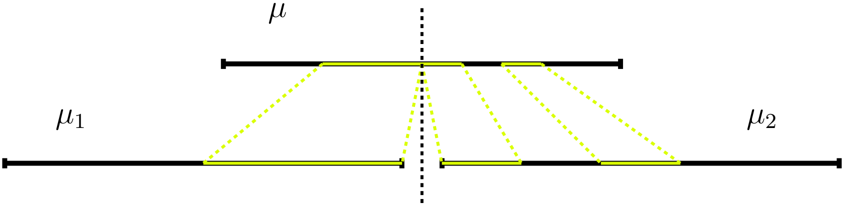
Uniwersytet Wrocławski

Local fluctuations of critical Mandelbrot cascades

Konrad Kolesko
joint with
D. Buraczewski and P. Dyszewski
Warwick, 18-22 May, 2015

Random measures



For given random variables X_1, X_2 s.t. $\mathbb{E}\left[e^{-X_1} + e^{-X_2}\right] = 1$ we are interested in random measures μ on $[0, 1)$ satisfying self similar property:

$$\mu(B) = e^{-X_1}\mu_1\big(2(B \cap [0, 1/2))\big) + e^{-X_2}\mu_2\big(2(B \cap [1/2, 1) - 1)\big),$$

where $\mu_1 \perp\!\!\!\perp \mu_2 \perp\!\!\!\perp X_1, X_2$ and $\mathcal{L}\mu = \mathcal{L}\mu_1 = \mathcal{L}\mu_2$.

Goal: Understand local properties of μ

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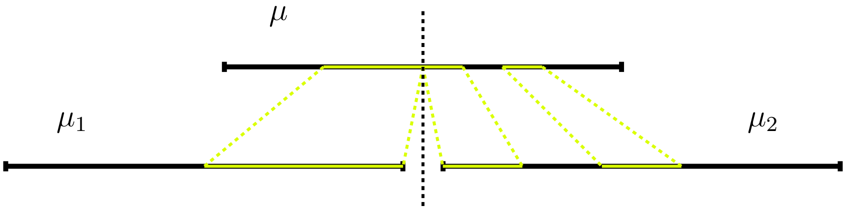

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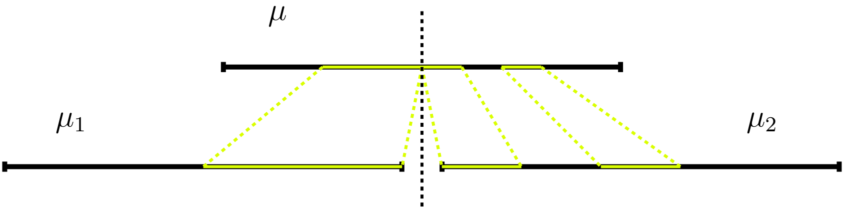

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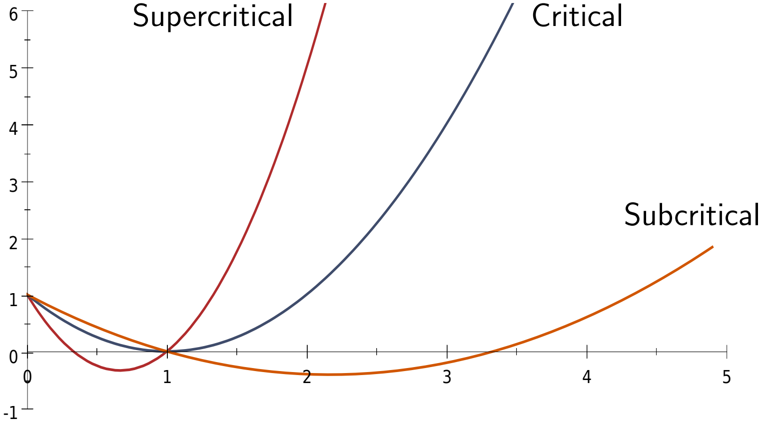
Phase transition

Define $\psi(t) := \log_2 \mathbb{E}\big(e^{-tX_1} + e^{-tX_2}\big)$.
By assumption $\psi(0) = 1$, $\psi(1) = 0$, $\psi \nearrow \infty$.



Subcritical case $\psi'(1) = -m < 0$

Theorem (Liu)

Suppose that $\psi'(1) = -m < 0$*, then for any* $\delta > 0$*, almost all realization of* μ*,* μ*-almost all* x *and sufficiently large* n

$$\mu(B(x,2^{-n})) \geq 2^{-nm-(1+\delta)\sqrt{2\sigma^2 n \log\log n}}$$
$$\mu(B(x,2^{-n})) \leq 2^{-nm+(1+\delta)\sqrt{2\sigma^2 n \log\log n}},$$

where $\sigma^2 = \psi''(1) - \psi(1)$*. Moreover*

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Supercritical & critical case

Barral, Rhodes, Vargas

When $\psi'(1) > 0$ then μ is purely atomic

Barral, Kupiainen, Nikula, Saksman, Webb

If $\psi'(1) = 0$, then the random measure μ almost surely has no atoms. Moreover for any k and $\delta > 0$, μ-a.e. x

- $\mu(B(x, 2^{-n})) \geq e^{-\sqrt{6 \log 2}\sqrt{n(\log n + (1/3+\delta) \log\log n)}}$ for sufficiently large n
- $\mu(B(x, 2^{-n})) \geq e^{-(\sqrt{2 \log 2}+\delta)\sqrt{n \log n}}$ i.o.
- $\mu(B(x, 2^{-n})) \leq n^{-k}$ for sufficiently large n.

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Main theorem

Theorem (D. Buraczewski, P. Dyszewski, K.K.)

Let $\psi'(1) = 0$, $k \in \mathbb{N}$ *and* $\delta > 0$. *Then for almost all realizations of* μ, μ*-almost all* $x \in [0, 1)$ *and sufficiently large* n *we have*

$$\mu(B(x, 2^{-n})) \geq \exp\left(-(1+\delta)\sqrt{2\sigma^2 n \log\log n}\right)$$

$$\mu(B(x, 2^{-n})) \leq \exp\left(\frac{-\sqrt{n}}{\prod_{i=1}^{k} \log_{(i)} n \ (\log_{(k+1)} n)^2}\right)$$

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Notations

Dyadic intervals on [0,1) $\leftrightsquigarrow$ vertices of a binary tree T

$x \in [0,1)$ $\leftrightsquigarrow$ $\theta \in \partial T$.

$B(v) = \{\theta \in \partial T : v \in \overline{o\theta}\}$.

For a random measure μ_ω we are interested in a pointwise estimates of $\mu_\omega(B(\theta_n))$ on the enlarged measure space

$$\widetilde{\mathbb{P}}(d\omega, d\theta) := \mathbb{P}(d\omega)\mu_\omega(d\theta)$$

i.e. we are looking for deterministic ϕ_1, ϕ_2 s.t.

$$\phi_1(n) \leq \mu_\omega(B(\theta_n)) \leq \phi_2(n),$$

for $\widetilde{\mathbb{P}}$-almost all (ω, θ) and large n.
$\widetilde{\mathbb{P}}$ can be replaced by $\widehat{\mathbb{P}}(d\omega, d\theta) = \mathbb{P}(d\omega)\overline{\mu}_\omega(d\theta)$, where $\overline{\mu}_\omega$ is the normalized measure μ_ω.

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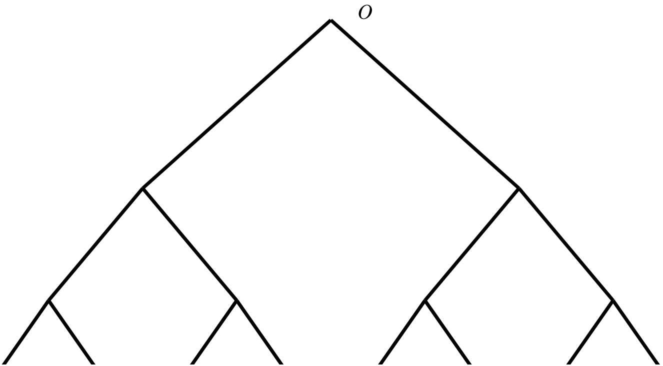
Branching random walk



$\mathbb{P}$ – probability measure on the set of labeled trees
$X(v)$ – the sum of random variables along the path between o and v $\quad (X(u) = X_2 + X_{21} + X_{212})$
$\{X(v)\}_{v \in T}$ – Branching random walk (BRW)

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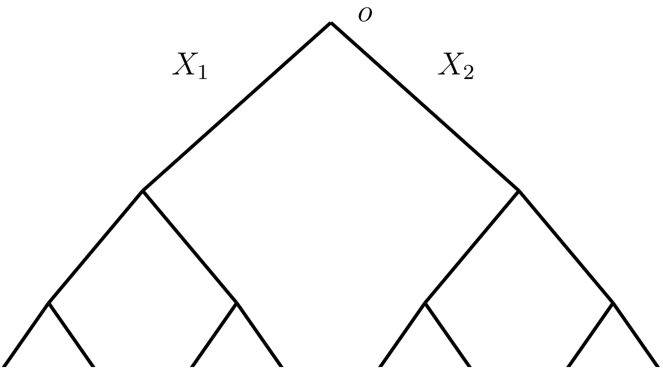

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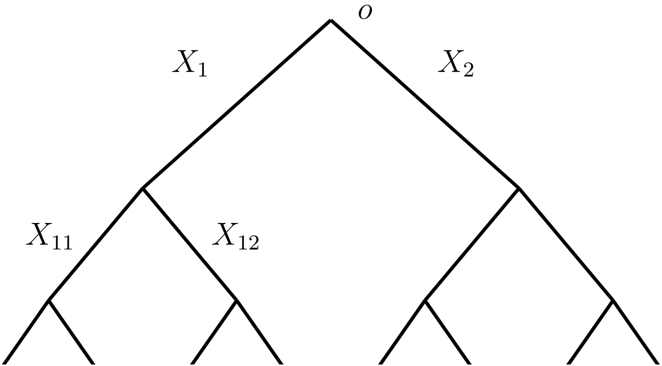

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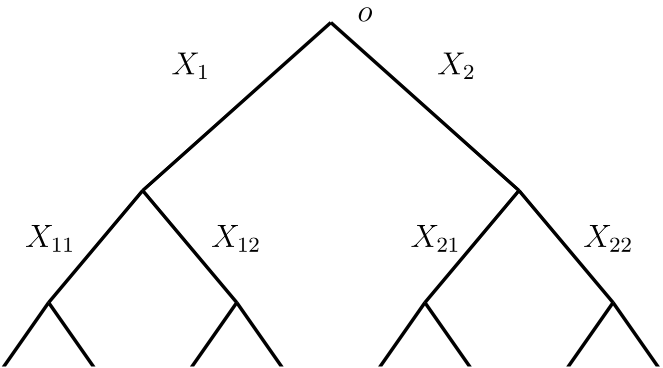

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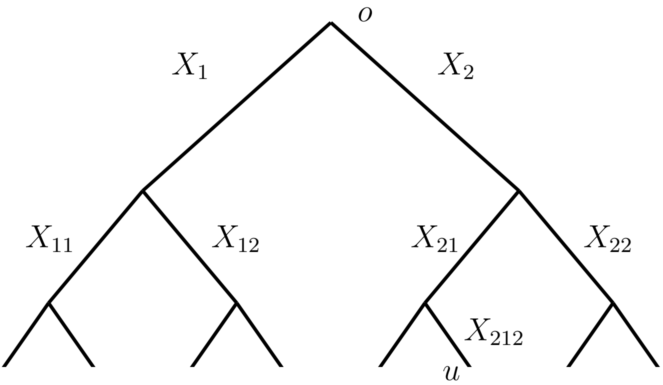

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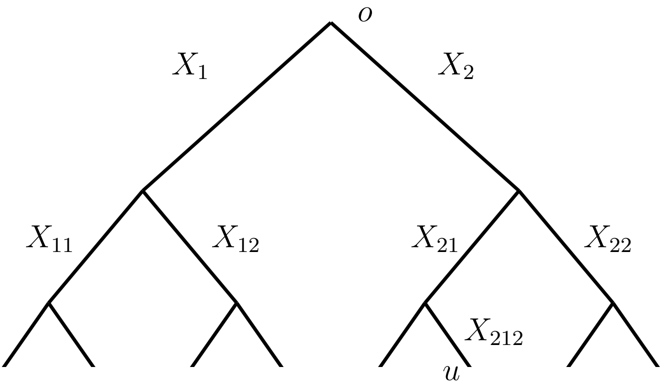

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Stopping line

Since $\mathbb{E}\sum_{|v|=1} e^{-X(v)} = 1$ and $\mathbb{E}\sum_{|v|=1} X(v)e^{-X(v)} = 0$ the equation

$$\mathbb{E}f(Y) := \mathbb{E}\sum_{|v|=1} f(X(v))e^{-X(v)}$$

defines distribution of a driftless r.v. Y

Let h be a harmonic function on some set A (a solution of a Dirichlet problem), $V_n = Y_1 + \cdots + Y_n$ and $\sigma = \min\{k : s + V_k \notin A\}$. Then the process $h(s + V_{\min(n,\sigma)})$ is a martingale.

Let $\tau = \{w : s + X(w) \notin A \text{ for the first time}\}$, $v_\tau = \min(v, \tau)$. Then

$$W_n^s = \sum_{|v|=n} h(s + X(v_\tau))e^{-X(v_\tau)}$$

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Martingales

Some natural martingales

- $h \equiv 1$: $W_n = \sum_{|v|=n} e^{-X(v)}$
- $h(x) = x$: $D_n = \sum_{|v|=n} X(v)e^{-X(v)}$
- $A = [0, \infty)$, $h(x) \approx x \vee 0$: $W_n^s = \sum_{|v|=n} h(s + X(v))1_{[s+X(v')>0, \text{ for } v' \leq v]} e^{-X(v)}$

$$\mu(B(v)) := \lim_n \sum_{|w|=n,\ w<v} X(w)e^{-X(w)}$$

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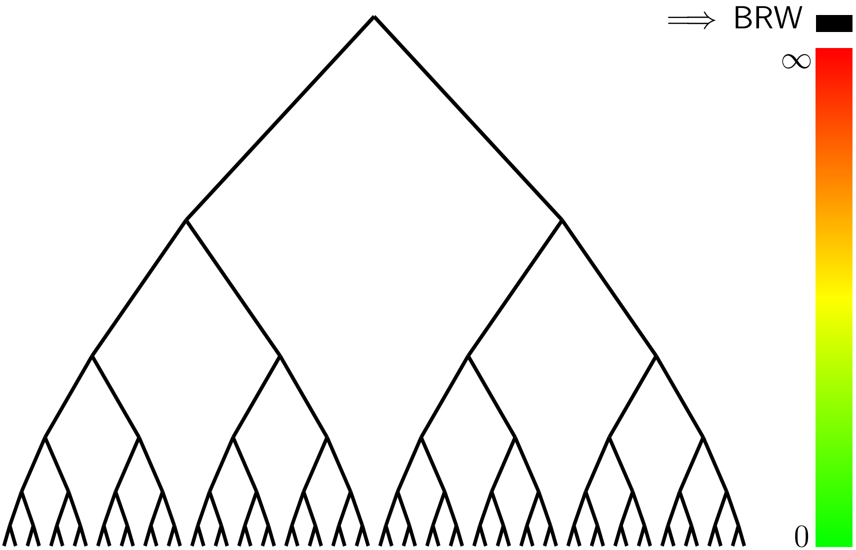
$$W_n^s \to W^s \qquad \mathbb{P}\text{-a.s. and } L^1$$

For any $s \in D$ define

$$\mathbb{P}^s := \frac{W^s}{h(s)} \cdot \mathbb{P}$$

$$\mathbb{P}[\operatorname{supp} W^s] \gtrsim 1 - 1/s$$


$\Longrightarrow$ BRW
∞
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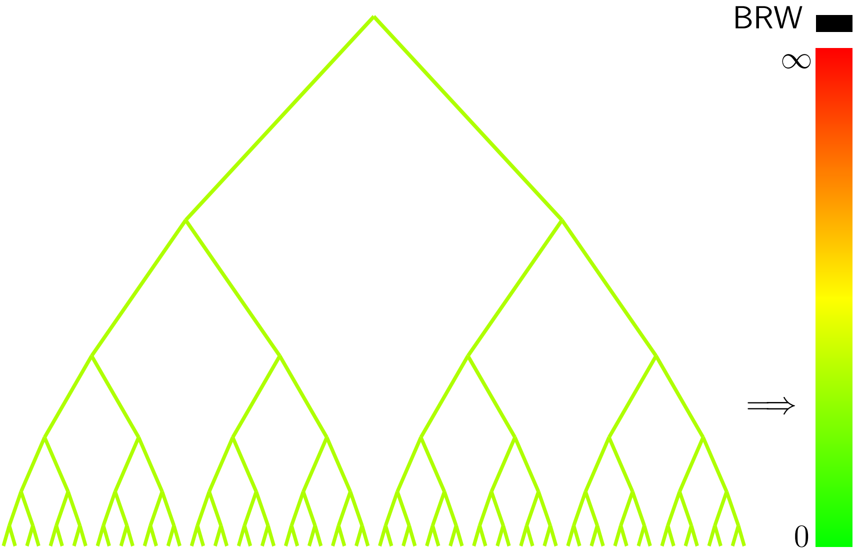
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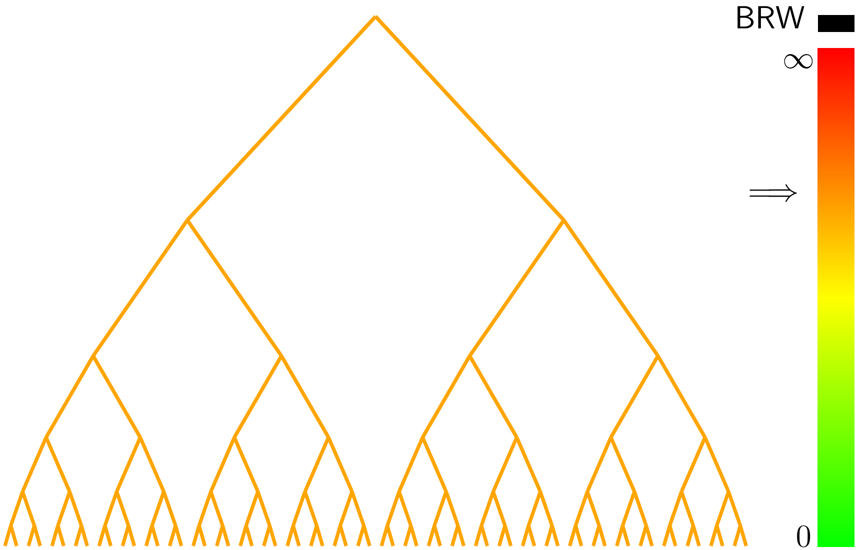
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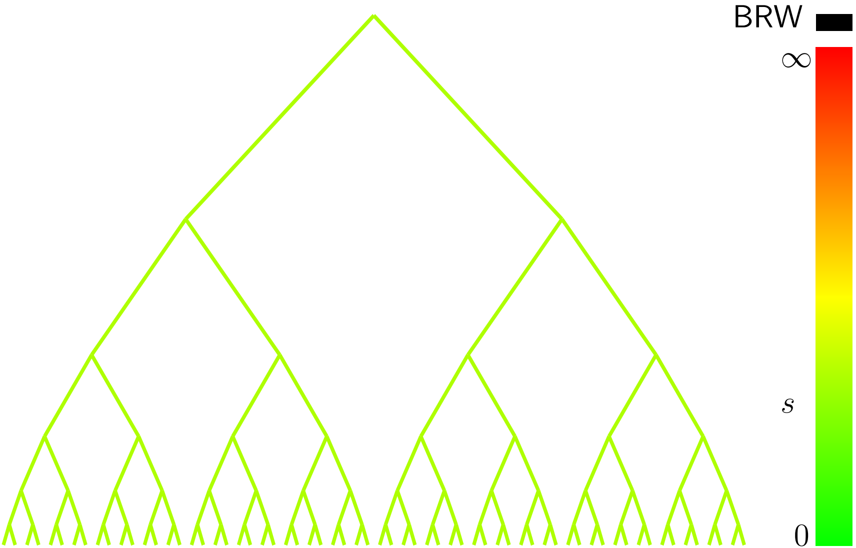
Spinal decomposition of $\mathbb{P}^s$



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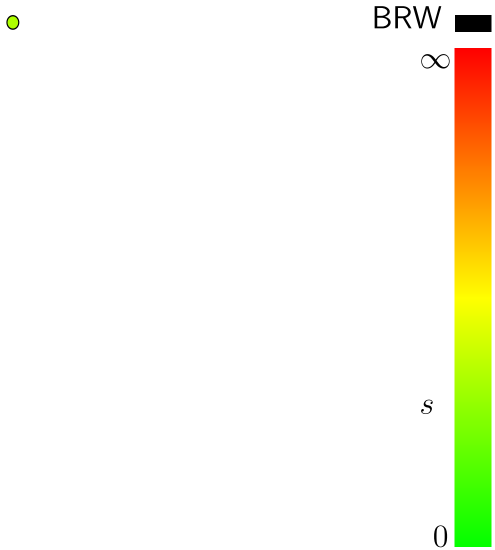

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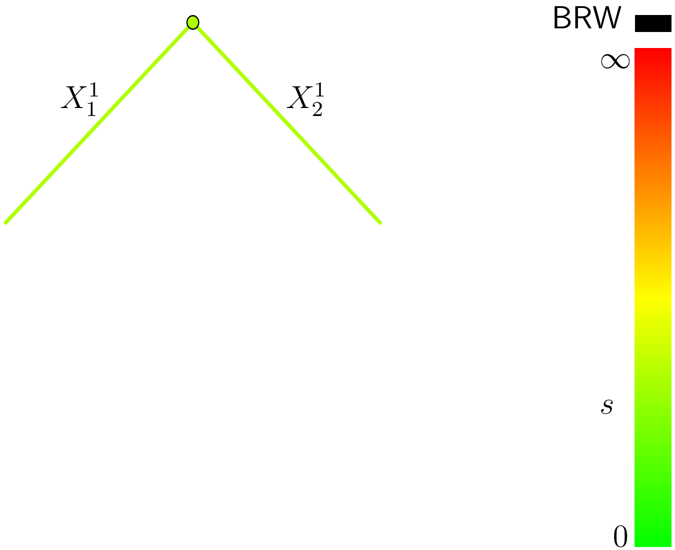

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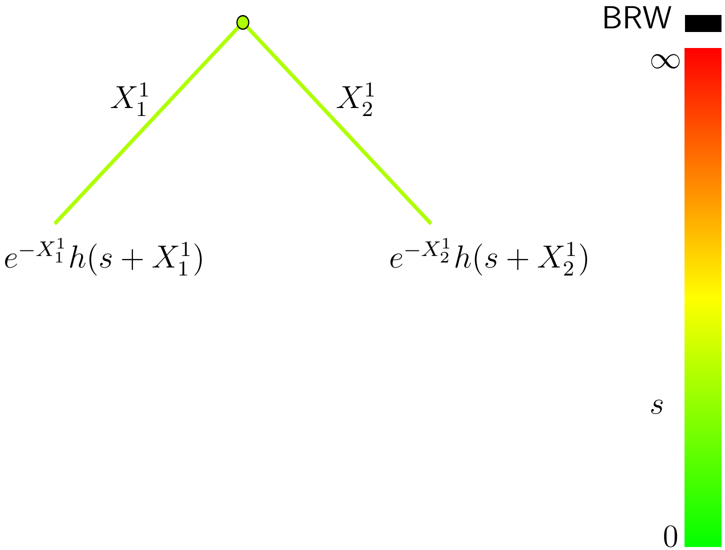

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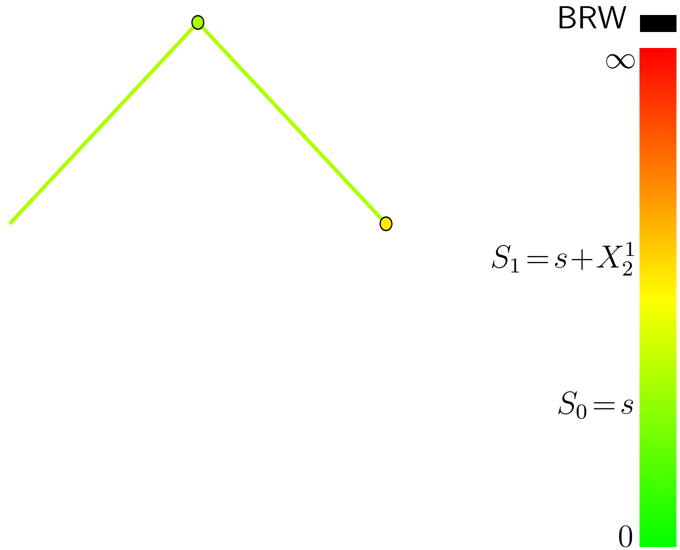

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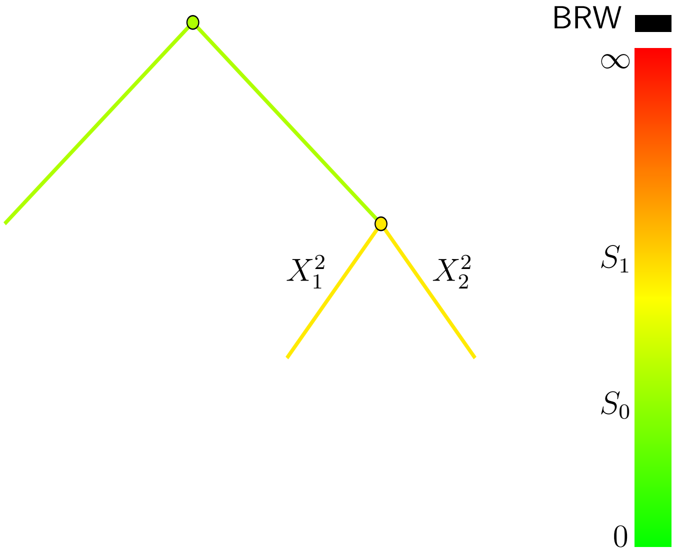

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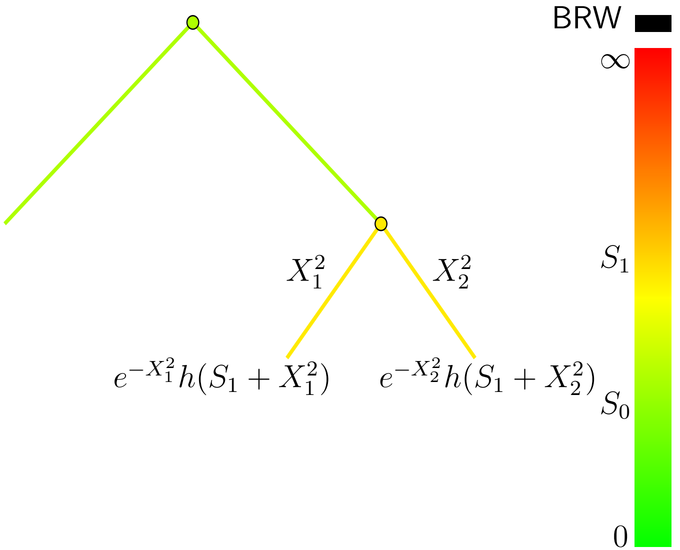

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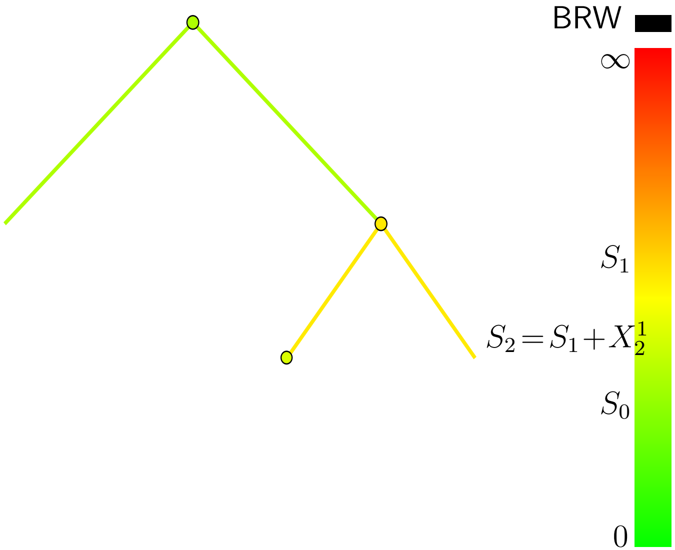

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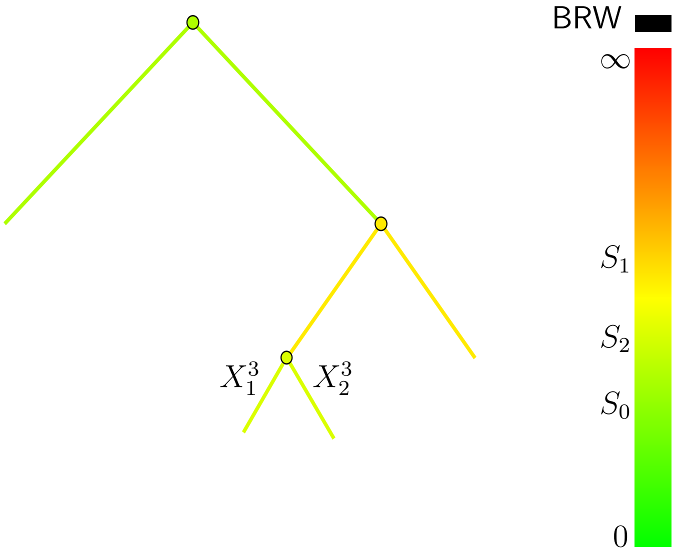

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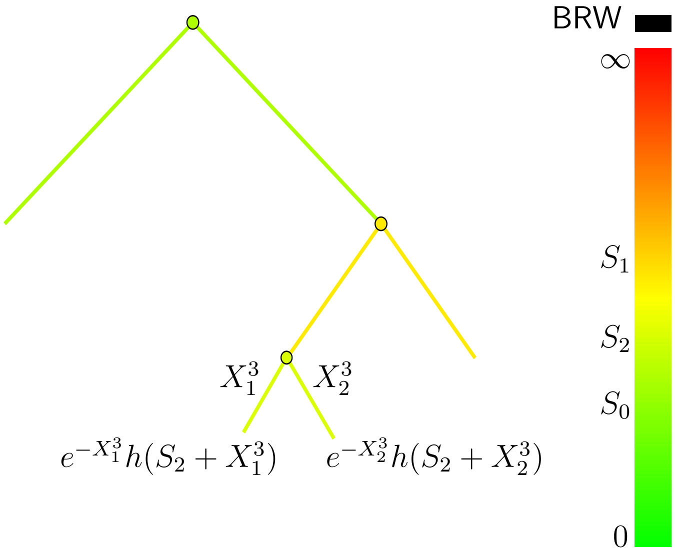

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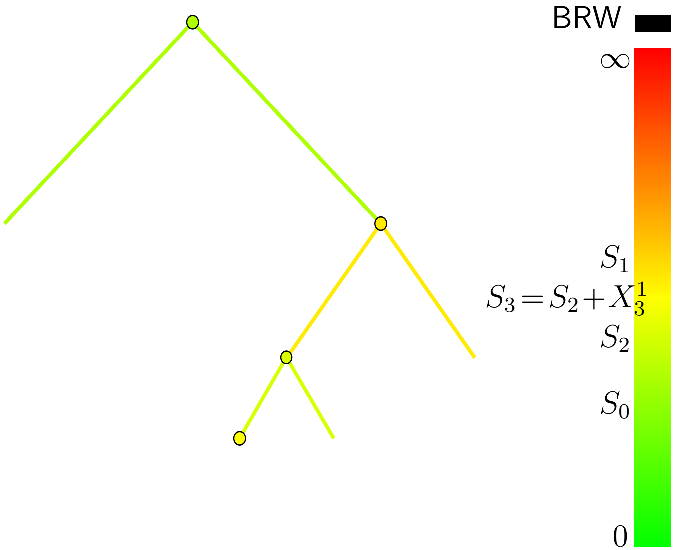

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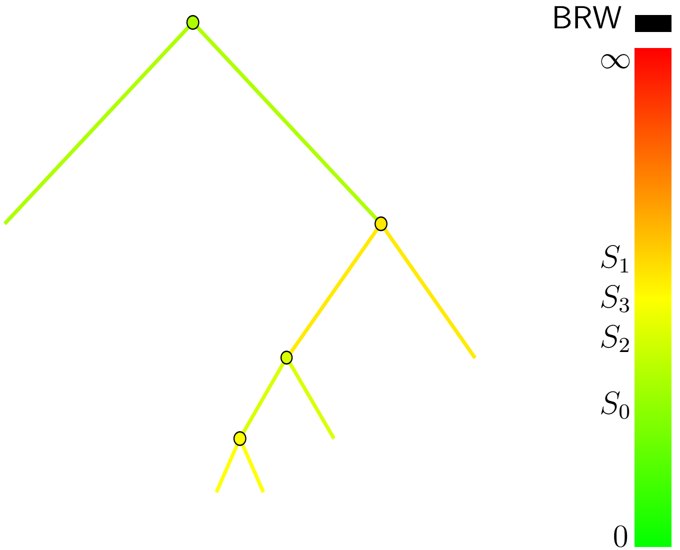

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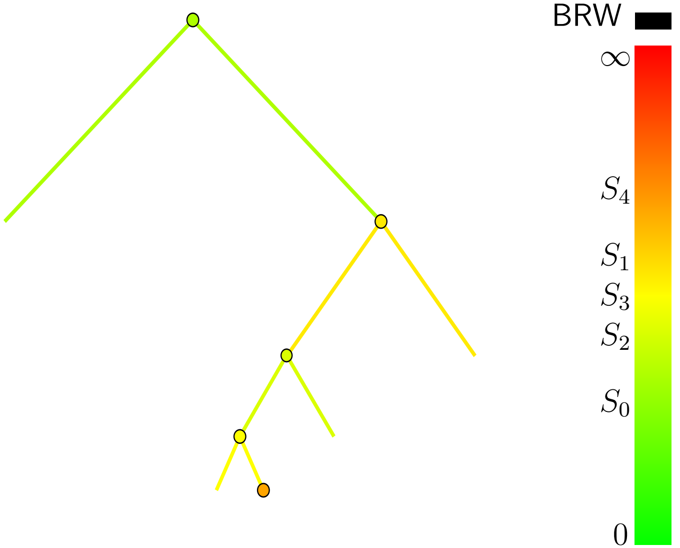

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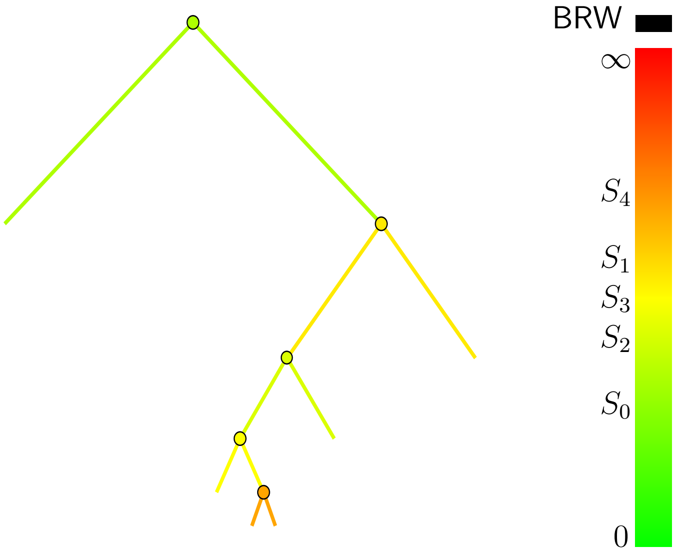

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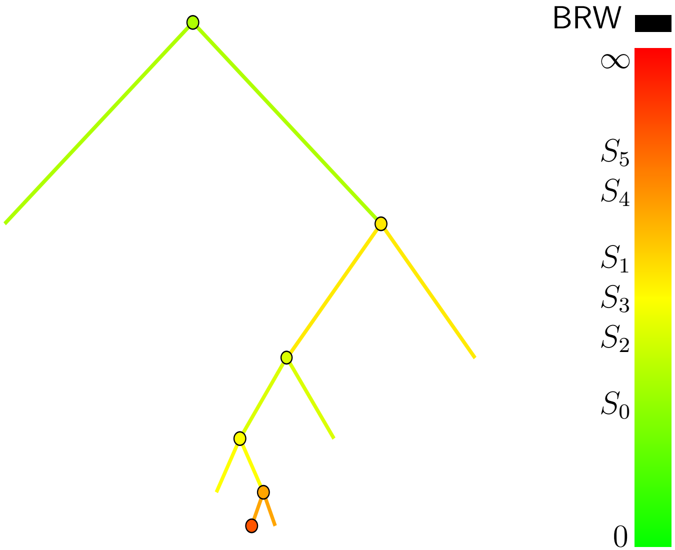

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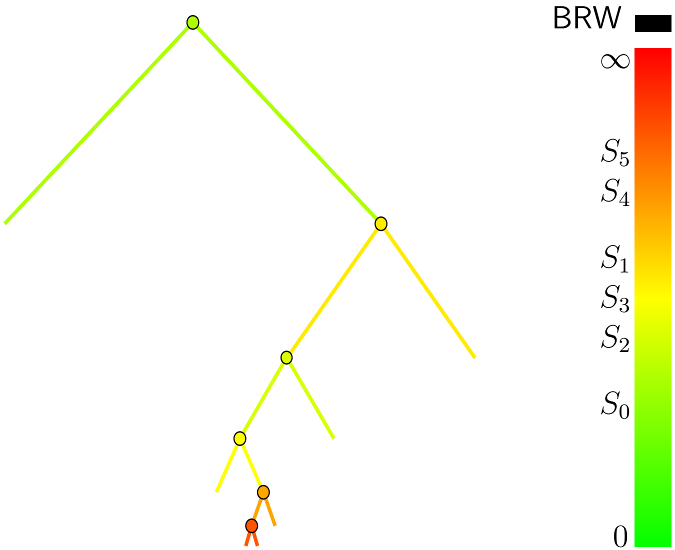

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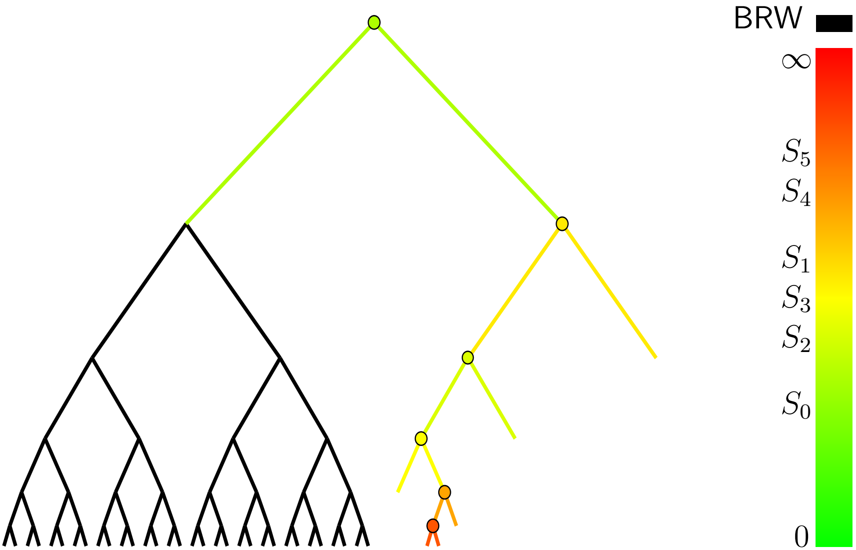

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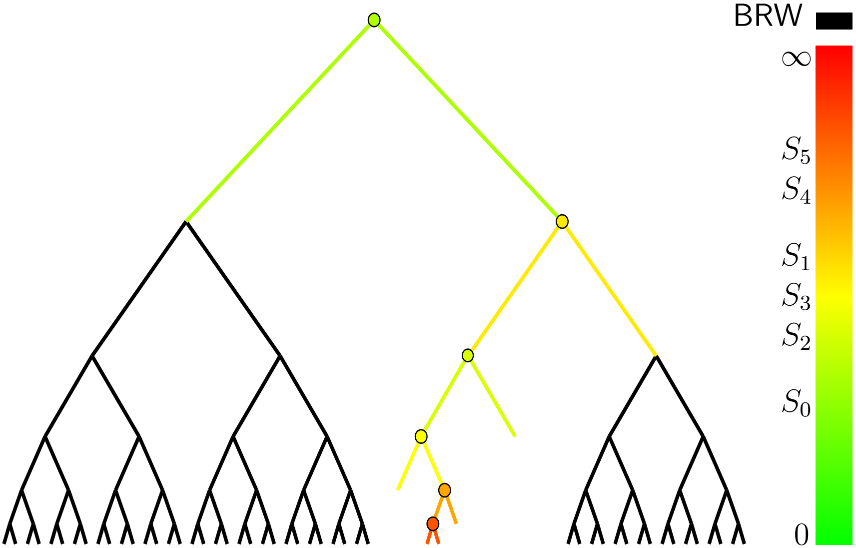

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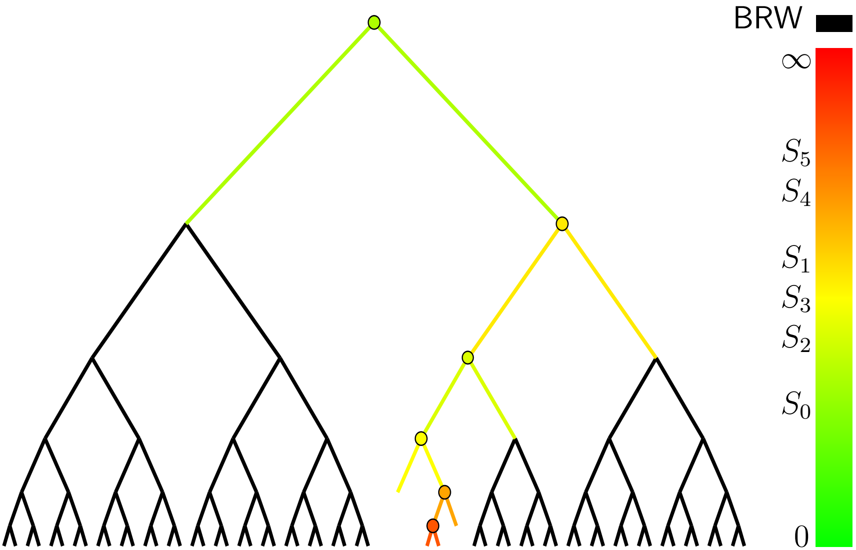

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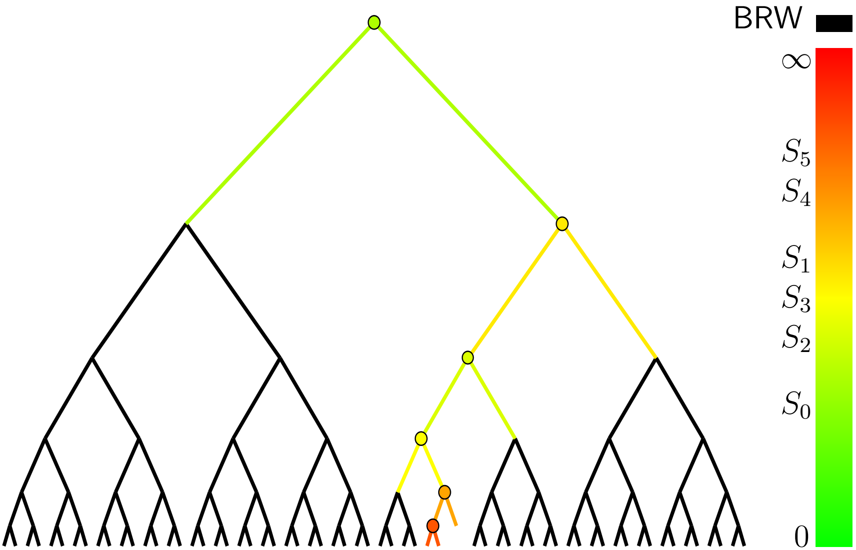
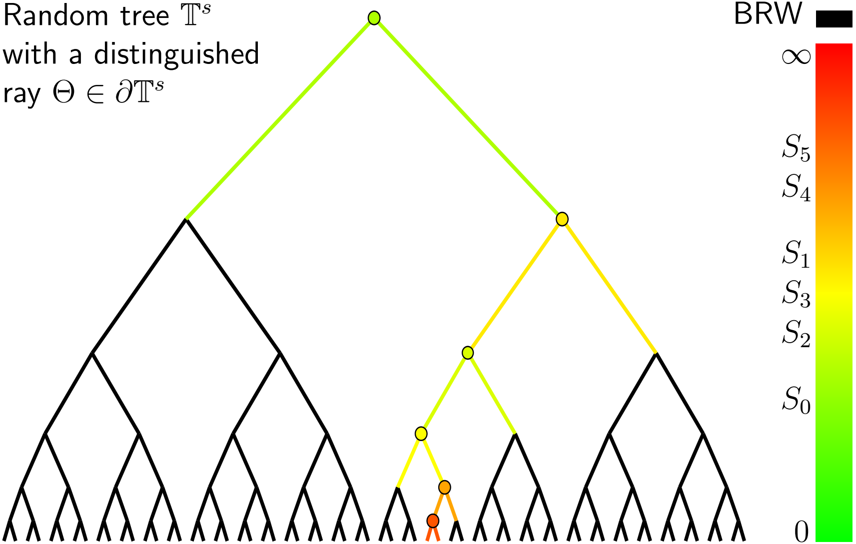


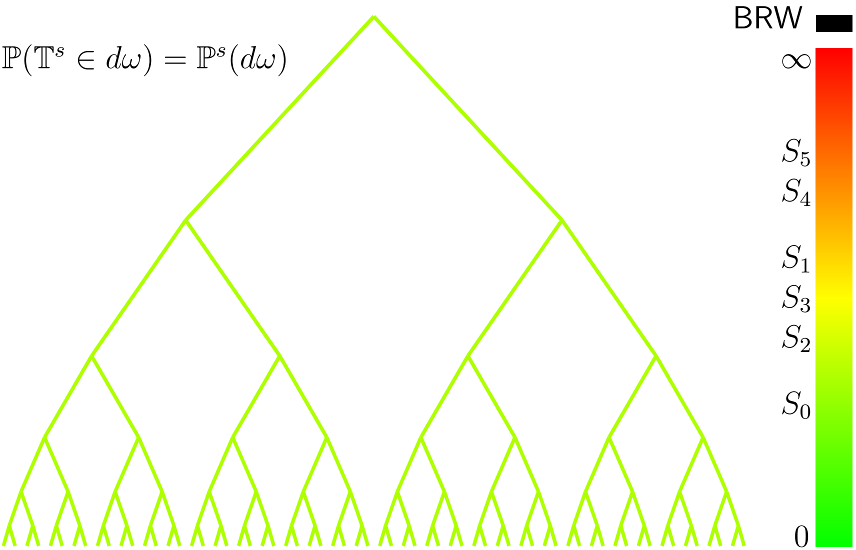
Spinal decomposition of $\mathbb{P}^s$

Random tree $\mathbb{T}^s$ with a distinguished ray $\Theta \in \partial\mathbb{T}^s$



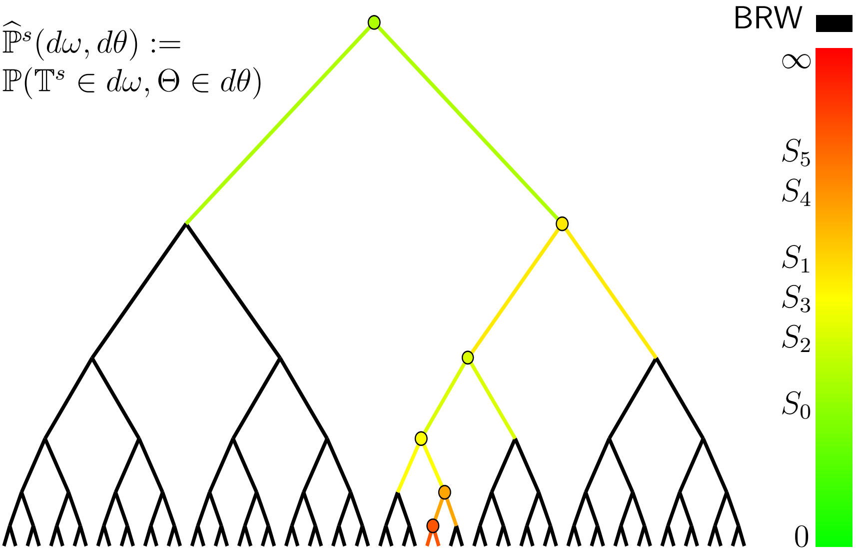
Spinal decomposition of $\mathbb{P}^s$

$$\mathbb{P}(\mathbb{T}^s \in d\omega) = \mathbb{P}^s(d\omega)$$



Spinal decomposition of $\mathbb{P}^s$

$\widehat{\mathbb{P}}^s(d\omega, d\theta) :=$
$\mathbb{P}(\mathbb{T}^s \in d\omega, \Theta \in d\theta)$



$\widehat{\mathbb{P}}^s$ vs. $\widehat{\mathbb{P}}$

We have

$$\widehat{\mathbb{P}}^s \ll \widehat{\mathbb{P}}.$$

The converse is not true, but for any set A

$$\widehat{\mathbb{P}}^s(A) = 1 \Rightarrow \widehat{\mathbb{P}}(A) \geq 1 - \epsilon(s),$$

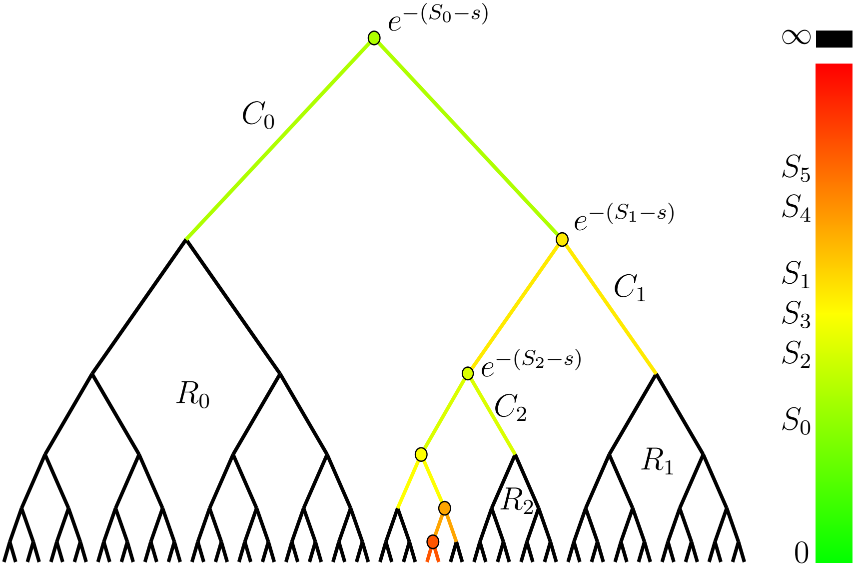
for some $\epsilon(s) \to 0$ as $s \to \infty$. In particular, if for any $s > 0$

$$\widehat{\mathbb{P}}^s(A) = 1$$

then

$$\widehat{\mathbb{P}}(A) = 1.$$

Probabilistic interpretation of local fluctuations



Probabilistic interpretation of local fluctuations

Under $\widehat{\mathbb{P}}^s$ the sequence $\mu_\omega(B(\theta_n))$ has the same law as

$$\sum_{k\geq n} e^{-(S_k-s)}C_kR_k,$$

where S_k is a random walk starting form s conditioned to stay positive, R_k independent random variables.

Useful information

We are looking for LIL for $\sum_{k\geq n} e^{-S_k+s} C_k R_k$,

Hambly, Kersting, Kyprianou

- $\int^{\infty} \frac{\psi(t)}{t} dt < \infty$ iff $S_n > \sqrt{n}\psi(n)$ eventually
- $\limsup_n \frac{S_n}{\sqrt{2n\sigma^2 \log\log n}} = 1$

$$\sqrt{n}\psi(n) < S_n < (1+\delta)\sqrt{2n\sigma^2 \log\log n}$$

$\mathbb{P}^s$-a.s. for n sufficiently large

Kyprianou

- $1 - \mathbb{E}e^{-tR} \sim tL(t)$ near 0 $\Longrightarrow \mathbb{E}R^{\gamma} < \infty$ for $\gamma < 1$

In particular, by Borel-Cantelli lemma, $R_n < n^2$ for all but finitely many n.

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Upper bound

Take any ψ such that $\int^{\infty} \frac{\psi(t)dt}{t} < \infty$. We have that

$$\sum_{k \geq n} e^{-S_k} R_k$$

is eventually bounded by

$$\sum_{k \geq n} e^{-\sqrt{k}\psi(k)} k^2.$$

On the other hand, if $\int^{\infty} \frac{\psi(t)dt}{t} = \infty$ then

$$\sum_{k \geq n} e^{-S_k} R_k \geq e^{-S_n} R_n \overset{i.o.}{\geq} e^{-\sqrt{n}\psi(n)} R_n \overset{i.o.}{\geq} \delta e^{-\sqrt{n}\psi(n)}$$

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Lower bound

Take $q > 1$ and by $N(n) := q^{\lceil \log_q n \rceil}$. Borel-Cantelli's lemma implies that for sufficiently large n, $\sup_{q^n < k \le q^{n+1}} R_k \ge \delta_0$.

$$\begin{aligned}
\sum_{k \ge n} e^{-S_k} R_k &\ge \sum_{k \ge n} e^{-(1+\delta)\sqrt{2k\sigma^2 \log\log k}} R_k \\
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&\ge \delta_0 e^{-(1+\delta)\sqrt{2qN(n)\sigma^2 \log\log(qN(n))}} \\
&\ge \delta_0 e^{-(1+\delta)\sqrt{2q^2 n\sigma^2 \log\log(q^2 n)}} \\
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