Electric network for non-reversible Markov chains

Joint work with Áron Folly

Márton Balázs

University of Bristol

Random walks on graphs and potential theory
University of Warwick, 20th May 2015.

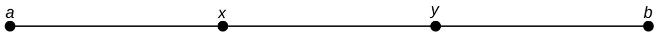
Reversible chains and resistors

Irreducible Markov chain: on Ω, $a \neq b$, $x \in \Omega$,

$$h_x := \mathbf{P}_x\{\tau_a < \tau_b\} \qquad (\tau \text{ is the hitting time})$$

is harmonic:

$$h_x = \sum_y P_{xy} h_y, \qquad h_a = 1, \quad h_b = 0.$$



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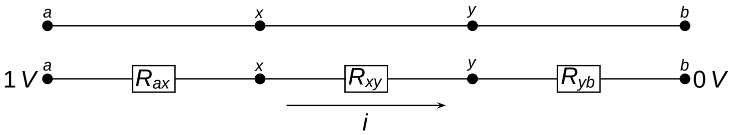

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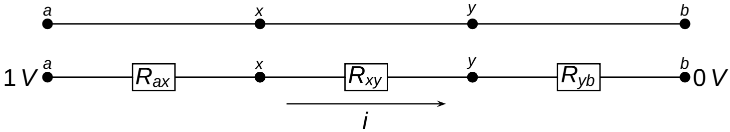

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Notice $\mu_x P_{xy} = C_{xy} = C_{yx} = \mu_y P_{yx}$, so the chain is reversible.

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Reversible chains and resistors

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$\mathbf{E}_a$(signed current $x \to y$ before absorbed in b)

$$= n_x P_{xy} - n_y P_{yx} = (u_x - u_y) C_{xy} = i_{xy} .$$ normalisation...

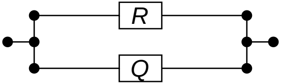
$P_{xy} = C_{xy} / C_x$ $\qquad$ $C_x = \mu_x$

Reducing a network

Series:

$$R_{\text{eff}} = R + Q$$

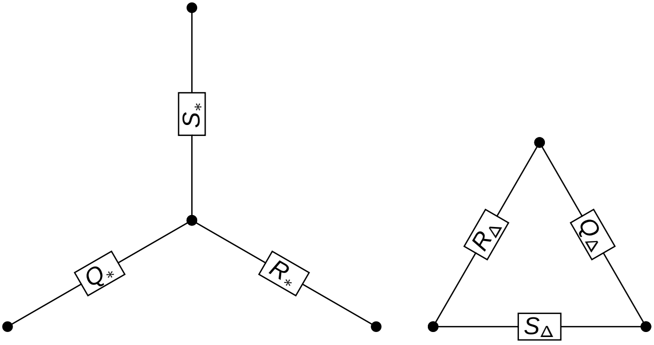
Parallel:



$$\frac{1}{R_{\text{eff}}} = \frac{1}{R} + \frac{1}{Q}$$

Reducing a network

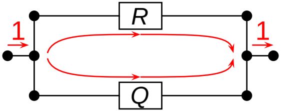
Star-Delta:



$$R_* = \frac{Q_\Delta S_\Delta}{R_\Delta + Q_\Delta + S_\Delta}, \qquad R_\Delta = \frac{R_* Q_* + R_* S_* + Q_* S_*}{R_*}.$$

Thomson, Dirichlet principles

Thomson principle:



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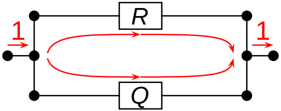

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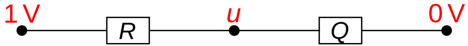

The physical voltage is the function that minimizes the ohmic power losses $\sum (\nabla u)^2 / R$.

Monotonicity, transience, recurrence

The monotonicity property:

Between any disjoint sets of vertices, the effective resistance is a non-decreasing function of the individual resistances.

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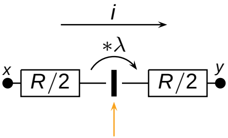
$\rightsquigarrow$ can be used to prove transience-recurrence by reducing the graph to something manageable in terms of resistor networks.

The part



Voltage amplifier: keeps the current, multiplies the potential.

$$(u_x - i \cdot \frac{R}{2}) \cdot \lambda - i \cdot \frac{R}{2} = u_y$$

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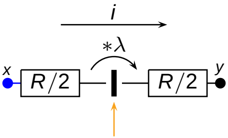

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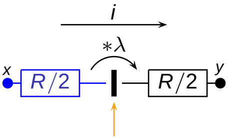

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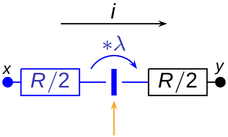

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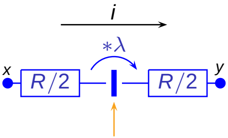

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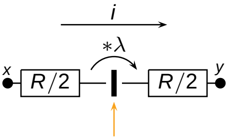

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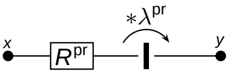

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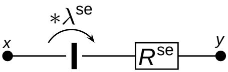

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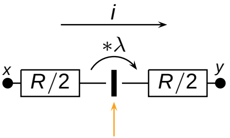

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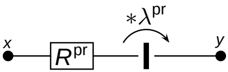

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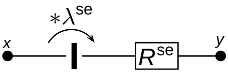

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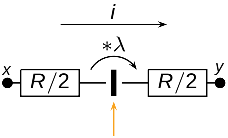

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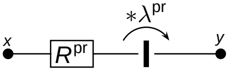

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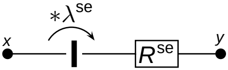

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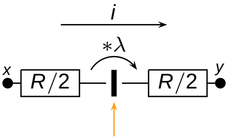

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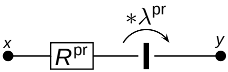

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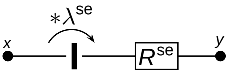

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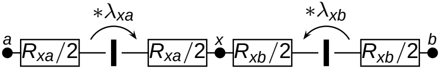

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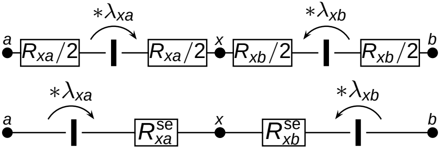

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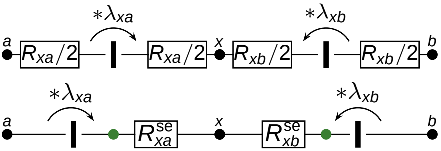

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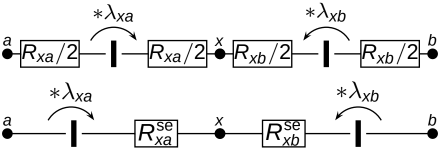

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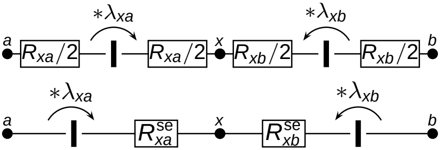

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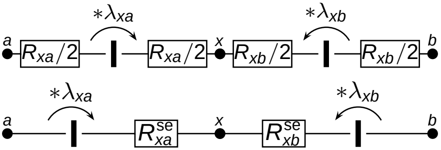

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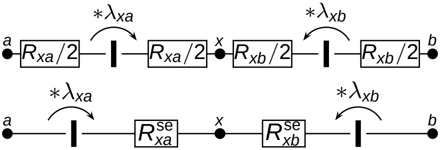

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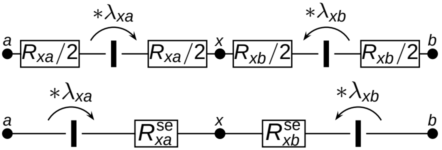

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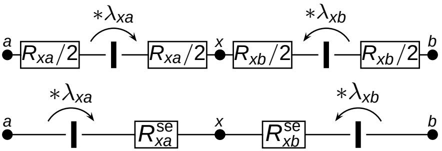

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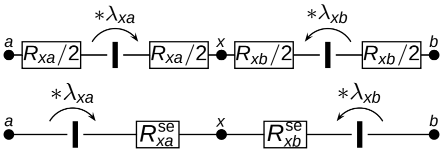
$R^{pr} = \frac{\lambda+1}{2\lambda} \cdot R$ $\qquad$ $R^{se} = \frac{\lambda+1}{2} \cdot R$

Harmonicity



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From network to chain

Irreducible Markov chain: on Ω, $a \neq b$, $x \in \Omega$,

$$h_x := \mathbf{P}_x\{\tau_a < \tau_b\} \qquad (\tau \text{ is the hitting time})$$

is harmonic:

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"Markovian" property

$$u_x = \sum_z P_{xz} u_z ; \qquad \sum_z P_{xz} = 1$$

$u_x \equiv$ const. is a solution of the network with no external sources. This is now nontrivial.

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$$P_{xy} = \frac{D_{xy}\gamma_{xy}}{D_x} = \frac{D_{xy}\gamma_{xy}}{\mu_x}$$

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Reversed chain: Replace P_{xy} by $\hat{P}_{xy} = P_{yx} \cdot \frac{\mu_y}{\mu_x}$.

$\rightsquigarrow$ D_{xy} stays, λ_{xy} reverses to λ_{yx}.

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normalisation...

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Effective resistance

Suppose u_a, u_b given, the solution is $\{u_x\}_{x\in\Omega}$ and $\{i_{xy}\}_{x\sim y\in\Omega}$. Current

$$i_a = \sum_{x\sim a} i_{ax}$$

flows in the network at a.

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$\leadsto$ In particular, i_a is proportional to u_a-u_b. We have effective resistance.

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Modulo normalisation. . .

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in the reversed network!

Theorem (Chandra, Raghavan, Ruzzo, Smolensky and Tiwari '96 for reversible)

Commute time $= R_{eff} \cdot$ *all conductances.*

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For all sets A, B, capacity$\sim$escape probability.

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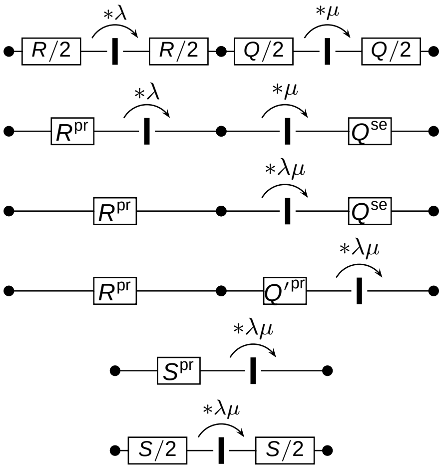
This is non-physical!

In particular, symmetrising the chain ($P_{xy} \to \frac{P_{xy} + \hat{P}_{xy}}{2}$) cannot increase escape probabilities:

- symmetrising leaves C_{xy} unchanged;
- the above sum is minimised by the symmetric voltages, not $\{u_x\}$ (Classical Dirichlet principle).

The electric network

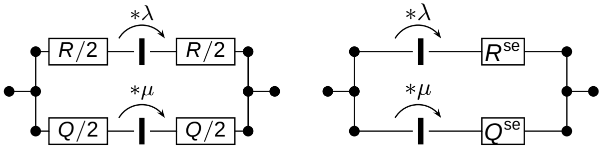
Series:



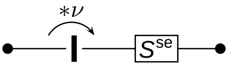
$$S = R\frac{(\lambda+1)\mu}{\lambda\mu+1} + Q\frac{\mu+1}{\lambda\mu+1}.$$

The electric network

Parallel:



Compare this with



$$S = \frac{RQ}{R+Q}$$

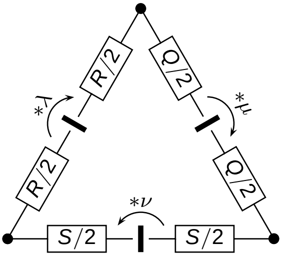
$$\nu = \frac{Q\lambda(\mu+1) + R\mu(\lambda+1)}{Q(\mu+1) + R(\lambda+1)}.$$

The electric network

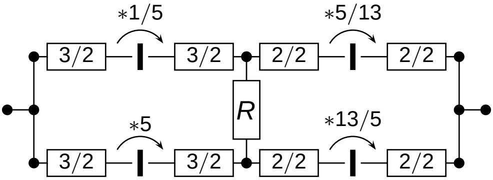
Star-Delta:

Star to Delta works,

Delta to Star only works if Delta does not produce a circular current by itself ($\lambda\mu\nu = 1$).



Nonmonotonicity



$$R^{\text{eff}} = \frac{27}{14} + \frac{1296}{1225R + 2268}.$$

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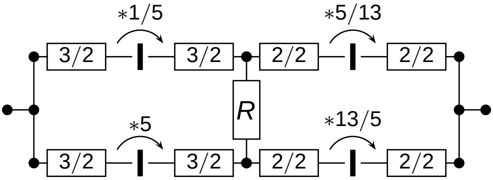

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Dirichlet principle

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$$(i^*_u)_{xy} = D_{xy} \cdot \big(\gamma_{xy} u(x) - \gamma_{yx} u(y)\big),$$

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Thank you.

A trivial statement

Theorem (Well Known Theorem)

A Markov chain is reversible if and only if for every closed cycle $x_0, x_1, x_2, \ldots, x_n = x_0$ in Ω we have

$$P_{x_0 x_1} \cdot P_{x_1 x_2} \cdots P_{x_{n-1} x_0} = P_{x_0 x_{n-1}} \cdot P_{x_{n-1} x_{n-2}} \cdots P_{x_1 x_0}.$$

In particular, any Markov chain on a finite connected tree G is necessarily reversible.

A trivial statement

Electrical proof.

Plug in

$$P_{xy} = \frac{D_{xy}\gamma_{xy}}{D_x}, \qquad D_{xy} \text{ symmetric:}$$

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- Total multiplication factor along any loop is one.

A trivial statement

Electrical proof.

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Repeat for trees:

- There are no loops.

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Second thank you.