= \langle \nabla g \mid \tau_{\delta_x, P_x^{(n)}} \rangle \\ &= \langle \nabla g \mid \tau_{\delta_x, P_x^{(n)}} \rangle_{|B_{n_\varepsilon}} + \langle \nabla g \mid \tau_{\delta_x, P_x^{(n)}} \rangle_{|B^c_{n_\varepsilon}} \end{aligned}$$

2nd term: ∇g is small, and there is an absolute bound on τ: $\sum_{i \geq 0} \|P^{(i)}\|$

1st term: ∇g is not small, but the τ will be small:
can replace $\tau_{\delta_x, P_x^{(n)}}$ by $\tau_{P^{(n_\varepsilon)}, P_x^{(n)}}$ because it takes at least n_ε steps to go from x to B_{n_ε}.

$\pi(g)=$cst $\implies$ constant at ∞

from $\pi(g)\equiv 0$, to prove: for some $K>0$

$$\forall\varepsilon>0,\ \exists n_\varepsilon \text{ such that } x\notin B_{3n_\varepsilon} \implies |g(x)|<K\varepsilon.$$

Given $\varepsilon>0$, define n_ε to be so that

- $\|\nabla g\|_{B^c_{n_\varepsilon}} < \varepsilon$
- $\sup_{x\in V} \sum_{i\geq n_\varepsilon} \|P_x^{(n)}\|_{\ell^{p'}} < \varepsilon$

Massage 2$^{\text{nd}}$ term:

$$\begin{aligned}
|P^{(n)}g(x)-g(x)| &\leq |\langle\nabla g \mid \tau_{\delta_x,P_x^{(n)}}\rangle_{|B_{n_\varepsilon}}| + |\langle\nabla g \mid \tau_{\delta_x,P_x^{(n)}}\rangle_{|B^c_{n_\varepsilon}}| \\
&\leq |\langle\nabla g \mid \tau_{\delta_x,P_x^{(n)}}\rangle_{|B_{n_\varepsilon}}| + \|\nabla g\|_{\ell^p(B^c_{n_\varepsilon})} \|\tau_{\delta_x,P_x^{(n)}}\|_{\ell^{p'}(B^c_{n_\varepsilon})} \\
&\leq |\langle\nabla g \mid \tau_{\delta_x,P_x^{(n)}}\rangle_{|B_{n_\varepsilon}}| + \quad \varepsilon \quad K_1 \textstyle\sum_{i\geq 0} \|P_x^{(i)}\|_{\ell^{p'}} \\
&\leq |\langle\nabla g \mid \tau_{\delta_x,P_x^{(n)}}\rangle_{|B_{n_\varepsilon}}| + \quad \varepsilon \quad K_2
\end{aligned}$$

$\pi(g) = $ cst $\implies$ constant at ∞

from $\pi(g) \equiv 0$, to prove: for some $K > 0$

$$\forall \varepsilon > 0, \exists n_\varepsilon \text{ such that } x \notin B_{3n_\varepsilon} \implies |g(x)| < K\varepsilon.$$

Given $\varepsilon > 0$, define n_ε to be so that

- $\|\nabla g\|_{B^c_{n_\varepsilon}} < \varepsilon$
- $\sup_{x \in V} \sum_{i \geq n_\varepsilon} \|P_x^{(n)}\|_{\ell^{p'}} < \varepsilon$

Massage 1st term:

$$\begin{aligned}
|P^{(n)}g(x) - g(x)| &\leq |\langle \nabla g \mid \tau_{\delta_x, P_x^{(n)}} \rangle_{|B_{n_\varepsilon}}| &+\varepsilon K_2 \\
&\leq |\langle \nabla g \mid \tau_{P_x^{(n_\varepsilon)}, P_x^{(n)}} \rangle_{|B_{n_\varepsilon}}| &+\varepsilon K_2 \\
&\leq \|\nabla g\|_{\ell^p} \|\tau_{P_x^{(n_\varepsilon)}, P_x^{(n)}}\|_{\ell^{p'}(B_{n_\varepsilon})} &+\varepsilon K_2 \\
&\leq \|\nabla g\|_{\ell^p} \sum_{i \geq n_\varepsilon} \|P_x^{(i)}\|_{\ell^{p'}} &+\varepsilon K_2 \\
&\leq \|\nabla g\|_{\ell^p} \quad \varepsilon &+\varepsilon K_2
\end{aligned}$$

$\pi(g) = $ cst $\implies$ constant at ∞

from $\pi(g) \equiv 0$, to prove: for some $K > 0$

$$\forall \varepsilon > 0, \exists n_\varepsilon \text{ such that } x \notin B_{3n_\varepsilon} \implies |g(x)| < K\varepsilon.$$

Given $\varepsilon > 0$, define n_ε to be so that

- $\|\nabla g\|_{B^c_{n_\varepsilon}} < \varepsilon$
- $\sup\limits_{x \in V} \sum_{i \geq n_\varepsilon} \|P_x^{(n)}\|_{\ell^{p'}} < \varepsilon$

Up to now: $\forall x \notin B_{3n_\varepsilon}$,

$$|P^{(n)}g(x) - g(x)| \leq K_3\varepsilon$$

Letting $n \to \infty$:

$$|g(x)| \leq K_3\varepsilon$$

as claimed.

Some questions

Q1: Is there an amenable group with a non-constant (bounded or not) harmonic functions whose gradient is in c_0?

Q2: If G is the Cayley graph of an amenable group (with $\mathcal{H}_b(G) \neq \{\text{constants}\}$, what is the maximal d so that G has a connected spanning subgraph G' with IS_d and $\mathcal{H}_b(G') = \{\text{constants}\}$?

Q3.a: Is there an explicit and more efficient transport pattern from δ_x to $P_x^{(n)}$?
Q3.b: In an amenable Cayley graph, is there a Følner sequence and an explicit (not too unefficient) transport pattern from δ_e to $\mathbb{1}_{F_n}$?

Q.4: For two infinite connected sets A and B let

$n_{A,B}(\ell) =$ number of mutually disjoint paths of length $\leq \ell$ from A to B.

Estimates for the growth (in ℓ) of $n_{A,B}(\ell)$ in an amenable Cayley graph (of exponential growth)