Local fluctuations of critical Mandelbrot cascades

Konrad Kolesko
joint with
D. Buraczewski and P. Dyszewski
Warwick, 18-22 May, 2015
For given random variables $X_1, X_2$ s.t. $\mathbb{E}[e^{-X_1} + e^{-X_2}] = 1$ we are interested in random measures $\mu$ on $[0, 1)$ satisfying self similar property:

$$\mu(B) = e^{-X_1} \mu_1\left(2(B \cap [0, 1/2])\right) + e^{-X_2} \mu_2\left(2(B \cap [1/2, 1) - 1)\right),$$

where $\mu_1 \perp \mu_2 \perp X_1, X_2$ and $\mathcal{L}\mu = \mathcal{L}\mu_1 = \mathcal{L}\mu_2$.

Goal: Understand local properties of $\mu$
For given random variables $X_1$, $X_2$ s.t. $\mathbb{E}[e^{-X_1} + e^{-X_2}] = 1$ we are interested in random measures $\mu$ on $[0,1)$ satisfying self similar property:

$$
\mu(B) = e^{-X_1}\mu_1\left(2(B \cap [0,1/2]\right)) + e^{-X_2}\mu_2\left(2(B \cap [1/2,1) - 1)\right),
$$

where $\mu_1 \perp \mu_2 \perp X_1, X_2$ and $\mathcal{L}\mu = \mathcal{L}\mu_1 = \mathcal{L}\mu_2$.

Goal: Understand local properties of $\mu$.
For given random variables $X_1, X_2$ s.t. $\mathbb{E}[e^{-X_1} + e^{-X_2}] = 1$ we are interested in random measures $\mu$ on $[0, 1)$ satisfying self similar property:

$$
\mu(B) = e^{-X_1}\mu_1(2(B \cap [0, 1/2))) + e^{-X_2}\mu_2(2(B \cap [1/2, 1)) - 1)),$$

where $\mu_1 \perp \mu_2 \perp X_1, X_2$ and $\mathcal{L}\mu = \mathcal{L}\mu_1 = \mathcal{L}\mu_2$.

**Goal:** Understand local properties of $\mu$.
Define \( \psi(t) := \log_2 \mathbb{E}(e^{-tX_1} + e^{-tX_2}) \).

By assumption \( \psi(0) = 1, \psi(1) = 0, \psi \uparrow \infty \).
Theorem (Liu)

Suppose that \( \psi'(1) = -m < 0 \), then for any \( \delta > 0 \), almost all realization of \( \mu \), \( \mu \)-almost all \( x \) and sufficiently large \( n \)

\[
\mu(B(x, 2^{-n})) \geq 2^{-nm - (1+\delta)\sqrt{2\sigma^2 n \log \log n}}
\]

\[
\mu(B(x, 2^{-n})) \leq 2^{-nm + (1+\delta)\sqrt{2\sigma^2 n \log \log n}}
\]

where \( \sigma^2 = \psi''(1) - \psi(1) \). Moreover

\[
\mu(B(x, 2^{-n})) \leq 2^{-nm - (1-\delta)\sqrt{2\sigma^2 n \log \log n}} \quad i.o.
\]

\[
\mu(B(x, 2^{-n})) \geq 2^{-nm + (1-\delta)\sqrt{2\sigma^2 n \log \log n}} \quad i.o.
\]
Subcritical case $\psi'(1) = -m < 0$

**Theorem (Liu)**

Suppose that $\psi'(1) = -m < 0$, then for any $\delta > 0$, almost all realization of $\mu$, $\mu$-almost all $x$ and sufficiently large $n$

\[
\mu(B(x, 2^{-n})) \geq 2^{-nm-(1+\delta)\sqrt{2\sigma^2 n \log \log n}}
\]
\[
\mu(B(x, 2^{-n})) \leq 2^{-nm+(1+\delta)\sqrt{2\sigma^2 n \log \log n}},
\]

where $\sigma^2 = \psi''(1) - \psi(1)$. Moreover

\[
\mu(B(x, 2^{-n})) \leq 2^{-nm-(1-\delta)\sqrt{2\sigma^2 n \log \log n}} \quad \text{i.o.}
\]
\[
\mu(B(x, 2^{-n})) \geq 2^{-nm+(1-\delta)\sqrt{2\sigma^2 n \log \log n}} \quad \text{i.o.}
\]
Barral, Rhodes, Vargas

When $\psi'(1) > 0$ then $\mu$ is purely atomic

Barral, Kupiainen, Nikula, Saksman, Webb

If $\psi'(1) = 0$, then the random measure $\mu$ almost surely has no atoms. Moreover for any $k$ and $\delta > 0$, $\mu$-a.e. $x$

- $\mu(B(x, 2^{-n})) \geq e^{-\sqrt{6\log 2}\sqrt{n(\log n + (1/3+\delta) \log \log n)}}$ for sufficiently large $n$
- $\mu(B(x, 2^{-n})) \geq e^{-(\sqrt{2\log 2+\delta})\sqrt{n \log n}}$ i.o.
- $\mu(B(x, 2^{-n})) \leq n^{-k}$ for sufficiently large $n$. 
Barral, Rhodes, Vargas

When $\psi'(1) > 0$ then $\mu$ is purely atomic

Barral, Kupiainen, Nikula, Saksman, Webb

If $\psi'(1) = 0$, then the random measure $\mu$ almost surely has no atoms. Moreover for any $k$ and $\delta > 0$, $\mu$-a.e. $x$

- $\mu(B(x, 2^{-n})) \geq e^{-\sqrt{6 \log 2} \sqrt{n (\log n + (1/3+\delta) \log \log n)}}$ for sufficiently large $n$
- $\mu(B(x, 2^{-n})) \geq e^{-\sqrt{2 \log 2 + \delta} \sqrt{n \log n}}$ i.o.
- $\mu(B(x, 2^{-n})) \leq n^{-k}$ for sufficiently large $n$. 
When $\psi'(1) > 0$ then $\mu$ is purely atomic

If $\psi'(1) = 0$, then the random measure $\mu$ almost surely has no atoms. Moreover for any $k$ and $\delta > 0$, $\mu$-a.e. $x$

- $\mu(B(x, 2^{-n})) \geq e^{-\sqrt{6 \log 2} \sqrt{n (\log n + (1/3 + \delta) \log \log n)}}$ for sufficiently large $n$
- $\mu(B(x, 2^{-n})) \geq e^{- (\sqrt{2 \log 2} + \delta) \sqrt{n \log n}}$ i.o.
- $\mu(B(x, 2^{-n})) \leq n^{-k}$ for sufficiently large $n$. 
Barral, Rhodes, Vargas

When $\psi'(1) > 0$ then $\mu$ is purely atomic

Barral, Kupiainen, Nikula, Saksman, Webb

If $\psi'(1) = 0$, then the random measure $\mu$ almost surely has no atoms. Moreover for any $k$ and $\delta > 0$, $\mu$-a.e. $x$

- $\mu(B(x, 2^{-n})) \geq e^{-\sqrt{6\log 2}\sqrt{n(\log n+(1/3+\delta)\log\log n)}}$ for sufficiently large $n$
- $\mu(B(x, 2^{-n})) \geq e^{-(\sqrt{2\log 2}\delta)\sqrt{n}\log n}$ i.o.
- $\mu(B(x, 2^{-n})) \leq n^{-k}$ for sufficiently large $n$. 
**Theorem (D. Buraczewski, P. Dyszewski, K.K.)**

Let \( \psi'(1) = 0, \ k \in \mathbb{N} \) and \( \delta > 0 \). Then for almost all realizations of \( \mu \), \( \mu \)-almost all \( x \in [0, 1) \) and sufficiently large \( n \) we have

\[
\mu(B(x, 2^{-n})) \geq \exp \left( -(1 + \delta) \sqrt{2\sigma^2 n \log \log n} \right)
\]

\[
\mu(B(x, 2^{-n})) \leq \exp \left( \frac{-\sqrt{n}}{\prod_{i=1}^{k} \log(i) n \left( \log(k+1) n \right)^2} \right)
\]
Theorem (D. Buraczewski, P. Dyszewski, K.K.)

Let \( \psi'(1) = 0 \), \( k \in \mathbb{N} \) and \( \delta > 0 \). Then for almost all realizations of \( \mu \), \( \mu \)-almost all \( x \in [0, 1) \) and sufficiently large \( n \) we have

\[
\mu(B(x, 2^{-n})) \geq \exp \left( - (1 + \delta) \sqrt{2\sigma^2 n \log \log n} \right)
\]

\[
\mu(B(x, 2^{-n})) \leq \exp \left( - \frac{\sqrt{n}}{\prod_{i=1}^{k} \log(i)n (\log(k+1)n)^2} \right)
\]
Dyadic intervals on $[0,1)$ ↔ vertices of a binary tree $T$

$x \in [0,1)$ ↔ $\theta \in \partial T$.

$B(v) = \{ \theta \in \partial T : v \in o\theta \}.$

For a random measure $\mu_\omega$ we are interested in a pointwise estimates of $\mu_\omega(B(\theta_n))$ on the enlarged measure space

$$\tilde{P}(d\omega, d\theta) := P(d\omega)\mu_\omega(d\theta)$$

i.e. we are looking for deterministic $\phi_1, \phi_2$ s.t.

$$\phi_1(n) \leq \mu_\omega(B(\theta_n)) \leq \phi_2(n),$$

for $\tilde{P}$-almost all $(\omega, \theta)$ and large $n$.

$\tilde{P}$ can be replaced by $\hat{P}(d\omega, d\theta) = P(d\omega)\bar{\mu}_\omega(d\theta)$, where $\bar{\mu}_\omega$ is the normalized measure $\mu_\omega$. 
Dyadic intervals on $[0,1)$ ↔ vertices of a binary tree $T$

$x \in [0,1)$ ↔ $\theta \in \partial T$.

$B(v) = \{ \theta \in \partial T : v \in \overline{o\theta} \}$.

For a random measure $\mu_\omega$ we are interested in a pointwise estimates of $\mu_\omega(B(\theta_n))$ on the enlarged measure space

$$\widetilde{\mathbb{P}}(d\omega, d\theta) := \mathbb{P}(d\omega)\mu_\omega(d\theta)$$

i.e. we are looking for deterministic $\phi_1, \phi_2$ s.t.

$$\phi_1(n) \leq \mu_\omega(B(\theta_n)) \leq \phi_2(n),$$

for $\widetilde{\mathbb{P}}$-almost all $(\omega, \theta)$ and large $n$.

$\widetilde{\mathbb{P}}$ can be replaced by $\widehat{\mathbb{P}}(d\omega, d\theta) = \mathbb{P}(d\omega)\overline{\mu}_\omega(d\theta)$, where $\overline{\mu}_\omega$ is the normalized measure $\mu_\omega$. 
Notations

Dyadic intervals on $[0,1)$ ⟷ vertices of a binary tree $T$
$x \in [0,1)$ ⟷ $\theta \in \partial T$.

$B(v) = \{\theta \in \partial T : v \in o\theta\}$.

For a random measure $\mu_\omega$ we are interested in a pointwise estimates of $\mu_\omega(B(\theta_n))$ on the enlarged measure space

$$\tilde{\mathbb{P}}(d\omega, d\theta) := \mathbb{P}(d\omega)\mu_\omega(d\theta)$$

i.e. we are looking for deterministic $\phi_1, \phi_2$ s.t.

$$\phi_1(n) \leq \mu_\omega(B(\theta_n)) \leq \phi_2(n),$$

for $\tilde{\mathbb{P}}$-almost all $(\omega, \theta)$ and large $n$.

$\tilde{\mathbb{P}}$ can be replaced by $\hat{\mathbb{P}}(d\omega, d\theta) = \mathbb{P}(d\omega)\overline{\mu}_\omega(d\theta)$, where $\overline{\mu}_\omega$ is the normalized measure $\mu_\omega$. 
\( \mathbb{P} \) – probability measure on the set of labeled trees

\( X(v) \) – the sum of random variables along the path between \( o \) and \( v \) \( (X(u) = X_2 + X_{21} + X_{212}) \)

\( \{X(v)\}_{v \in T} \) – Branching random walk (BRW)
Branching random walk

\[ P \] – probability measure on the set of labeled trees
\[ X(v) \] – the sum of random variables along the path between \( o \) and \( v \) \( (X(u) = X_2 + X_{21} + X_{212}) \)
\{X(v)\}_{v \in T} – Branching random walk (BRW)
Branching random walk

$\mathbb{P}$ – probability measure on the set of labeled trees

$X(v)$ – the sum of random variables along the path between $o$ and $v$ 

$(X(u) = X_2 + X_{21} + X_{212})$

$\{X(v)\}_{v \in T}$ – Branching random walk (BRW)
\( \mathbb{P} \) – probability measure on the set of labeled trees
\( X(v) \) – the sum of random variables along the path between \( o \) and \( v \) \( (X(u) = X_2 + X_{21} + X_{212}) \)
\( \{X(v)\}_{v \in T} \) – Branching random walk (BRW)
$\mathbb{P}$ – probability measure on the set of labeled trees

$X(v)$ – the sum of random variables along the path between $o$ and $v$ 

$(X(u) = X_2 + X_{21} + X_{212})$

$\{X(v)\}_{v \in T}$ – Branching random walk (BRW)
$\mathbb{P}$ – probability measure on the set of labeled trees

$X(v)$ – the sum of random variables along the path between $o$ and $v$ ($X(u) = X_2 + X_{21} + X_{212}$)

$\{X(v)\}_{v \in T}$ – Branching random walk (BRW)
Since $\mathbb{E} \sum |v|=1 e^{-X(v)} = 1$ and $\mathbb{E} \sum |v|=1 X(v)e^{-X(v)} = 0$ the equation

$$\mathbb{E} f(Y) := \mathbb{E} \sum_{|v|=1} f(X(v))e^{-X(v)}$$

defines distribution of a driftless r.v. $Y$

Let $h$ be a harmonic function on some set $A$ (a solution of a Dirichlet problem), $V_n = Y_1 + \cdots + Y_n$ and $\sigma = \min\{k : s + V_k \notin A\}$. Then the process $h(s + V_{\min(n,\sigma)})$ is a martingale.

Let $\tau = \{w : s + X(w) \notin A \text{ for the first time}\}$, $v_\tau = \min(v, \tau)$. Then

$$W_n^s = \sum_{|v|=n} h(s + X(v_\tau))e^{-X(v_\tau)}$$

is a martingale.
Since $\mathbb{E} \sum_{|v|=1} e^{-X(v)} = 1$ and $\mathbb{E} \sum_{|v|=1} X(v) e^{-X(v)} = 0$ the equation
\[
\mathbb{E} f(Y) := \mathbb{E} \sum_{|v|=1} f(X(v)) e^{-X(v)}
\]
defines distribution of a driftless r.v. $Y$

Let $h$ be a harmonic function on some set $A$ (a solution of a Dirichlet problem), $V_n = Y_1 + \cdots + Y_n$ and $\sigma = \min\{k : s + V_k \notin A\}$. Then the process $h(s + V_{\min(n,\sigma)})$ is a martingale.

Let $\tau = \{w : s + X(w) \notin A \text{ for the first time}\}$, $v_\tau = \min(v, \tau)$. Then
\[
W^s_n = \sum_{|v|=n} h(s + X(v_\tau)) e^{-X(v_\tau)}
\]
is a martingale.
Since $\mathbb{E} \sum_{|v|=1} e^{-X(v)} = 1$ and $\mathbb{E} \sum_{|v|=1} X(v)e^{-X(v)} = 0$ the equation

$$\mathbb{E} f(Y) := \mathbb{E} \sum_{|v|=1} f(X(v))e^{-X(v)}$$

defines distribution of a driftless r.v. $Y$

Let $h$ be a harmonic function on some set $A$ (a solution of a Dirichlet problem), $V_n = Y_1 + \cdots + Y_n$ and $\sigma = \min\{k : s + V_k \notin A\}$. Then the process $h(s + V_{\min(n,\sigma)})$ is a martingale.

Let $\tau = \{w : s + X(w) \notin A \text{ for the first time}\}$, $v_\tau = \min(v, \tau)$. Then

$$W_n^s = \sum_{|v|=n} h(s + X(v_\tau))e^{-X(v_\tau)}$$

is a martingale.
Some natural martingales

- $h \equiv 1$: $W_n = \sum_{|v|=n} e^{-X(v)}$
- $h(x) = x$: $D_n = \sum_{|v|=n} X(v) e^{-X(v)}$
- $A = [0, \infty)$, $h(x) \approx x \lor 0$: $W_n^s = \sum_{|v|=n} h(s + X(v)) 1_{[s + X(v') > 0, \text{for } v' \leq v]} e^{-X(v)}$

$$\mu(B(v)) := \lim_n \sum_{|w|=n, w<v} X(w) e^{-X(w)}$$
Some natural martingales

- $h \equiv 1$: $W_n = \sum_{|v|=n} e^{-X(v)} \to 0$
- $h(x) = x$: $D_n = \sum_{|v|=n} X(v) e^{-X(v)} \to D > 0$
- $A = [0, \infty), h(x) \approx x \vee 0$: $W_n^s = \sum_{|v|=n} h(s + X(v)) 1[s+X(v')>0, \text{for } v'\leq v] e^{-X(v)}$

$$
\mu(B(v)) := \lim_{n} \sum_{|w|=n, w<v} X(w) e^{-X(w)}
$$
Some natural martingales

- $h \equiv 1$: $W_n = \sum_{|v|=n} e^{-X(v)} \to 0$
- $h(x) = x$: $D_n = \sum_{|v|=n} X(v) e^{-X(v)} \to D > 0$
- $A = [0, \infty)$, $h(x) \approx x \vee 0$: $W^s_n = \sum_{|v|=n} h(s + X(v)) 1_{s+X(v')>0, \text{ for } v' \leq v} e^{-X(v)}$

$$\mu(B(v)) := \lim_{n} \sum_{|w|=n, w<v} X(w) e^{-X(w)}$$
Some natural martingales

- $h \equiv 1$: $W_n = \sum_{|v|=n} e^{-X(v)} \to 0$
- $h(x) = x$: $D_n = \sum_{|v|=n} X(v)e^{-X(v)} \to D > 0$
- $A = [0, \infty)$, $h(x) \approx x \lor 0$:
  $W^s_n = \sum_{|v|=n} h(s + X(v))1_{[s+X(v') > 0, \text{for } v' \leq v]}e^{-X(v)}$

$$
\mu(B(v)) := \lim_n \sum_{|w|=n, w < v} X(w)e^{-X(w)}
$$
\[ W_n^s = \sum_{|v|=n} h(s + X(v))1_{[s+X(v')>0, \text{ for } v'\leq v]}e^{-X(v)} \]

\[ W_n^s \to W^s \quad \mathbb{P}\text{-a.s. and } L^1 \]

For any \( s \in D \) define

\[ \mathbb{P}^s := \frac{W^s}{h(s)} \cdot \mathbb{P} \]

\[ \mathbb{P}[^{\text{supp}} W^s] \gtrsim 1 - 1/s \]
Spinal decomposition of $\mathbb{P}^s$
Spinal decomposition of $\mathbb{P}^s_{BRW}$
Spinal decomposition of $P^s$

$X_1^1$, $X_2^1$
Spinal decomposition of $P^s$

$$e^{-x^1_1} h(s + X^1_1) \quad e^{-x^1_2} h(s + X^1_2)$$
Spinal decomposition of $P^s$

$S_0 = s$

$S_1 = s + X_2^1$

$\infty$

$0$
Spinal decomposition of $P_s$
Spinal decomposition of $\mathbb{P}^s$

\[ e^{-X_1^2} h(S_1 + X_1^2) \quad e^{-X_2^2} h(S_1 + X_2^2) \]
$S_2 = S_1 + X_2^1$
Spinal decomposition of $\mathbb{P}_s$
Spinal decomposition of $P_s$:

\begin{align*}
X_1^3 & e^{-X_1^3} h(S_2 + X_1^3) \\
X_2^3 & e^{-X_2^3} h(S_2 + X_2^3)
\end{align*}
Spinal decomposition of $P^s$ 

$S_0 = S_1 + X_3^1$

$S_2 = S_3 + X_2$

$S_3 = S_2 + X_1$

$S_3 = S_2 + X_1$

Konrad Kolesko

Local fluctuations of critical Mandelbrot cascades

18-22 May, 2015
Spinal decomposition of $P_s$
Spinal decomposition of $\mathcal{P}^S$

Konrad Kolesko

Local fluctuations of critical Mandelbrot cascades

18-22 May, 2015

13 / 19
Spinal decomposition of $\mathbb{P}^s_{BRW}$

Konrad Kolesko

Local fluctuations of critical Mandelbrot cascades

18-22 May, 2015

13 / 19
Spinal decomposition of $\mathcal{P}^s$
Spinal decomposition of $\mathbb{P}^s$
Spinal decomposition of $P_s$
Spinal decomposition of $\mathcal{P}_s$

Konrad Kolesko

Local fluctuations of critical Mandelbrot cascades

18-22 May, 2015
Spinal decomposition of $\mathbb{P}^s$
Spinal decomposition of $\mathcal{P}_s$
Spinal decomposition of $P^s$

Random tree $T^s$ with a distinguished ray $\Theta \in \partial T^s$
\( \mathbb{P}(T^s \in d\omega) = \mathbb{P}^s(d\omega) \)
\[ \hat{P}_s(d\omega, d\theta) := P(T^s \in d\omega, \Theta \in d\theta) \]
We have
\[ \hat{P}^s \ll \hat{P}. \]

The converse is not true, but for any set \( A \)
\[ \hat{P}^s(A) = 1 \Rightarrow \hat{P}(A) \geq 1 - \epsilon(s), \]
for some \( \epsilon(s) \to 0 \) as \( s \to \infty \). In particular, if for any \( s > 0 \)
\[ \hat{P}^s(A) = 1 \]
them
\[ \hat{P}(A) = 1. \]
Probabilistic interpretation of local fluctuations

$e^{-(S_0-s)}$

$e^{-(S_1-s)}$

$e^{-(S_2-s)}$

$e^{-(S_3-s)}$

$e^{-(S_4-s)}$

$e^{-(S_5-s)}$

Konrad Kolesko

Local fluctuations of critical Mandelbrot cascades

18-22 May, 2015 15 / 19
Under $\hat{P}^s$ the sequence $\mu_\omega(B(\theta_n))$ has the same law as

$$\sum_{k \geq n} e^{-(S_k - s)} C_k R_k,$$

where $S_k$ is a random walk starting from $s$ conditioned to stay positive, $R_k$ independent random variables.
We are looking for LIL for $\sum_{k \geq n} e^{-S_k + s} C_k R_k$.

**Hambly, Kersting, Kyprianou**

- $\int_{\infty}^{\infty} \frac{\psi(t)}{t} dt < \infty$ iff $S_n > \sqrt{n} \psi(n)$ eventually
- $\limsup_n \frac{S_n}{\sqrt{2n\sigma^2 \log \log n}} = 1$
- $\sqrt{n} \psi(n) < S_n < (1 + \delta) \sqrt{2n\sigma^2 \log \log n}$

$\mathbb{P}^s$-a.s. for $n$ sufficiently large

**Kyprianou**

- $1 - \mathbb{E}e^{-tR} \sim tL(t)$ near 0 $\Rightarrow$ $\mathbb{E}R^\gamma < \infty$ for $\gamma < 1$

In particular, by Borel-Cantelli lemma, $R_n < n^2$ for all but finitely many $n$. 
We are looking for LIL for $\sum_{k\geq n} e^{-S_k R_k}$.

**Hambly, Kersting, Kyprianouou**

1. $\int_{\infty}^{\infty} \frac{\psi(t)}{t} dt < \infty$ iff $S_n > \sqrt{n\psi(n)}$ eventually
2. $\limsup_n \frac{S_n}{\sqrt{2n\sigma^2 \log \log n}} = 1$

$$\sqrt{n\psi(n)} < S_n < (1 + \delta) \sqrt{2n\sigma^2 \log \log n}$$

$\mathbb{P}^s$-a.s. for $n$ sufficiently large

**Kyprianouou**

1. $1 - \mathbb{E}e^{-tR} \sim tL(t)$ near 0 $\Rightarrow \mathbb{E}R^\gamma < \infty$ for $\gamma < 1$

In particular, by Borel-Cantelli lemma, $R_n < n^2$ for all but finitely many $n$. 
We are looking for LIL for $\sum_{k \geq n} e^{-S_k} R_k$,

**Hambly, Kersting, Kyprianou**

- $\int_{\infty}^{\infty} \frac{\psi(t)}{t} dt < \infty$ iff $S_n > \sqrt{n} \psi(n)$ eventually
- $\limsup_n \frac{S_n}{\sqrt{2n\sigma^2 \log \log n}} = 1$

$$\sqrt{n} \psi(n) < S_n < (1 + \delta) \sqrt{2n\sigma^2 \log \log n}$$

$\mathbb{P}^s$-a.s. for $n$ sufficiently large

**Kyprianou**

- $1 - \mathbb{E}e^{-tR} \sim tL(t)$ near $0$ $\Rightarrow$ $\mathbb{E}R^\gamma < \infty$ for $\gamma < 1$

In particular, by Borel-Cantelli lemma, $R_n < n^2$ for all but finitely many $n$. 
We are looking for LIL for $\sum_{k \geq n} e^{-S_k} R_k$.

Hambly, Kersting, Kyprianou

- $\int_0^\infty \frac{\psi(t)}{t} dt < \infty$ iff $S_n > \sqrt{n\psi(n)}$ eventually
- $\limsup_n \frac{S_n}{\sqrt{2n\sigma^2 \log \log n}} = 1$

$$\sqrt{n\psi(n)} < S_n < (1 + \delta) \sqrt{2n\sigma^2 \log \log n}$$

$\mathbb{P}^s$-a.s. for $n$ sufficiently large

Kyprianou

- $1 - \mathbb{E}e^{-tR} \sim tL(t)$ near 0 $\implies \mathbb{E}R^\gamma < \infty$ for $\gamma < 1$

In particular, by Borel-Cantelli lemma, $R_n < n^2$ for all but finitely many $n$. 
Take any $\psi$ such that $\int_\infty^{\infty} \frac{\psi(t)dt}{t} < \infty$. We have that

$$\sum_{k \geq n} e^{-S_k} R_k$$

is eventually bounded by

$$\sum_{k \geq n} e^{-\sqrt{k} \psi(k)} k^2.$$

On the other hand, if $\int_\infty^{\infty} \frac{\psi(t)dt}{t} = \infty$ then

$$\sum_{k \geq n} e^{-S_k} R_k \geq e^{-S_n} R_n \overset{i.o.}{\geq} e^{-\sqrt{n} \psi(n)} R_n \overset{i.o.}{\geq} \delta e^{-\sqrt{n} \psi(n)}$$
Take any \( \psi \) such that \( \int_0^\infty \frac{\psi(t) dt}{t} < \infty \). We have that

\[
\sum_{k \geq n} e^{-S_k} R_k
\]

is eventually bounded by

\[
\sum_{k \geq n} e^{-\sqrt{k} \psi(k)}.
\]

On the other hand, if \( \int_0^\infty \frac{\psi(t) dt}{t} = \infty \) then

\[
\sum_{k \geq n} e^{-S_k} R_k \geq e^{-S_n} R_n \overset{i.o.}{\geq} e^{-\sqrt{n} \psi(n)} R_n \overset{i.o.}{\geq} \delta e^{-\sqrt{n} \psi(n)}
\]
Take any $\psi$ such that $\int_{t=0}^{\infty} \frac{\psi(t)dt}{t} < \infty$. We have that

$$\sum_{k \geq n} e^{-S_k R_k}$$

is eventually bounded by

$$\sum_{k \geq \sqrt{n\psi}(n)} e^{-k} \cdot \text{polynomial}(k).$$

On the other hand, if $\int_{t=0}^{\infty} \frac{\psi(t)dt}{t} = \infty$ then

$$\sum_{k \geq n} e^{-S_k R_k} \geq e^{-S_n R_n} \overset{i.o.}{\geq} e^{-\sqrt{n\psi}(n)} R_n \overset{i.o.}{\geq} \delta e^{-\sqrt{n\psi}(n)}$$
Take any $\psi$ such that $\int_{\infty}^{\infty} \frac{\psi(t)dt}{t} < \infty$. We have that

$$\sum_{k\geq n} e^{-S_k} R_k$$

is eventually bounded by

$$\sum_{k\geq \sqrt{n}\psi(n)} e^{-k}.$$

On the other hand, if $\int_{\infty}^{\infty} \frac{\psi(t)dt}{t} = \infty$ then

$$\sum_{k\geq n} e^{-S_k} R_k \geq e^{-S_n} R_n \overset{i.o.}{\geq} e^{-\sqrt{n}\psi(n)} R_n \overset{i.o.}{\geq} \delta e^{-\sqrt{n}\psi(n)}$$
Take any $\psi$ such that $\int_\infty^\infty \frac{\psi(t)dt}{t} < \infty$. We have that
\[ \sum_{k \geq n} e^{-S_k} R_k \]
is eventually bounded by
\[ e^{-\sqrt{n}\psi(n)}. \]

On the other hand, if $\int_\infty^\infty \frac{\psi(t)dt}{t} = \infty$ then
\[ \sum_{k \geq n} e^{-S_k} R_k \geq e^{-S_n} R_n \geq e^{-\sqrt{n}\psi(n)} R_n \geq \delta e^{-\sqrt{n}\psi(n)} \]
Take any $\psi$ such that $\int_{\infty}^{\infty} \frac{\psi(t)dt}{t} < \infty$. We have that

$$\sum_{k \geq n} e^{-S_k} R_k$$

is eventually bounded by

$$e^{-\sqrt{n}\psi(n)}.$$

On the other hand, if $\int_{\infty}^{\infty} \frac{\psi(t)dt}{t} = \infty$ then

$$\sum_{k \geq n} e^{-S_k} R_k \geq e^{-S_n} R_n \overset{i.o.}{\geq} e^{-\sqrt{n}\psi(n)} R_n \overset{i.o.}{\geq} \delta e^{-\sqrt{n}\psi(n)}$$
Take $q > 1$ and by $N(n) := q^{\lceil \log_q n \rceil}$. Borel-Cantelli’s lemma implies that for sufficiently large $n$, \[ \sup_{q^n < k \leq q^{n+1}} R_k \geq \delta_0. \]

\[
\sum_{k \geq n} e^{-S_k} R_k \geq \sum_{k \geq n} e^{-(1+\delta)\sqrt{2k\sigma^2 \log \log k}} R_k
\]
\[
\geq \sum_{k=N(n)}^{qN(n)} e^{-(1+\delta)\sqrt{2k\sigma^2 \log \log k}} R_k
\]
\[
\geq \delta_0 e^{-(1+\delta)\sqrt{2qN(n)\sigma^2 \log \log(qN(n))}}
\]
\[
\geq \delta_0 e^{-(1+\delta)\sqrt{2q^2n\sigma^2 \log \log(q^2n)}}
\]
\[
\geq e^{-(1+2\delta)\sqrt{2n\sigma^2 \log \log n}},
\]
for some $q$.
Take $q > 1$ and by $N(n) := q^{\lceil \log_q n \rceil}$. Borel-Cantelli’s lemma implies that for sufficiently large $n$, \( \sup_{q^n < k \leq q^{n+1}} R_k \geq \delta_0. \)

\[
\sum_{k \geq n} e^{-S_k} R_k \geq \sum_{k \geq n} e^{-(1+\delta)\sqrt{2k\sigma^2 \log \log k}} R_k \\
\geq \sum_{k = N(n)} e^{-(1+\delta)\sqrt{2k\sigma^2 \log \log k}} R_k \\
\geq \delta_0 e^{-(1+\delta)\sqrt{2qN(n)\sigma^2 \log \log (qN(n))}} \\
\geq \delta_0 e^{-(1+\delta)\sqrt{2q^2 n\sigma^2 \log \log (q^2n)}} \\
\geq e^{-(1+2\delta)\sqrt{2n\sigma^2 \log \log n}},
\]

for some $q$. 
Take \( q > 1 \) and by \( N(n) := q^{\lceil \log_q n \rceil} \). Borel-Cantelli’s lemma implies that for sufficiently large \( n \), \( \sup_{q^n < k \leq q^{n+1}} R_k \geq \delta_0 \).

\[
\sum_{k \geq n} e^{-S_k} R_k \geq \sum_{k \geq n} e^{-(1+\delta) \sqrt{2k\sigma^2 \log \log k}} R_k
\]

\[
\geq \sum_{k = N(n)} e^{-(1+\delta) \sqrt{2k\sigma^2 \log \log k}} R_k
\]

\[
\geq \delta_0 e^{-(1+\delta) \sqrt{2qN(n)\sigma^2 \log \log (qN(n))}}
\]

\[
\geq \delta_0 e^{-(1+\delta) \sqrt{2q^2n\sigma^2 \log \log (q^2n)}}
\]

\[
\geq e^{-(1+2\delta) \sqrt{2n\sigma^2 \log \log n}},
\]

for some \( q \).