Electric network for non-reversible Markov chains

Joint work with Áron Folly

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*Random walks on graphs and potential theory*

University of Warwick, 20th May 2015.
Reversible chains and resistors
  Reducing a network
  Thomson, Dirichlet principles
  Monotonicity, transience, recurrence

Irreversible chains and electric networks
  The part
  From network to chain
  From chain to network
  Effective resistance
  What works

The electric network
  Reducing the network
  Nonmonotonicity
  Dirichlet principle
Reversible chains and resistors

Irreducible Markov chain: on $\Omega$, $a \neq b$, $x \in \Omega$,

$$h_x := \mathbb{P}_x\{\tau_a < \tau_b\} \quad (\tau \text{ is the hitting time})$$

is harmonic:

$$h_x = \sum_y P_{xy} h_y, \quad h_a = 1, \quad h_b = 0.$$
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Electric resistor network: the voltage $u$ is harmonic ($C = 1/R$):

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Stationary distribution:

\[ \mu_x = \sum_y \mu_y P_{yx} = \sum_y \mu_y \frac{C_{xy}}{C_y} \]

\[ C_x = \sum_y C_y \frac{C_{xy}}{C_y} \]

Thus, \( P_{xy} = C_{xy} / C_x \)
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Notice \( \mu_x P_{xy} = C_{xy} = C_{yx} = \mu_y P_{yx} \), so the chain is reversible.

\[ P_{xy} = \frac{C_{xy}}{C_x} \]

\[ C_x = \mu_x \]
Reversible chains and resistors

Let \( n_x = E_a(\text{number of visits to } x \text{ before absorbed in } b) \). Then

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n_x = \sum_y n_y p_{yx} = \sum_y \frac{C_{xy}}{C_y} n_y
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$$u_x = \sum_y \frac{C_{xy}}{C_x} \cdot u_y$$

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\[\leadsto u_x C_x = n_x.\]

\[
E_a \text{(signed current } x \rightarrow y \text{ before absorbed in } b) = n_x P_{xy} - n_y P_{yx} = (u_x - u_y) C_{xy} = i_{xy}. \quad \text{normalisation...}
\]

\[
P_{xy} = \frac{C_{xy}}{C_x}
\]

\[
C_x = \mu_x
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Reducing a network

Series:

\[ R_{\text{eff}} = R + Q \]

Parallel:

\[ \frac{1}{R_{\text{eff}}} = \frac{1}{R} + \frac{1}{Q} \]
Reducing a network

Star-Delta:

\[ R_\star = \frac{Q_\Delta S_\Delta}{R_\Delta + Q_\Delta + S_\Delta}, \]

\[ R_\Delta = \frac{R_\star Q_\star + R_\star S_\star + Q_\star S_\star}{R_\star}. \]
Thomson, Dirichlet principles

Thomson principle:

The physical unit current is the unit flow that minimizes the sum of the ohmic power losses $\sum i^2 R$. 
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The physical unit current is the unit flow that minimizes the sum of the ohmic power losses $\sum i^2 R$.

Dirichlet principle:

The physical voltage is the function that minimizes the ohmic power losses $\sum (\nabla u)^2 / R$. 
Monotonicity, transience, recurrence

The monotonicity property:
Between any disjoint sets of vertices, the effective resistance is a non-decreasing function of the individual resistances.
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Between any disjoint sets of vertices, the effective resistance is a non-decreasing function of the individual resistances.

\( \sim \) can be used to prove transience-recurrence by reducing the graph to something manageable in terms of resistor networks.
The part

Voltage amplifier: keeps the current, multiplies the potential.

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(u_x - i \cdot \frac{R}{2}) \cdot \lambda - i \cdot \frac{R}{2} = u_y
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\[(u_x - i \cdot \frac{R}{2}) \cdot \lambda - i \cdot \frac{R}{2} = u_y\]

Equivalent:

\[(u_x - i \cdot R^{pr}) \cdot \lambda^{pr} = u_y\]

\[u_x \cdot \lambda^{se} - R^{se} \cdot i = u_y\]
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Equivalent:

\[\lambda^{pr} = \lambda\]

\[R^{pr} = \frac{\lambda + 1}{2\lambda} \cdot R\]
The part

![Diagram of a voltage amplifier]

**Voltage amplifier**: keeps the current, multiplies the potential.

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Harmonicity

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\[ u_x = \sum_y \frac{C_{xy}^{\text{se}}}{\sum_z C_{xz}^{\text{se}}} \cdot \lambda_{xy} u_y \]

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\[ u_x = \sum_y \frac{C_{xy}^{se}}{\sum_z C_{xz}^{se}} \cdot \lambda_{xy} u_y = \sum_y \frac{C_{xy}}{\lambda_{xy} + 1} \cdot \lambda_{xy} u_y \]

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with \( \gamma_{xy} = \sqrt{\lambda_{xy}} = \frac{1}{\gamma_{yx}} \), \( D_{xy} = \frac{2\gamma_{xy} C_{xy}}{\lambda_{xy} + 1} = D_{yx} \).

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From network to chain

Irreducible Markov chain: on $\Omega$, $a \neq b$, $x \in \Omega$,

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\gamma_{xy} &= \sqrt{\lambda_{xy}} \\
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"Markovian" property

\[ u_x = \sum_z P_{xz} u_z; \quad \sum_z P_{xz} = 1 \]

\( u_x \equiv \text{const.} \) is a solution of the network with no external sources. This is now nontrivial.

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From chain to network

\[ P_{xy} = \frac{D_{xy} \gamma_{xy}}{D_x} = \frac{D_{xy} \gamma_{xy}}{\mu_x} \]

\[ \mu_x P_{xy} \cdot \mu_y P_{yx} = D_{xy}^2; \]

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\[ \mu_x P_{xy} \cdot \mu_y P_{yx} = D^2_{xy}; \]

\[ \frac{\mu_x P_{xy}}{\mu_y P_{yx}} = \gamma^2_{xy} = \lambda_{xy}. \]

Reversed chain: Replace \( P_{xy} \) by \( \hat{P}_{xy} = P_{yx} \cdot \frac{\mu_y}{\mu_x}. \)

\[ \Rightarrow D_{xy} \text{ stays, } \lambda_{xy} \text{ reverses to } \lambda_{yx}. \]

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u_x D_x = \sum_y \frac{D_{xy} \gamma_{xy}}{D_y} \cdot u_y D_y
\]

\[\leadsto \hat{u}_x D_x = n_x\]

in the reversed chain.

\[\gamma_{xy} = \sqrt{\lambda_{xy}}\]

\[D_x = \sum_z D_{xz} \gamma_{zx} = \sum_z D_{xz} \gamma_{xz}\]

\[D_{xy} = 2 \gamma_{xy} C_{xy} / (\lambda_{xy} + 1)\]

\[P_{xy} = D_{xy} \gamma_{xy} / D_x\]
From chain to network

Let $n_x = E_a($number of visits to $x$ before absorbed in $b$). Then

$$\sim \hat{u}_x D_x = n_x$$

in the reversed chain.

$E_a($signed current $x \rightarrow y$ before absorbed in $b)$

$$= n_x P_{xy} - n_y P_{yx} = (\hat{u}_x \gamma_{xy} - \hat{u}_y \gamma_{yx}) D_{xy} = \hat{i}_{xy}. \text{ normalisation...}$$

\[ \gamma_{xy} = \sqrt{\lambda_{xy}} \quad D_x = \sum_z D_{xz} \gamma_{zx} = \sum_z D_{xz} \gamma_{xz} \quad D_{xy} = 2 \gamma_{xy} C_{xy} / (\lambda_{xy} + 1) \]

\[ P_{xy} = D_{xy} \gamma_{xy} / D_x \]
Effective resistance

Suppose $u_a, u_b$ given, the solution is $\{u_x\}_{x \in \Omega}$ and $\{i_{xy}\}_{x \sim y \in \Omega}$. Current

$$i_a = \sum_{x \sim a} i_{ax}$$

flows in the network at $a$. 
Effective resistance

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The "Markovian" property has another solution: constant $u_b$ potentials with zero external currents.
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The difference of these two: $\{u_x - u_b\}_{x \in \Omega}$ is a solution too, with $i_a$ flowing in the network.
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The difference of these two: $\{u_x - u_b\}_{x \in \Omega}$ is a solution too, with $i_a$ flowing in the network.

Going backwards from $u_b - u_b = 0$ at $b$, all currents and potentials are proportional to $u_a - u_b$ at $a$. 
Effective resistance

Suppose \( u_a, u_b \) given, the solution is \( \{ u_x \}_{x \in \Omega} \) and \( \{ i_{xy} \}_{x \sim y \in \Omega} \). Current

\[
i_a = \sum_{x \sim a} i_{ax}
\]

flows in the network at \( a \).

\( \leadsto \) The "Markovian" property has another solution: constant \( u_b \) potentials with zero external currents.

\( \leadsto \) The difference of these two: \( \{ u_x - u_b \}_{x \in \Omega} \) is a solution too, with \( i_a \) flowing in the network.

\( \leadsto \) Going backwards from \( u_b - u_b = 0 \) at \( b \), all currents and potentials are proportional to \( u_a - u_b \) at \( a \).

\( \leadsto \) In particular, \( i_a \) is proportional to \( u_a - u_b \). We have effective resistance.
What works

... the analogy with $P\{\tau_a < \tau_b\}$. 
What works

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Modulo normalisation... 

$E_a($signed current $x \rightarrow y$ before absorbed in $b) = \hat{i}_{xy}$. 

in the reversed network!
What works

... the analogy with $P\{\tau_a < \tau_b\}$.

Modulo normalisation...

$E_a(\text{signed current } x \rightarrow y \text{ before absorbed in } b) = \hat{i}_{xy}$.

in the reversed network!

Theorem (Chandra, Raghavan, Ruzzo, Smolensky and Tiwari ’96 for reversible)

*Commut time* = $R_{\text{eff}} \cdot \text{all conductances.}$
What works

For all sets $A, B$, capacity $\sim$ escape probability.

$$\text{cap}(A, B) = C_{AB}^{\text{eff}} = \frac{1}{R_{AB}^{\text{eff}}}$$
What works

For all sets $A$, $B$, capacity $\sim$ escape probability.

$$\text{cap}(A, B) = C_{AB}^{\text{eff}} = \frac{1}{R_{AB}^{\text{eff}}} = \frac{1}{2} \sum_{x \sim y \in V} C_{xy}(u_x - u_y)^2.$$ 

This is non-physical!
What works

For all sets $A$, $B$, capacity $\sim$ escape probability.

$$\text{cap}(A, B) = C_{AB}^{\text{eff}} = \frac{1}{R_{AB}^{\text{eff}}} = \frac{1}{2} \sum_{x \sim y \in V} C_{xy}(u_x - u_y)^2.$$

This is non-physical!

In particular, symmetrising the chain $(P_{xy} \to \frac{P_{xy} + \hat{P}_{xy}}{2})$ cannot increase escape probabilities:

- symmetrising leaves $C_{xy}$ unchanged;
- the above sum is minimised by the symmetric voltages, not $\{u_x\}$ (Classical Dirichlet principle).
The electric network

Series:

\[
S = R \frac{(\lambda + 1)\mu}{\lambda \mu + 1} + Q \frac{\mu + 1}{\lambda \mu + 1}.
\]
The electric network

Parallel:

\[ \begin{align*}
R/2 & \quad \ast \lambda \quad R/2 \\
Q/2 & \quad \ast \mu \quad Q/2
\end{align*} \]

\[ \begin{align*}
R^\text{se} & \quad \ast \lambda \quad R^\text{se} \\
Q^\text{se} & \quad \ast \mu \quad Q^\text{se}
\end{align*} \]

Compare this with

\[ S = \frac{RQ}{R + Q} \]

\[ \nu = \frac{Q \lambda (\mu + 1) + R \mu (\lambda + 1)}{Q (\mu + 1) + R (\lambda + 1)} \]
The electric network

Star-Delta:

Star to Delta works,

Delta to Star only works if Delta does not produce a circular current by itself ($\lambda \mu \nu = 1$).
Nonmonotonicity

\[ R_{\text{eff}} = \frac{27}{14} + \frac{1296}{1225R + 2268}. \]
Nonmonotonicity

\[ R_{\text{eff}} = \frac{27}{14} + \frac{1296}{1225R + 2268}. \]
Dirichlet principle
Classical case:
Dirichlet principle

Classical case:

\[(i_u)_{xy} = C_{xy} \cdot (u(x) - u(y)),\]
\[E_{\text{Ohm}}(i_u) = \sum_{x \sim y} (i_u)_{xy}^2 \cdot R_{xy}.\]
Dirichlet principle

Classical case:

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Classical case:

$$R_{ab}^{\text{eff}} = \min_{u: u(a) = 1, u(b) = 0} E_{\text{Ohm}}(i_u),$$

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Irreversible case (A. Gaudillièere, C. Landim / M. Slowik):

\[ (i^*_u)_{xy} = D_{xy} \cdot (\gamma_{xy} u(x) - \gamma_{yx} u(y)), \]

\[ E_{\text{Ohm}}(i^*_u - \psi) = \sum_{x \sim y} (i^*_u - \psi_{xy})^2 \cdot R_{xy}. \]
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Classical case:

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R_{ab}^{\text{eff}} = \min_{u: u(a) = 1, u(b) = 0} E_{\text{Ohm}}(i_u),
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\]

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\]

\[
E_{\text{Ohm}}(i_u^* - \psi) = \sum_{x \sim y} (i_u^* - \psi_{xy})^2 \cdot R_{xy}.
\]
Dirichlet principle

Classical case:

\[ R_{\text{eff}}^{ab} = \min_{u: u(a) = 1, u(b) = 0} E_{\text{Ohm}}(i_u), \]

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Irreversible case (A. Gaudillièrè, C. Landim / M. Slowik):

\[ R_{\text{eff}}^{ab} = \min_{\Psi: \text{flow}} E_{\text{Ohm}}(i_u^* - \Psi), \]

\[ (i_u^*)_{xy} = D_{xy} \cdot (\gamma_{xy} u(x) - \gamma_{yx} u(y)), \]

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Dirichlet principle

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E_{\text{Ohm}}(i_u^* - \psi) = \sum_{x \sim y} (i_u^* - \psi_{xy})^2 \cdot R_{xy}.
\]

Thank you.
A trivial statement

Theorem (Well Known Theorem)

A Markov chain is reversible if and only if for every closed cycle $x_0, x_1, x_2, \ldots, x_n = x_0$ in $\Omega$ we have

$$P_{x_0x_1} \cdot P_{x_1x_2} \cdots P_{x_{n-1}x_0} = P_{x_0x_{n-1}} \cdot P_{x_{n-1}x_{n-2}} \cdots P_{x_1x_0}.$$

In particular, any Markov chain on a finite connected tree $G$ is necessarily reversible.
A trivial statement

Electrical proof.

Plug in

\[ P_{xy} = \frac{D_{xy} \gamma_{xy}}{D_x}, \quad D_{xy} \text{ symmetric:} \]

\[ P_{x_0 x_1} \cdot P_{x_1 x_2} \cdots P_{x_{n-1} x_0} = P_{x_0 x_{n-1}} \cdot P_{x_{n-1} x_{n-2}} \cdots P_{x_1 x_0} \]

\[ \gamma_{x_0 x_1} \cdot \gamma_{x_1 x_2} \cdots \gamma_{x_{n-1} x_0} = \gamma_{x_0 x_{n-1}} \cdot \gamma_{x_{n-1} x_{n-2}} \cdots \gamma_{x_1 x_0}, \text{ or} \]

\[ \lambda_{x_0 x_1} \cdot \lambda_{x_1 x_2} \cdots \lambda_{x_{n-1} x_0} = 1. \]

- Total multiplication factor along any loop is one.
A trivial statement

Electrical proof.
Plug in

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\[ \gamma_{x_0x_1} \cdot \gamma_{x_1x_2} \cdots \gamma_{x_{n-1}x_0} = \gamma_{x_0x_{n-1}} \cdot \gamma_{x_{n-1}x_{n-2}} \cdots \gamma_{x_1x_0} \]
\[ \lambda_{x_0x_1} \cdot \lambda_{x_1x_2} \cdots \lambda_{x_{n-1}x_0} = 1. \]

- Total multiplication factor along any loop is one.
- Zero current and free vertices is a solution.
A trivial statement

Electrical proof.

Plug in

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\]

\[
\gamma_{x_0x_1} \cdot \gamma_{x_1x_2} \cdots \gamma_{x_{n-1}x_0} = \gamma_{x_0x_{n-1}} \cdot \gamma_{x_{n-1}x_{n-2}} \cdots \gamma_{x_1x_0}, \quad \text{or}
\]

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- Total multiplication factor along any loop is one.
- Zero current and free vertices is a solution.
- It’s the only solution.
A trivial statement

Electrical proof.

Plug in

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\[ P_{x_0 x_1} \cdot P_{x_1 x_2} \cdots P_{x_{n-1} x_0} = P_{x_0 x_{n-1}} \cdot P_{x_{n-1} x_{n-2}} \cdots P_{x_1 x_0} \]

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- Total multiplication factor along any loop is one.
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- It’s the only solution.
- The network is ”Markovian”: potential is constant.
A trivial statement

Electrical proof. Plug in

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P_{x_0x_1} \cdot P_{x_1x_2} \cdots P_{x_{n-1}x_0} &= P_{x_0x_{n-1}} \cdot P_{x_{n-1}x_{n-2}} \cdots P_{x_1x_0} \\
\gamma_{x_0x_1} \cdot \gamma_{x_1x_2} \cdots \gamma_{x_{n-1}x_0} &= \gamma_{x_0x_{n-1}} \cdot \gamma_{x_{n-1}x_{n-2}} \cdots \gamma_{x_1x_0} \\ 
\lambda_{x_0x_1} \cdot \lambda_{x_1x_2} \cdots \lambda_{x_{n-1}x_0} &= 1.
\end{align*}
\]

- Total multiplication factor along any loop is one.
- Zero current and free vertices is a solution.
- It’s the only solution.
- The network is ”Markovian”: potential is constant.
- All \( \lambda \)'s are 1, and the chain is reversible.
A trivial statement

**Electrical proof.**

Repeat for trees:

- There are no loops.
A trivial statement

Electrical proof.
Repeat for trees:

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A trivial statement

Electrical proof.
Repeat for trees:
  ▶ There are no loops.
  ▶ Zero current and free vertices is a solution.
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Second thank you.