Asymptotics for spectra and heat kernels for some random fractals

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Motivation

- In the 1970s De Gennes suggested the ‘ant in the labyrinth’ to investigate the transport properties of percolation cluster models for random media.
- Early toy models for clusters were structures with exact self-similarity which enabled explicit renormalization group calculations such as the Sierpinski gasket.
- This introduces some fixed scaling parameters and periodic behaviour is seen in the asymptotics of many quantities associated with random walks on such fractal graphs.

Aim: To examine the asymptotics of quantities such as the eigenvalue counting function for some simple random fractals.

Spectral asymptotics

- The standard Laplacian on a bounded domain $D \subseteq \mathbb{R}^d$ with Dirichlet boundary conditions has a discrete spectrum consisting of eigenvalues $0 < \lambda_1^D < \lambda_2^D \leq \ldots$. That is λ_i satisfies for some u

$$\begin{cases} -\Delta u = \lambda_i u & \text{in } D \\ u = 0 & \text{on } \partial D \end{cases}$$

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- Weyl's Theorem of 1912 states that the eigenvalue counting function

$$N(\lambda) = |\{\lambda_i : \lambda_i \leq \lambda\}|$$

satisfies

$$\lim_{\lambda \to \infty} \frac{N(\lambda)}{\lambda^{d/2}} = \frac{B_d}{(2\pi)^d} |D|$$

where $|D|$ is the Lebesgue measure of D. and B_d the volume of the unit ball in $\mathbb{R}^d$.

The connection to the heat kernel

Consider the Dirichlet heat kernel on the domain. Mercer's theorem gives

$$p_t^a(x,y) = \sum_{i=1}^{\infty} e^{-\lambda_i t}\phi_i(x)\phi_i(y),$$

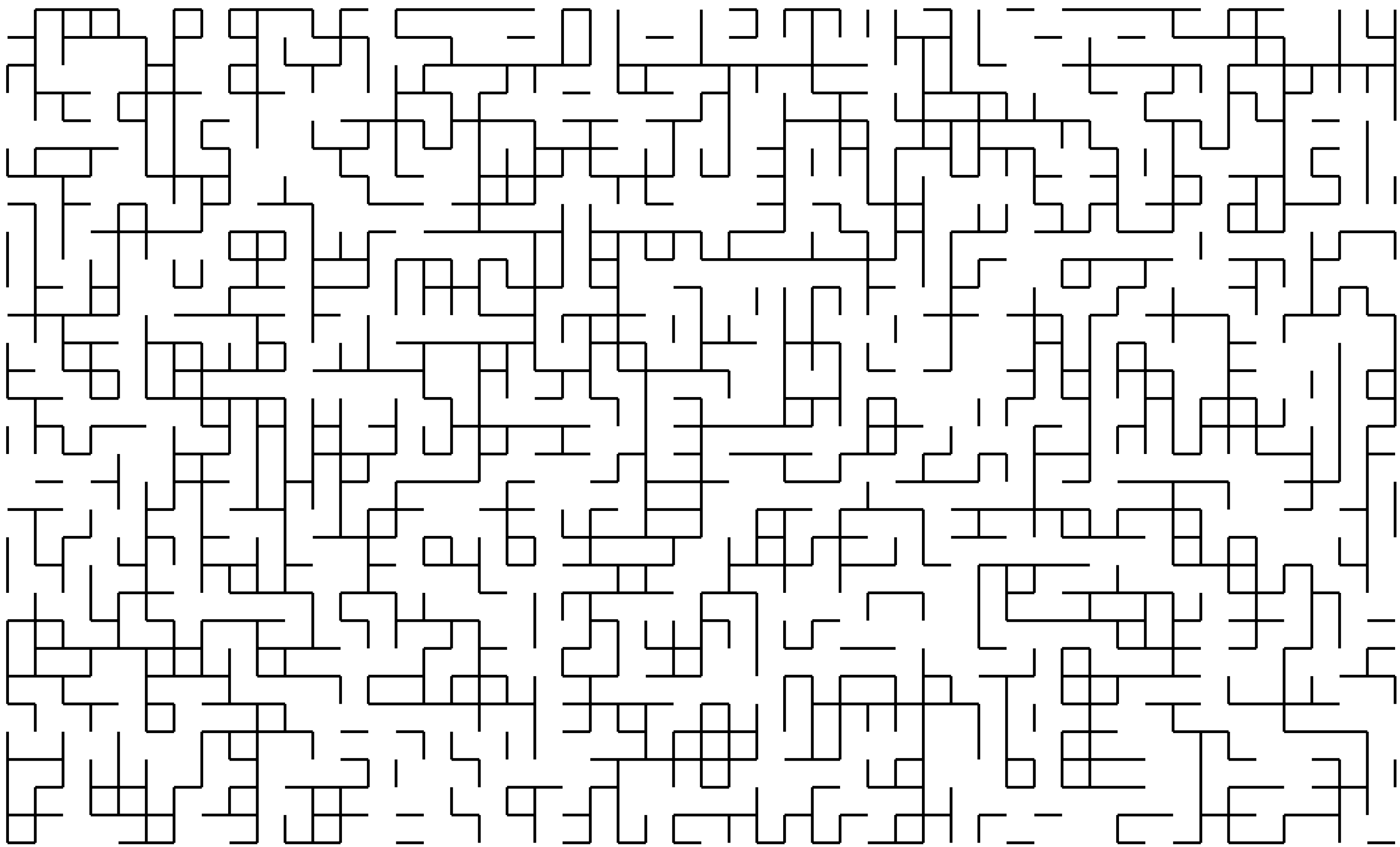
where ϕ_i are an orthonormal set of eigenfunctions, eigenvalue λ_i.

The trace of the heat semigroup, or the partition function, satisfies

$$\int_D p_t^a(x,x)dx = \sum_{i=1}^{\infty} e^{-\lambda_i t} = \int_0^{\infty} e^{-st}N(ds).$$

Thus information about the spectrum can be recovered from Tauberian theorems, if we understand the short time heat kernel asymptotics, and vice versa.

Percolation on $\mathbb{Z}^2$



$p = 0.5$

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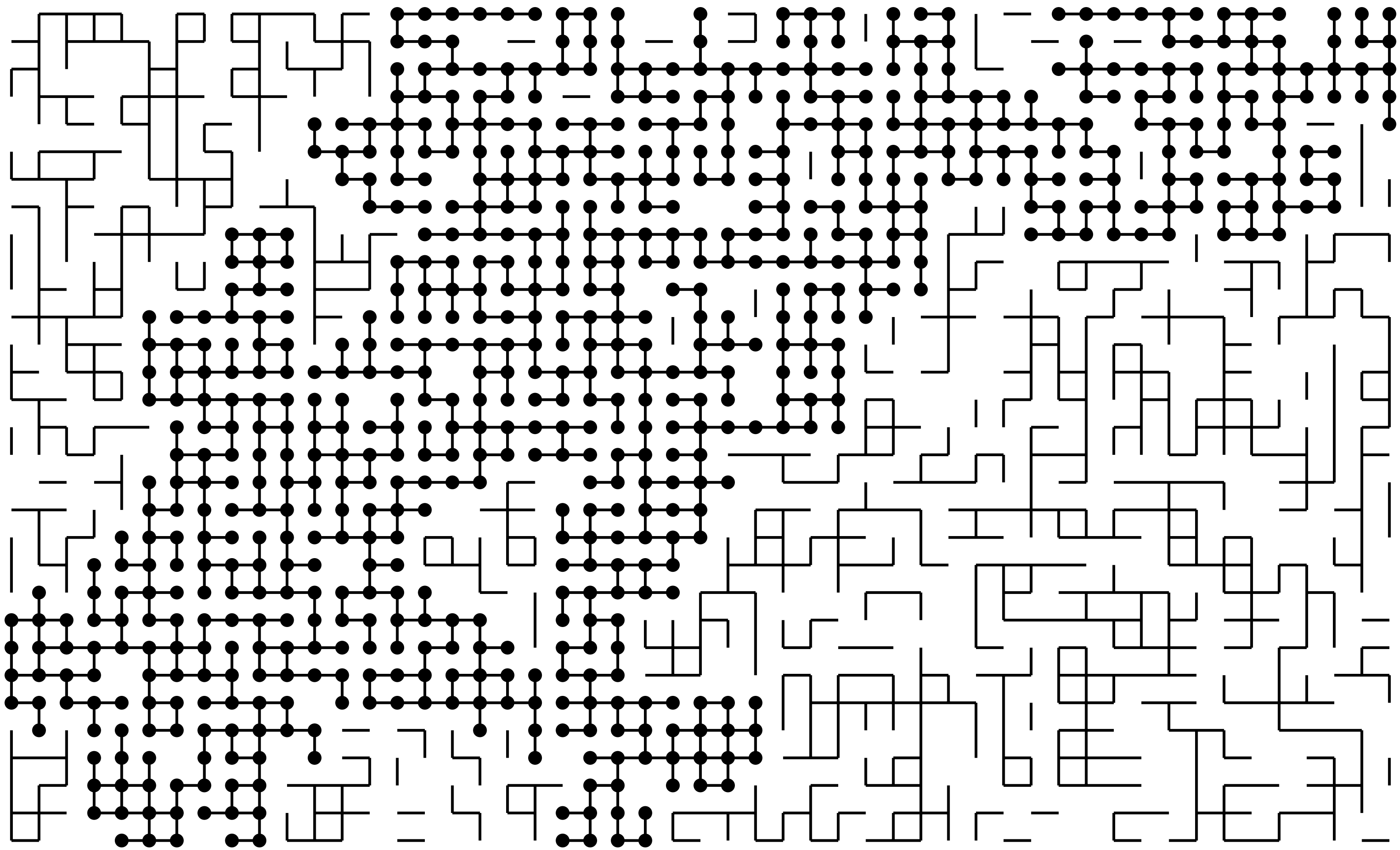
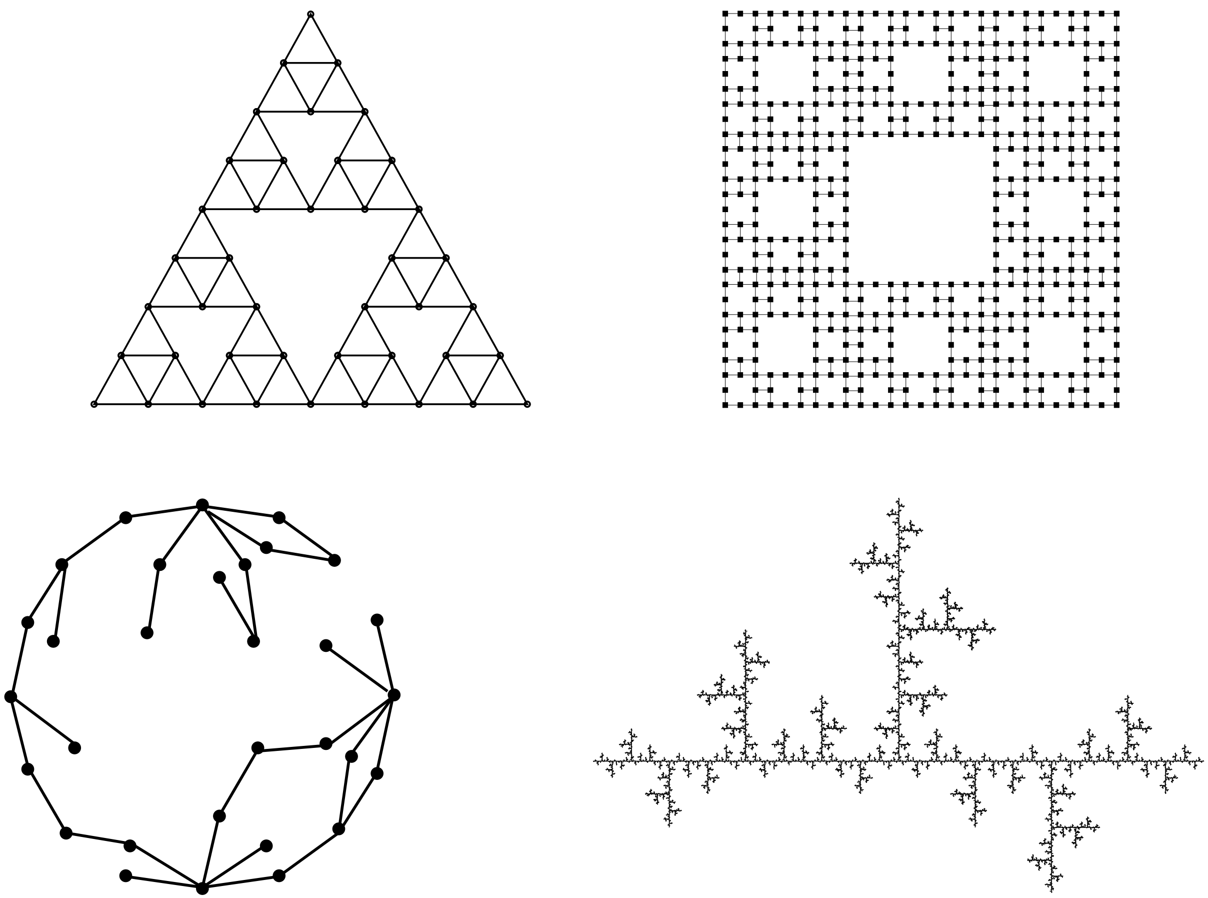


$p = 0.5$

Fractals

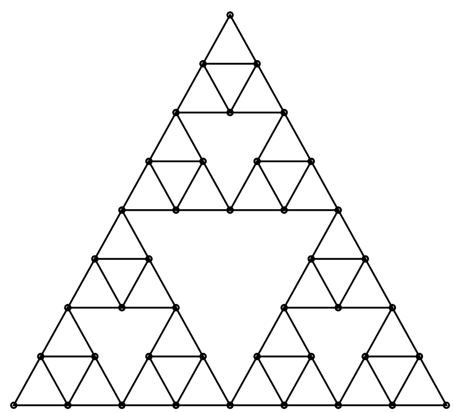


Random walks on SG

The random walk on the Sierpinski gasket graph shows sub-Gaussian heat kernel estimates (Jones)

$$p_n(x,y) \asymp c_1 n^{-d_f/d_w} \exp\left(-c_2\left(\frac{d(x,y)^{d_w}}{n}\right)^{1/(d_w-1)}\right).$$

where $d_f = \log 3/\log 2, d_w = \log 5/\log 2$ and $d(.,.)$ is the shortest path metric.

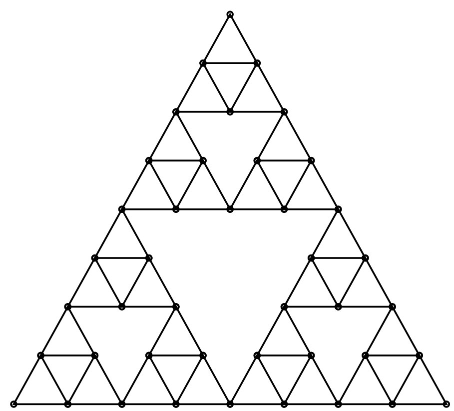


Random walks on SG

There are small oscillations in the long time heat kernel, so the constants in the upper and lower bounds are different (Grabner-Woess)

$$p_n(0,0) = n^{-d_f/d_w} F(\log n)(1 + o(1)),$$

where F is a non-constant log 5 periodic function.



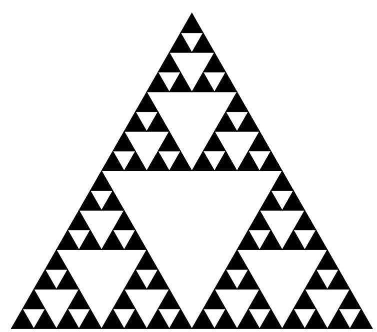
Spectral asymptotics for SG

The scaling limit of the random walks is a Brownian motion. Its generator is the Laplacian and we can examine its eigenvalues.

For the Sierpinski gasket (and other nested fractals) we have

$$N(\lambda) = \lambda^{d_S/2}(G(\log \lambda) + o(1)), \qquad \text{as } \lambda \to \infty,$$

where $d_s = 2\log 3/\log 5 = 2d_f/d_w$ and G is a periodic function (Fukushima-Shima, Barlow-Kigami).



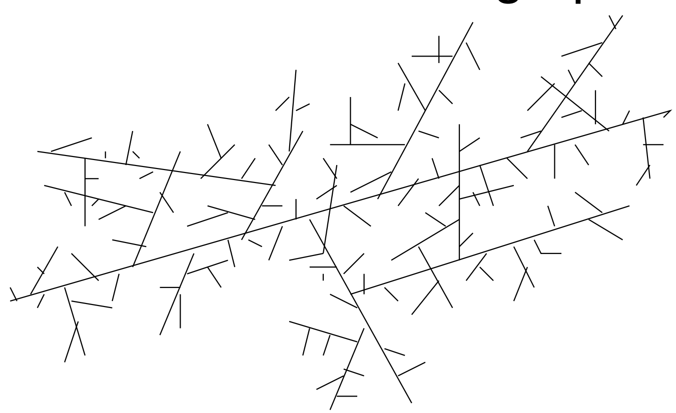
There exist strictly localized eigenfunctions.

The continuum random tree

The continuum random tree, initially constructed by Aldous, arises as

- the scaling limit of uniform random trees on n vertices.
- a random real tree defined as the contour process of Brownian excursion.
- A third view is that it is a random recursive self-similar set.

It is closely related to mean field limits for critical percolation on graphs, in particular high dimensional critical percolation on $\mathbb{Z}^d$ and limit models arising in the critical window of the random graph model.

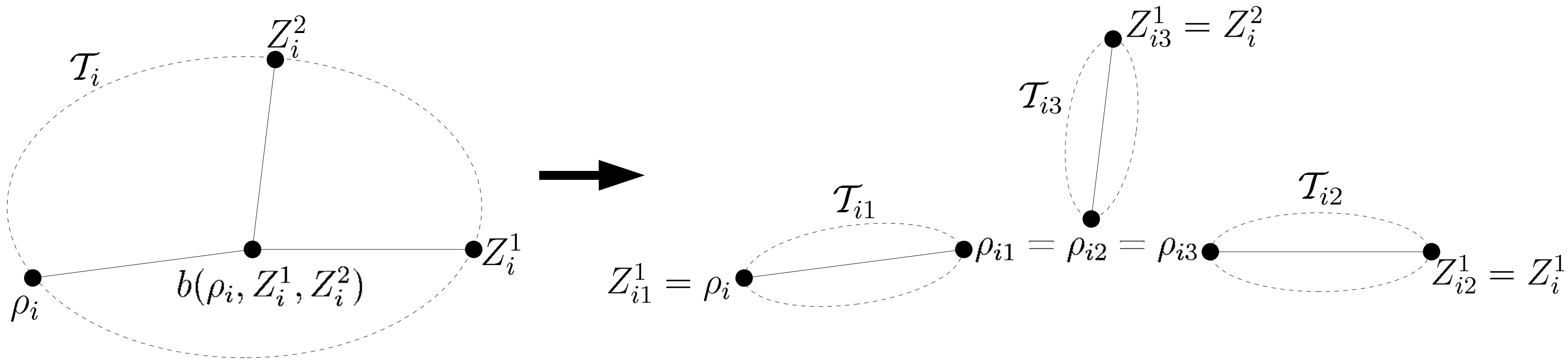


Self-similar decomposition

Let Z^1, Z^2 be two $\mu_{\mathcal{T}}$-random vertices of $\mathcal{T}$. There exists a unique branch-point $b^{\mathcal{T}}(\rho, Z^1, Z^2) \in \mathcal{T}$ of these three vertices. Let $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ the components containing ρ, Z^1 and Z^2. For $i = 1, 2, 3$, we define a metric $d_{\mathcal{T}_i}$ and probability measure $\mu_{\mathcal{T}_i}$ on $\mathcal{T}_i$ by setting

$$d_{\mathcal{T}_i} := \Delta_i^{-1/2} d_{\mathcal{T}}|_{\mathcal{T}_i \times \mathcal{T}_i}, \qquad \mu_{\mathcal{T}_i}(\cdot) := \Delta_i^{-1} \mu(\cdot \cap \mathcal{T}_i),$$

where $\Delta_i := \mu_{\mathcal{T}}(\mathcal{T}_i)$.

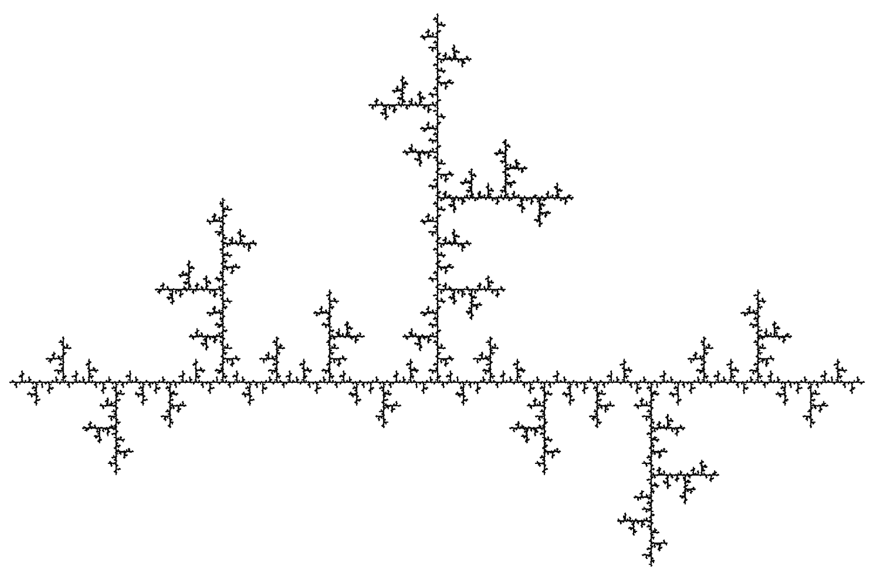


Random recursive fractal

Lemma

The collections $(\mathcal{T}_i, d_{\mathcal{T}_i}, \mu_{\mathcal{T}_i}, \rho_i, Z_i^1, Z_i^2)$, $i = 1, 2, 3$, are independent copies of $(\mathcal{T}, d_{\mathcal{T}}, \mu_{\mathcal{T}}, \rho, Z^1, Z^2)$, and the entire family of random variables is independent of $(\Delta_i)_{i=1}^3$, which has a Dirichlet-$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ distribution.

The CRT is isomorphic to a deterministic self-similar set with a random metric



The Dirichlet form

The natural Laplace operator on $\mathcal{T}$ is defined via its Dirichlet form.

$\mathbf{P}$-a.s. there exists a local regular Dirichlet form $(\mathcal{E}_{\mathcal{T}}, \mathcal{F}_{\mathcal{T}})$ on $L^2(\mathcal{T}, \mu)$, which is associated with the Laplace operator $\mathcal{L}_{\mathcal{T}}$ via for $f, g \in \mathcal{F}_{\mathcal{T}}$

$$\mathcal{E}_{\mathcal{T}}(f, g) = -(\mathcal{L}_{\mathcal{T}} f, g).$$

and the metric $d_{\mathcal{T}}$ through, for every $x \neq y$,

$$d_{\mathcal{T}}(x, y)^{-1} = \inf\{\mathcal{E}_{\mathcal{T}}(f, f) : \ f \in \mathcal{F}_{\mathcal{T}}, \ f(x) = 0, \ f(y) = 1\}.$$

A Neumann eigenvalue λ with eigenfunction u satisfies $\mathcal{E}_{\mathcal{T}}(f, u) = \lambda(f, u)$ for all $f \in \mathcal{F}_{\mathcal{T}}$.

We work with the eigenvalue counting function defined from $(\mathcal{E}_{\mathcal{T}}, \mathcal{F}_{\mathcal{T}}, \mu)$.

Scalings

We have the following relationships:

For the Dirichlet form

$$\mathcal{E}_{\mathcal{T}}(f,f) = \sum_{i=1}^{3} \Delta_i^{-1/2} \mathcal{E}_{\mathcal{T}_i}(f \circ \phi_i, f \circ \phi_i),$$

where ϕ_i is the map from $\mathcal{T} \to \mathcal{T}_i$.

For the measure

$$\int_{\mathcal{T}} f \; d\mu_{\mathcal{T}} = \sum_{i=1}^{3} \Delta_i \int_{\mathcal{T}_i} f \circ \phi_i \; d\mu_{\mathcal{T}_i}.$$

Dirichlet-Neumann bracketing

There is a simple relationship between the Dirichlet and Neumann counting functions. For trees we have

$$N^D(\lambda) \leq N^N(\lambda) \leq N^D(\lambda) + 2.$$

For the CRT we can compare eigenvalues and use the scalings:
For a Neumann eigenvalue λ of $\mathcal{T}$ we have

$$\sum_{i=1}^{3} \Delta_i^{-1/2} \mathcal{E}_{\mathcal{T}_i}(f \circ \phi_i, u \circ \phi_i) = \mathcal{E}_{\mathcal{T}}(f, u) = \lambda(f, u) = \lambda \sum_{i=1}^{3} \Delta_i (f \circ \phi_i, u \circ \phi_i).$$

Thus $\lambda \Delta_i^{3/2}$ is a Neumann eigenvalue of $\mathcal{T}_i$. Hence

$$\sum_{i=1}^{3} N_i^D(\lambda \Delta_i^{3/2}) \leq N^D(\lambda) \leq N^N(\lambda) \leq \sum_{i=1}^{3} N_i^N(\lambda \Delta_i^{3/2}).$$

The renewal equation

The key tool for studying the behaviour is a renewal equation. Let $X(t) = N^D(e^t)$ and $\eta(t) = N^D(e^t) - \sum_{i=1}^3 N_i^D(e^t \Delta_i^{3/2})$. Then

$$X(t) = \eta(t) + \sum_{i=1}^{3} X_i(t + \frac{3}{2}\log\Delta_i).$$

If $m(t) = e^{-2t/3} EX(t), u(t) = e^{-2t/3} E\eta(t)$, then

$$m(t) = u(t) + \int_0^\infty e^{-s} m(t-s) ds.$$

Theorem [Croydon + H (2008)] There exists a deterministic constant

$$C_0 = m(\infty) = \int_{-\infty}^{\infty} u(t)dt \in (0, \infty)$$

such that

- $\lambda^{-2/3}\mathbf{E}N(\lambda) \to C_0$, as $\lambda \to \infty$.
- $\lambda^{-2/3}N(\lambda) \to C_0$, as $\lambda \to \infty$, $\mathbf{P}$-a.s.

This second result is proved in a similar manner to Nerman's proof of the almost sure convergence of the general branching process.

The annealed heat kernel

We have been working directly with the eigenvalue counting function. We can use the trace theorem and Tauberian theorems to obtain results on the partition function.

Firstly from the trace theorem and the property that the continuum random tree is invariant under random rerooting

$$Ep_t(\rho,\rho) = E\int_{\mathcal{T}} p_t(x,x)\mu_{\mathcal{T}}(dx) = E\int_0^\infty e^{-st}N^N(ds).$$

Thus an application of a Tauberian Theorem gives

Corollary

Let Γ be the standard gamma function, then

$$t^{2/3}\mathbf{E}p_t(\rho,\rho) \to C_0\Gamma(5/3) \text{ as } t \to 0,$$

Note that Croydon obtained quenched and annealed heat kernel bounds for the CRT:

$$C_1 \le t^{2/3}\mathbf{E}p_t(\rho,\rho) \le C_2, \quad 0 < t < 1.$$

and $\mathbb{P}$-a.s.

$$C_3|\log t|^{a'} \le t^{2/3}\sup_{x\in\mathcal{T}} p_t(x,x) \le C_4|\log t|^{a}, \quad 0 < t < 1.$$

CRT results

Theorem

Suppose $(N_{\mathcal{T}}(\lambda))_{\lambda\in\mathbb{R}}$ is the eigenvalue counting function for the natural Laplacian on the continuum random tree. As $\lambda \to \infty$:

- $$\mathbb{E}N_{\mathcal{T}}(\lambda) = C_0\lambda^{2/3} + O(1).$$

CRT results

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- $$\mathbb{E}N_{\mathcal{T}}(\lambda)=C_0\lambda^{2/3}+O(1).$$
- $\mathbb{P}$-a.s., for $\epsilon>0$,
$$N_{\mathcal{T}}(\lambda)=C_0\lambda^{2/3}+o(\lambda^{1/3+\epsilon}).$$

Scaling limit of the CRG

Let $G(N, p)$ be the Erdös-Renyi random graph. The critical window is $p = N^{-1} + \nu N^{-4/3}$ for a fixed $\nu \in (-\infty, \infty)$.

Addario-Berry, Broutin and Goldschmidt construct the scaling limit: Conditioned on the number of connections $J = j$ we have (for $j \geq 2$) that $\mathcal{M}$ is constructed by

- taking a random 3 regular graph on $2(j-1)$ vertices
- generate $(\alpha_1, \ldots, \alpha_{3(j-1)})$ according to a Dirichlet $(\frac{1}{2}, \ldots, \frac{1}{2})$ distribution.
- construct $3(j-1)$ size α_j CRTs with root plus a randomly chosen vertex.
- replace the edges in the graph with the trees linked at the roots and randomly chosen vertices.

Dirichlet-Neumann bracketing allows us to compare eigenvalues of $\mathcal{M}, \mathcal{T}$.

Theorem

Suppose $(N_{\mathcal{M}}(\lambda))_{\lambda \in \mathbb{R}}$ is the eigenvalue counting function for the natural Laplacian on the scaling limit of the giant component of the critical random graph $\mathcal{M}$, and Z_1 is the mass of $\mathcal{M}$ with respect to its canonical measure $\mu_{\mathcal{M}}$. Then, as $\lambda \to \infty$:

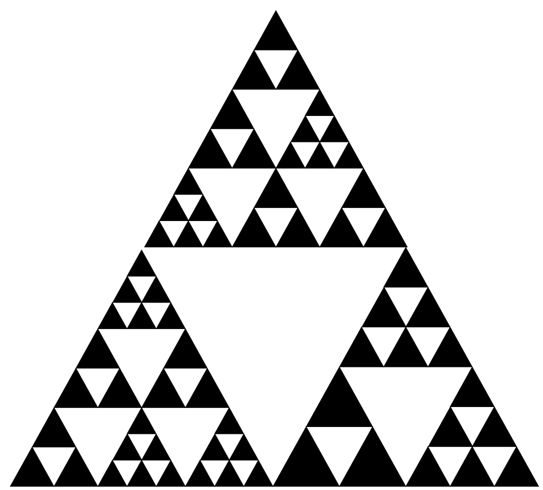
- $\mathbb{E}N_{\mathcal{M}}(\lambda) = C_0 \mathbb{E} Z_1 \lambda^{2/3} + O(1)$.
- $\lambda^{-2/3} N_{\mathcal{M}}(\lambda) \to C_0 Z_1. \ \mathbb{P} - a.s.$
- A CLT for the CRT will give a CLT here too.

Spectral asymptotics for random gaskets

For a random recursive Sierpinski gasket, where each 2, 3 side division is independently chosen with probability $p, 1-p$ for each triangle

$$\lim_{\lambda\to\infty}\frac{N(\lambda)}{\lambda^{d_s/2}} = W, \quad a.s.$$

where $d_s = 2\alpha/(\alpha+1)$ and α satisfies $p3(\frac{3}{5})^\alpha + (1-p)6(\frac{7}{15})^\alpha = 1$.



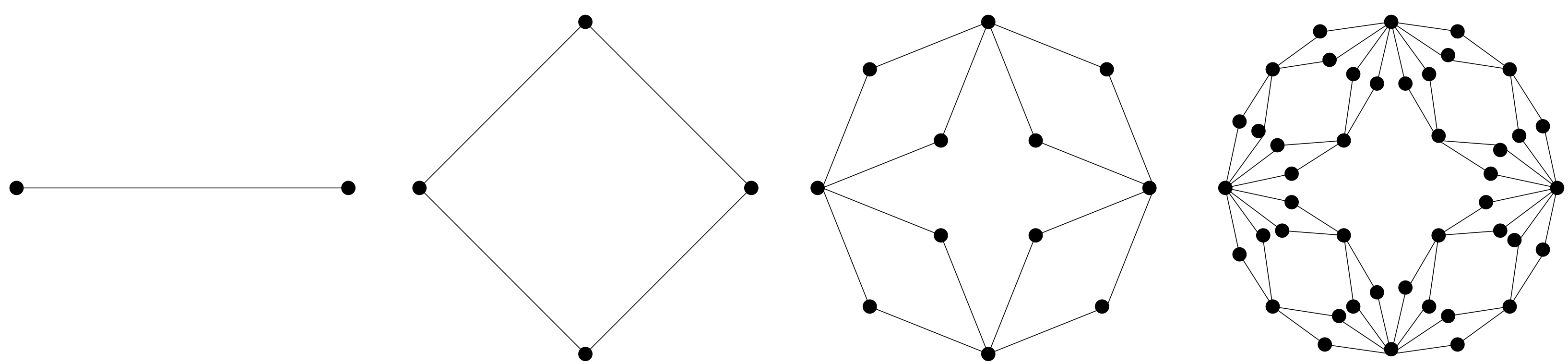
Is there a CLT in this case?

The diamond hierarchical lattice

A recursively constructed graph

1. $D_0 = (V_0, E_0)$, where V_0 consists of two vertices and E_0 an edge between them.

2. $D_{n+1} = (V_{n+1}, E_{n+1})$ with each edge in E_n replaced by a diamond: two sets of two edges in series in parallel.



Scaling limit

The scaling limit for the diamond lattice is a self-similar set.

- Let (K, d) be a compact metric space with two points labelled 0,1.
- Let $\{\psi_i : i = 1, 2, 3, 4\}$ be a set of 1/2-contractions $\psi_i : K \to K$. This defines the scaling limit of the diamond hierarchical lattice K as a self-similar set

$$K = \bigcup_{i=1}^{4} \psi_i(K).$$

The set is not finitely ramified. Its Hausdorff dimension is 2.

The diffusion on the scaling limit

- There is a local regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(K, \mu)$ with self-similarity,

$$\mathcal{E}(f, g) = \sum_{i=1}^{4} \mathcal{E}(f \circ \psi_i, g \circ \psi_i), \qquad \forall f, g \in \mathcal{F}.$$

- Let $N(x)$ be the Neumann eigenvalue counting function, then

$$0 < \liminf_{x \to \infty} \frac{N(x)}{x} < \limsup_{x \to \infty} \frac{N(x)}{x} < \infty. \tag{1}$$

- There exists a jointly continuous heat kernel $p_t(x, y)$, for all $t \in (0, 1), x, y \in K$. For μ-a.a $x \in K$

$$c_1 t^{-1} |\log t|^{-2-\epsilon} \leq p_t(x, x) \leq c_2 t^{-1}$$

Percolation on D_n

- Fix $p \in [0,1]$. For $e \in E_n$, let μ_e be independent random variables with $\mathbb{P}(\mu_e = 1) = p$, $\mathbb{P}(\mu_e = 0) = 1 - p$.
- Let E_n^p be the open edges in D_n.
- We say percolation occurs if there is a connected component of E_n^p containing 0 and 1 as n gets large.
- Let f be the map on the percolation probability obtained by considering whether a single diamond is open

$$f(p) = 1 - (1 - p^2)^2 = 2p^2 - p^4.$$

f has 3 fixed points in $[0, 1]$.

- 0 and 1 are attracting
- $p_c = (\sqrt{5} - 1)/2$ is repulsive.

Lemma 1 *If the lattice D_n is subject to Bernoulli bond percolation with $p = p_c$, then there is percolation in the sense that the end points of the lattice, 0 and 1, are connected with probability p_c. As $n \to \infty$*
If $p > p_c$, then $P($0 and 1 are connected in $D_n) \to 1$.
If $p < p_c$, then $P($0 and 1 are connected in $D_n) \to 0$.

Thus we can let $n \to \infty$ when $p = p_c$ to obtain an 'infinite lattice' under critical percolation $D_\infty^{p_c}$. Easier to think about the scaling limit.

‘Random walk on $D_\infty^{p_c}$’

- H and Kumagai (2010): We can define the scaling limit of the critical percolation cluster on D_∞ as a random recursive fractal and analyse the properties of the diffusion on this scaling limit.
- We have an explicit formula for the spectral exponent in that for $\mathbb{P}$-a.e. ω for $\mu_\omega - a.e.x \in \mathcal{C}$ we have

$$\lim_{\lambda\to\infty} \frac{\log N_\omega(\lambda)}{\log \lambda} = \lim_{t\to 0} \frac{\log q_t^\omega(x,x)}{-\log t} = \frac{\theta}{\theta+1}$$

where $\theta = 5.2654...$. Thus $d_s = 1.6807...$
- Sharper results mimic those on the diamond hierarchical lattice itself.
- The analysis can be extended to the random cluster model on the diamond lattice.

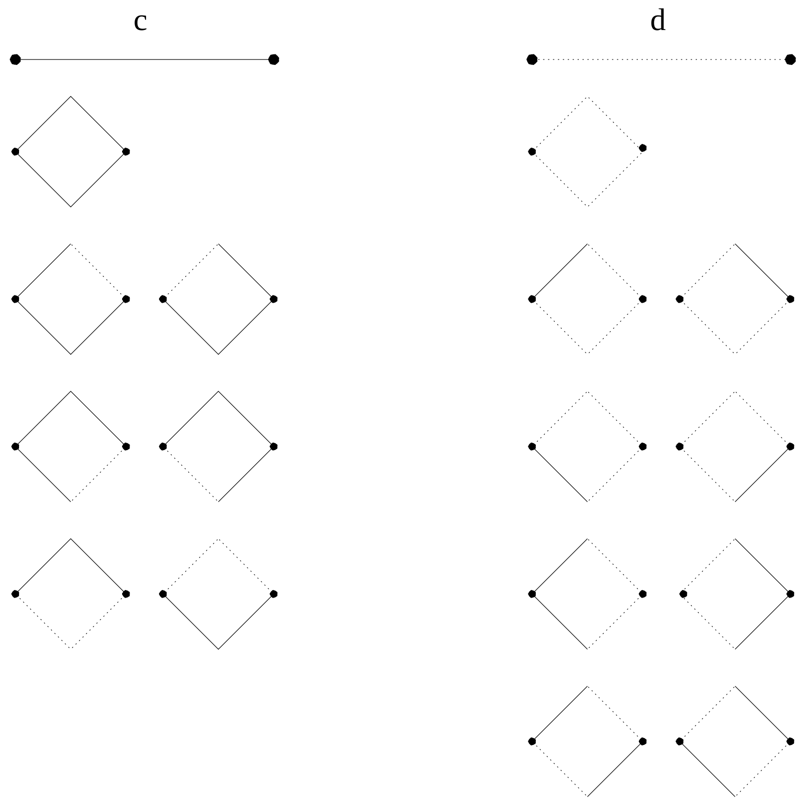
A tree description

We build a branching tree model of $(D_n^{p_c})_{n=0}^{\infty}$. For any graph D_n we label each edge by either c for connected or d for disconnected. To produce D_{n+1} we use the following reproduction rule:

1. If we have a c, the replacement graph is one of the 7 possible connected graph structures.
2. If we have a d, the replacement graph for that non-edge is chosen from the 9 possible disconnected configurations.

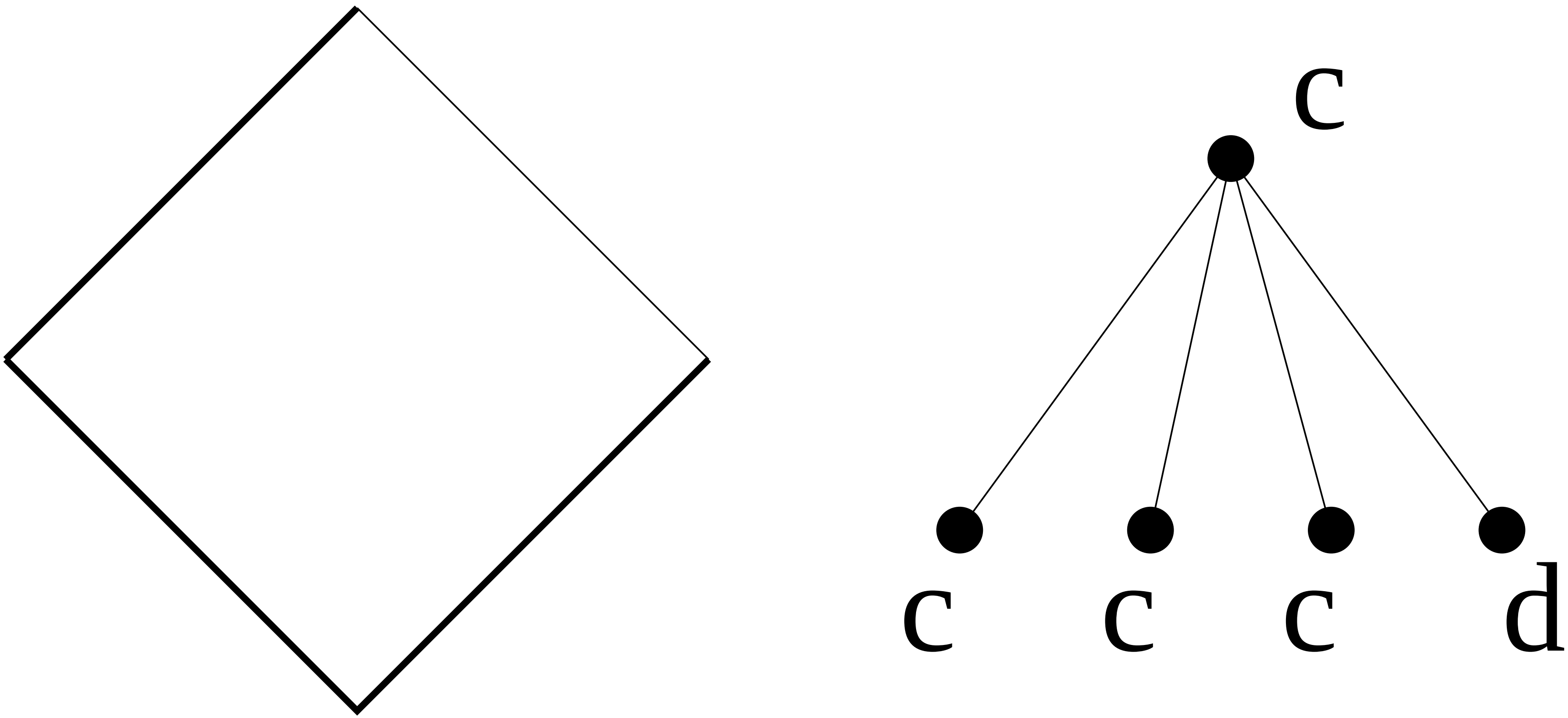
Thus we view our sequence of percolation configurations (G_n) as starting from the initial edge G_0, that is D_0 labelled with a c, and then each graph G_n is the subgraph of the labelled graph D_n where we only keep the edges with labels c.

The configurations

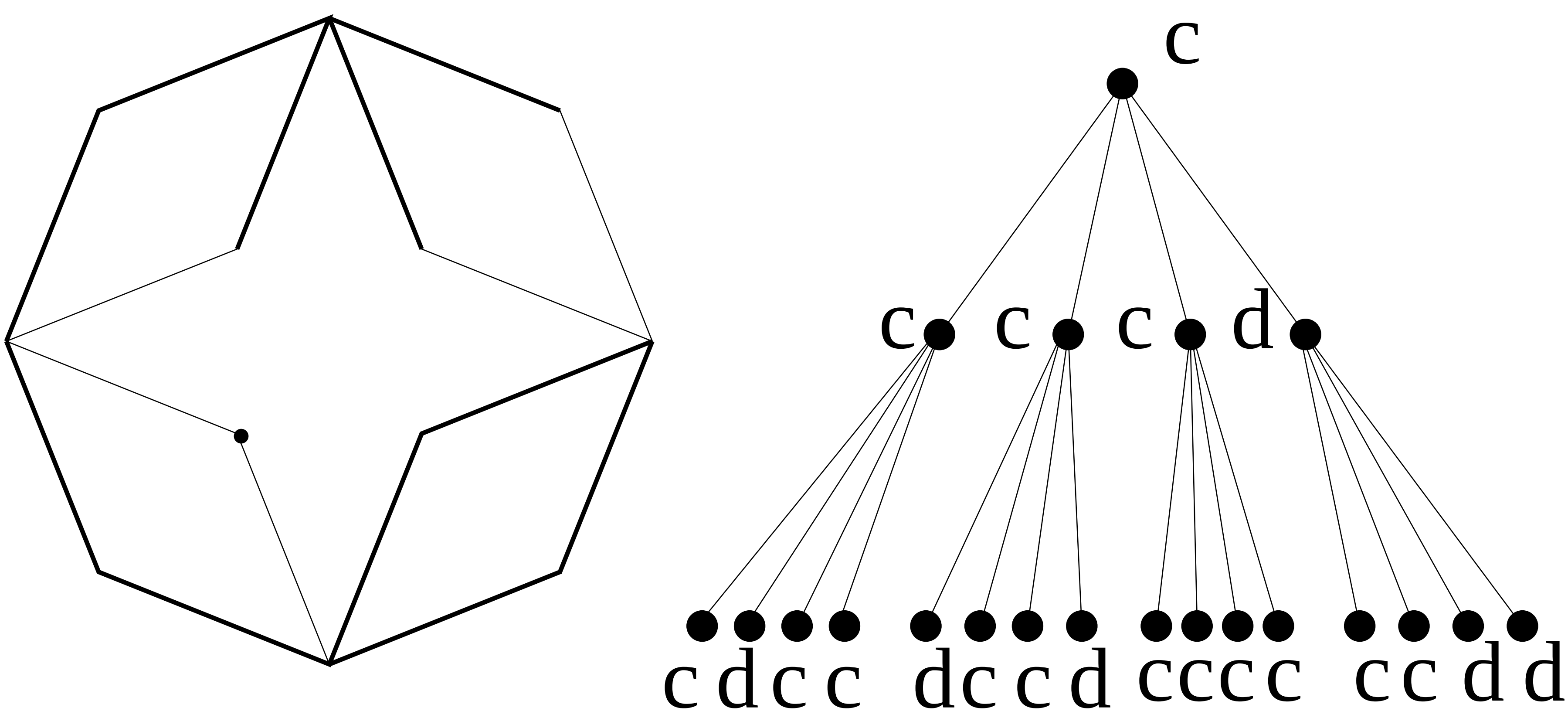


For D_1^p we have 7 combinations of edges which give a connection across D_0 and 9 which give a disconnection.

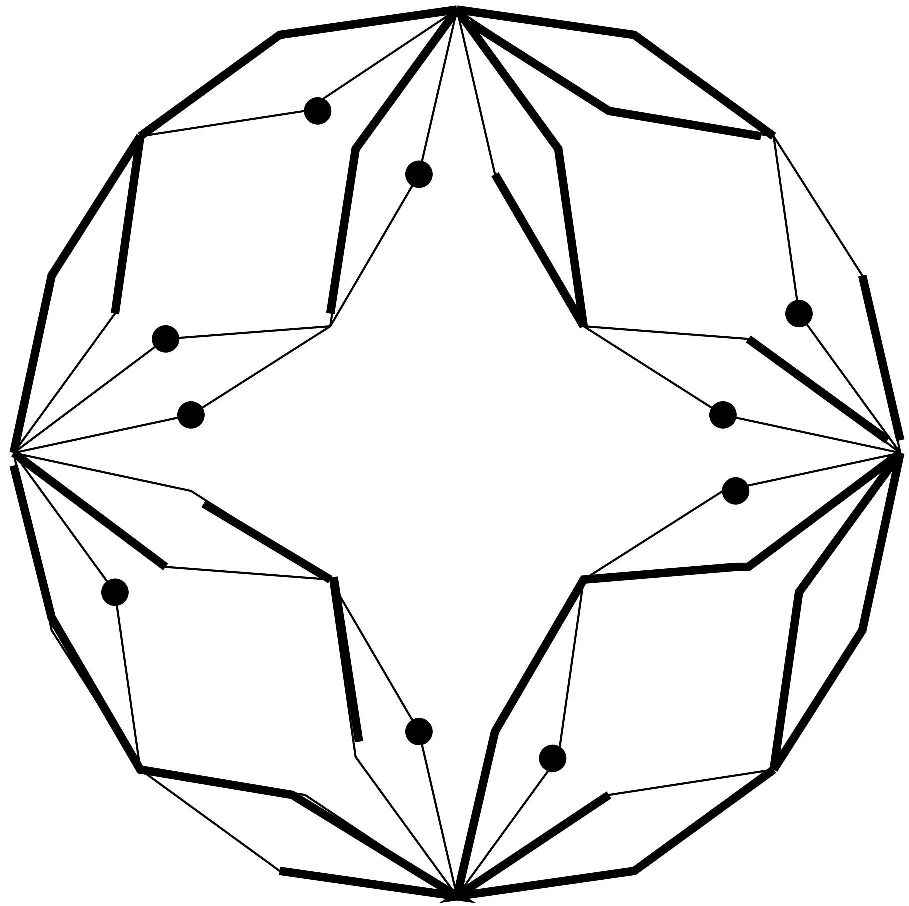
level 0



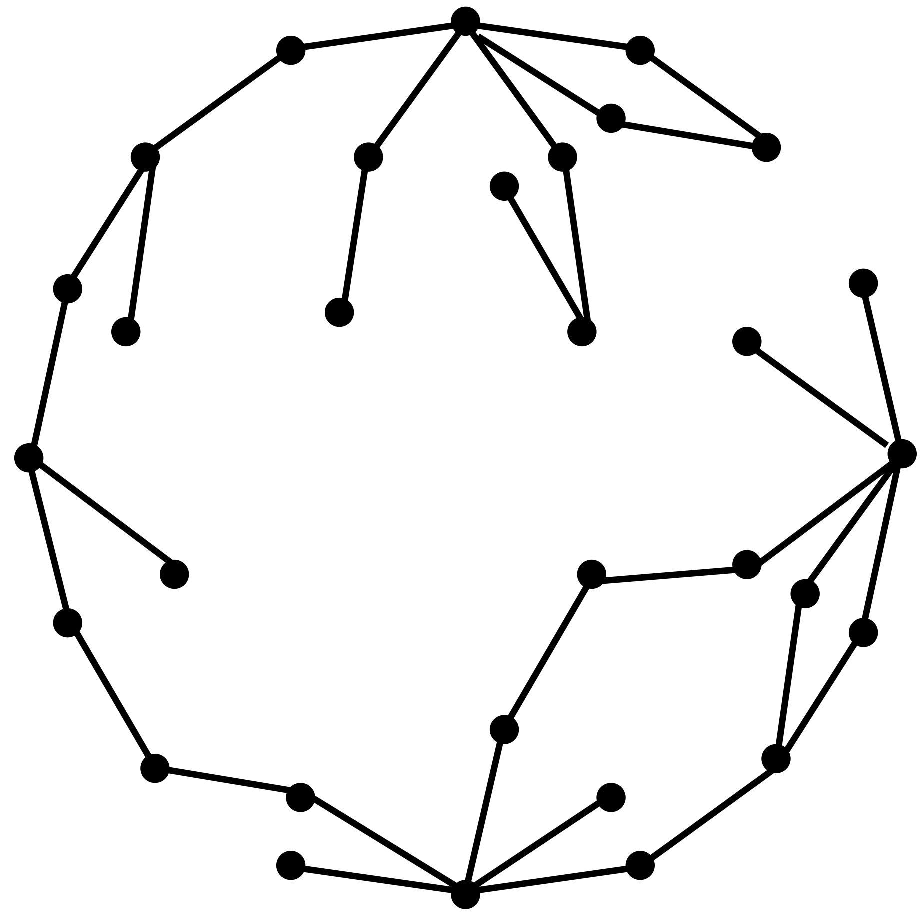
level 1



level 2



level 3



The infinite cluster at level 3

The critical cluster

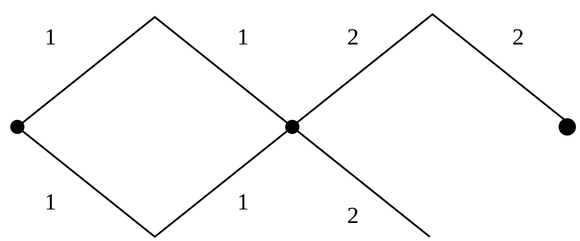
- The critical cluster is the connected component of $D_\infty^{p_c}$ containing 0 and 1.
- The existence of the critical cluster has positive probability, thus we can condition on its existence and work on a subset $\Omega^c \subset \Omega$ of our probability space.

The construction of $(D_{n,N}^{p_c})_{0 \le n \le N}$ can be extended to describe the infinite cluster at criticality. This is a sequence of subgraphs of $(D_{n,N}^{p_c})_{0 \le n \le N}$ which we label $(\mathcal{C}_n)_{0 \le n \le N}$ where we remove all the edges of the graph $D_N^{p_c}$ that are not connected to the vertices 0 and 1 and then apply the percolation construction.

Diffusion on the critical cluster

The key is to understand the electrical resistance. To ensure the effective resistance across the essentially two different configurations is one we take edge weights

$$\rho_{u_{\mathbf{i}}} = \begin{cases} 1 & u_{\mathbf{i}} = c, u_{\mathbf{i}j} = c, j = 1,2,3,4 \\ 2 & \text{otherwise} \end{cases} \tag{2}$$



The electrical resistance can be used as a metric and we can describe the cluster's properties in the resistance metric by a multitype branching random walk.

The Dirichlet form

We put resistances on each cell to ensure that the global resistance remains at 1.

Theorem 1 *There exists a Dirichlet form* $(\mathcal{E}^{(\omega)}, \mathcal{F}^{\omega})$ *on* $L^2(\mathcal{C}(\omega), \mu^{\omega})$ *for all* $\omega \in \Omega$ *with the self-similarity condition*

$$\mathcal{E}^{(\omega)}(f,g) = \sum_{i=1}^{4} \mathcal{E}^{(\sigma_i \omega)}(f \circ \psi_i, g \circ \psi_i)\rho_{u_\emptyset} \;\; \forall f, g \in \mathcal{F}^{\omega}.$$

The Hausdorff dimension in the resistance metric $d_f^r = \theta$, the Malthusian parameter of a branching process.

The spectral dimension $d_s = 2d_f^r/(d_f^r + 1) < 2$.

Spectral Properties

As before the Dirichlet and Neumann Laplacians have compact resolvent and we have an eigenvalue counting function $N(\lambda)$ for either. Using multidimensional renewal theory and branching random walk we have:

Theorem 2 *There is oscillation for the high frequency asymptotics of either Dirichlet or Neumann eigenvalues in that for each $\lambda \in [1, 2^{(1+\theta)})$, we have a random variable W such that*

$$\lim_{n\to\infty} \left| \frac{N^{u_\emptyset}(\lambda 2^{(1+\theta)n})}{(\lambda 2^{(1+\theta)n})^{\theta/(\theta+1)}} - m_\infty^{u_\emptyset}(\log \lambda) W \right| = 0, \quad a.s.$$

where $m_\infty^{u_\emptyset}(t)$ is the limit of the renewal equation for the mean.

Oscillations are inherited from those in the lattice despite the randomness.

Higher order terms

We try to examine the higher order terms in the spectral asymptotics.

1. The case of the continuum random tree there is a CLT in the spectrum but proving the non-triviality of the variance is difficult.

2. For the random recursive gaskets the LLN has a random limit at first order and checking CLT conditions is more challenging.

3. Percolation on the diamond lattice has a random multiple of a periodic function.

4. We look at a simpler model - random fractal strings.

A fractal string is a union of intervals with a boundary that is typically a Cantor set. The problem of the spectral asymptotics is more straightforward as we understand very well the eigenvalues for the interval.

The Brownian string

A natural random fractal string can be generated by Brownian motion.

Take Brownian motion started from 0 in $\mathbb{R}$ run for unit time. The path can be viewed as a sequence of excursions away from 0. The zero set is a Cantor set (perfect and nowhere dense) and so divides the time axis into a countable number of intervals. Thus we have a decomposition of the unit interval - a fractal string.

For the Dirichlet counting function

$$N(\lambda) = \frac{1}{\pi}\lambda^{1/2} - L\zeta(1/2)\lambda^{1/4} + o(\lambda^{1/8+\epsilon}).$$

where L is the local time at 0 of the Brownian motion and ζ is the Riemann zeta function (H-Lapidus).

We look at a family of examples:

Dirichlet strings

Let T_1, T_2 be chosen from a Beta(α, α) distribution, $\gamma \in (0,1)$.

Divide $[0,1]$ into three pieces $\psi_1([0,1]) = [0, T_1^{1/\gamma}], \psi_2([0,1]) = [T_2^{\gamma}, 1]$ and S_1, the middle open interval, the first piece of string of length $1 - T_1^{1/\gamma} - T_2^{1/\gamma}$.

By choosing ψ_1, ψ_2 independently according to the distribution we generate a random cantor set K defined by $K = \cup_i \psi_i(K)$.

This is the boundary of the string S and has dimension γ a.s.

The counting function

- For the boundary term in the asymptotics:

  Theorem:

  For the fractal string S, for all $\alpha \in \mathbb{N}, \gamma \in (0,1)$ we have

  $$\lambda^{-\gamma/2}\left[\pi^{-1}\lambda^{1/2} - N(\lambda)\right] \to C, \text{ a.s.,}$$

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- For the second term we use a Central Limit Theorem for the GBP (Charmoy, Croydon, H 2015)

  Theorem:

  For the string S, we have, provided $\alpha \leq 59$

  $$\lambda^{\gamma/4}\{\lambda^{-\gamma/2}\left[\frac{1}{\pi}\lambda^{1/2} - N(\lambda)\right] - C\} \to N(0,\sigma^2), \text{ in distribution,}$$

  as $\lambda \to \infty$, for some positive constant σ.

Theorem: (Charmoy, Croydon, H 2015)

There exists an $\tilde{\alpha} \geq 80$ and a $\gamma \in (0,1)$ such that: if $59 < \alpha \leq \tilde{\alpha}$, then there exists a constant $c_1(\gamma,\alpha) > 0$ such that

$$\mathbb{E}N_{\gamma,\alpha}(\lambda) = \frac{1}{\pi}\lambda^{1/2} - C(\gamma,\alpha)\lambda^{\gamma/2} + c_1(\gamma,\alpha)\lambda^{\gamma\eta(\alpha)/2} + o(\lambda^{\eta(\alpha)}),$$

where $\eta(\alpha) = \max\{\Re(\theta_0) \in (-\infty, 1) : P(\theta_0) = 0\}$,

$$P(\theta) := \prod_{i=0}^{\alpha-1}(\alpha+\theta+i) - \frac{(2\alpha)!}{\alpha!}$$

and, for this range of α we have $1/2 < \eta(\alpha) < 1$. In particular

$$\lambda^{\gamma/4}\left(\lambda^{-\gamma/2}\left(\frac{1}{\pi}\lambda^{1/2} - N_{\gamma,\alpha}(\lambda)\right) - C(\gamma,\alpha)\right)$$

does not converge in distribution as $\lambda \to \infty$.