Loop models on a fractal

Stephan Wagner
(joint work with Elmar Teufl)

Stellenbosch University

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Self-similar graphs

- Constructed recursively: X_n is obtained by "glueing" copies of X_{n-1} together.
- Finite approximations of fractals or finite parts of infinite graphs.

Some examples:

(modified) Koch curve

Sierpiński gasket

Lindstrøm snowflake

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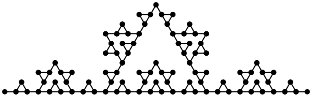

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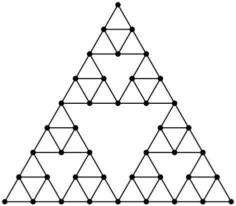

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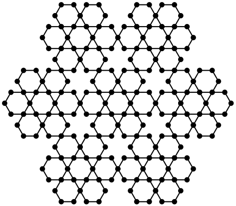

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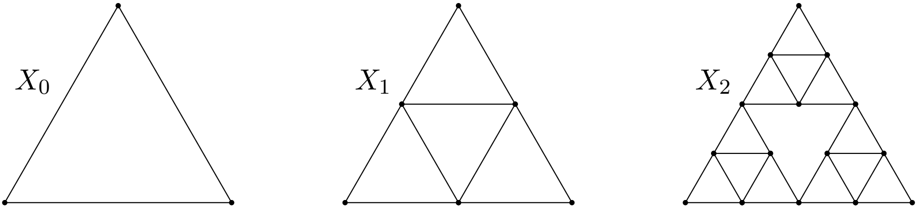
Sierpiński graphs

The classical sequence of Sierpiński graphs (finite approximations of the Sierpiński gasket, starting with a single triangle) will serve as a running example to illustrate the general idea:



Random loops

We study two different kinds of models to create a random partition into cycles:

- Partition the *edge set* randomly into cycles; this is only possible if the graph is Eulerian.
- Partition the *vertex set* randomly into cycles; more precisely, take a random 2-*factor* of the graph (a spanning subgraph whose connected components are cycles).

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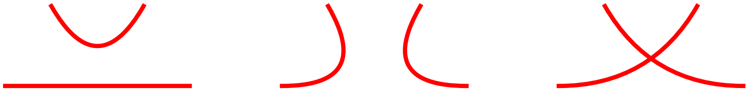
Random edge partitions

Choosing a partition of the edge set into cycles uniformly at random is equivalent to choosing, independently for each vertex, the way in which edges are "linked". In the example of the Sierpiński graph, we have three possibilities at each vertex (except for the corners, where we have no choice):



This means that for the n-th Sierpiński graph X_n, we have

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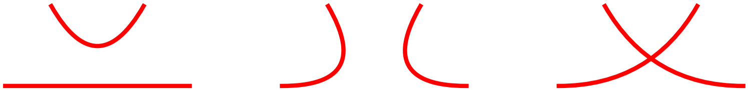

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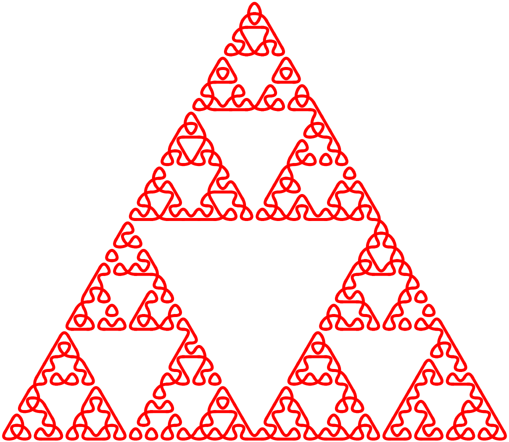
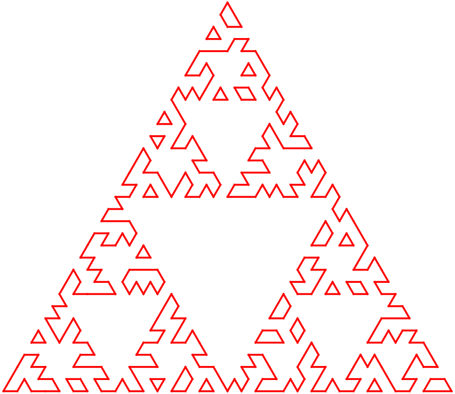
An example

The following picture shows a randomly generated instance for $n = 5$:



Random 2-factors

The random 2-factor model is a dual in some sense: instead of using every edge exactly once, we use every vertex exactly once. The picture shows a randomly generated instance on the Sierpiński graph X_5 again:



Counting 2-factors

Counting all possibilities (which will be necessary for a probabilistic analysis) is slightly more involved than in the previous model.

We notice that a 2-factor of X_n induces 2-factors on the three copies of X_{n-1}, with two exceptions:

- One or more corners may be left out.
- One of the components may be a path connecting two corners.

Thus we define a few auxiliary quantities first:

- $a_{i,n}$ denotes the number of 2-factors of X_n, from which i (fixed) corner vertices have been removed ($i \in \{0, 1, 2, 3\}$).
- $b_{i,n}$ denotes the number of spanning subgraphs of X_n, where all but one component are cycles, and the exceptional component is a path connecting the two bottom corners. For $i = 0$, the third corner is covered as well, for $i = 1$, we remove it first.

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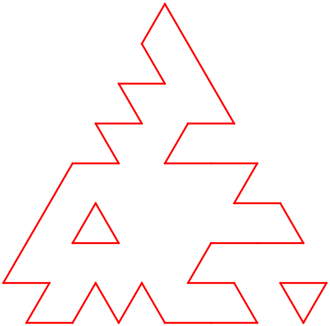
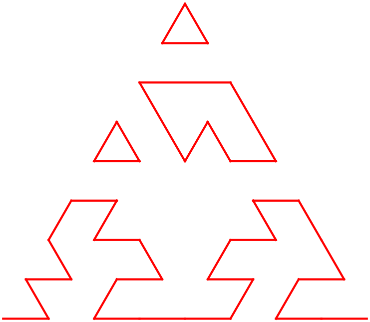
The auxiliary quantities

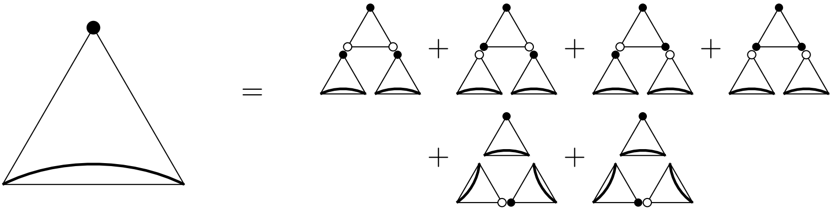
The following two examples show instances that are counted by $a_{2,3}$ and $b_{0,3}$ respectively.





Setting up a recursion

Now we only need to determine the number of ways of putting the pieces together. For instance:



This gives us

$$b_{0,n+1} = b_{0,n}^2 a_{2,n} + 2b_{0,n} b_{1,n} a_{1,n} + b_{1,n}^2 a_{0,n} + 2b_{0,n}^2 b_{1,n},$$

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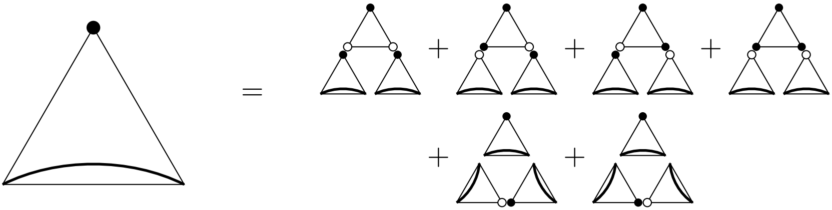

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The recursions

$$
\begin{aligned}
a_{0,n+1} &= 6a_{0,n}a_{1,n}a_{2,n} + 2a_{1,n}^3 + b_{0,n}^3,\\
a_{1,n+1} &= 2a_{0,n}a_{1,n}a_{3,n} + 2a_{0,n}a_{2,n}^2 + 4a_{1,n}^2a_{2,n} + b_{0,n}^2b_{1,n},\\
a_{2,n+1} &= 2a_{0,n}a_{2,n}a_{3,n} + 2a_{1,n}^2a_{3,n}^2 + 4a_{1,n}a_{2,n}^2 + b_{0,n}b_{1,n}^2,\\
a_{3,n+1} &= 6a_{1,n}a_{2,n}a_{3,n} + 2a_{2,n}^3 + b_{1,n}^3,\\
b_{0,n+1} &= b_{0,n}^2a_{2,n} + 2b_{0,n}b_{1,n}a_{1,n} + b_{1,n}^2a_{0,n} + 2b_{0,n}^2b_{1,n},\\
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with $a_{0,0} = a_{3,0} = b_{0,0} = b_{1,0} = 1,\ a_{1,0} = a_{2,0} = 0.$

We are particularly interested in $a_{0,n}$:

n	0	1	2	3	4
$a_{0,n}$	1	1	35	1072000	27269660672000000000

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Solving the recursions

One observes immediately that $a_{0,n} = a_{1,n} = a_{2,n} = a_{3,n}$ for $n \geq 1$ (proof by induction or by a simple combinatorial bijection) and $b_{0,n} = b_{1,n}$ for $n \geq 1$. This is not crucial per se, but it simplifies the calculations considerably.

We can set

$$a_n = a_{0,n} = a_{1,n} = a_{2,n} = a_{3,n}$$

and

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and obtain

$$a_{n+1} = 8a_n^3 + b_n^3, \quad b_{n+1} = 4a_n b_n^2 + 2b_n^3.$$

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Now consider the quotient: $q_n = a_n/b_n$, which satisfies

$$q_{n+1} = \frac{a_{n+1}}{b_{n+1}} = \frac{8a_n^3 + b_n^3}{4a_nb_n^2 + 2b_n^3} = \frac{8q_n^3 + 1}{4q_n + 2}$$

with $q_1 = \frac{1}{3}$. Thus q_n converges to the fixed point of the map $x \mapsto \frac{8x^3+1}{4x+1}$, which is $\frac{1}{2}$. Indeed,

$$q_n = \frac{1}{2} - \frac{1}{2n} + O\Big(\frac{\log n}{n^2}\Big).$$

It follows that

$$a_{n+1} = 16a_n^3(1 + O(n^{-1})).$$

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The asymptotic solution

We finally arrive at the following formula:

Theorem

The number of 2-factors of the nth Sierpiński graph X_n is asymptotically given by

$$a_n \sim \frac{1}{4} A^{3^n},$$

for a constant $A = 1.77019389\ldots$. Moreover, the number of "almost 2-factors" satisfies

$$b_n \sim \frac{1}{2} A^{3^n}.$$

The number of loops

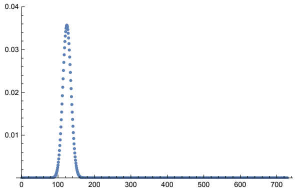
In both models, the number of loops satisfies a central limit theorem with mean and variance linear in the number of vertices of X_n:

Theorem

Let L_n denote the random number of cycles in the random edge partition model. The mean and variance of L_n are asymptotically given by

$$\mu_n \sim 0.169619 \cdot 3^n, \qquad \sigma_n^2 \sim 0.171443 \cdot 3^n.$$

The normalised random variable $\frac{L_n - \mu_n}{\sigma_n}$ converges weakly to a standard normal distribution.



The distribution for $n = 6$.

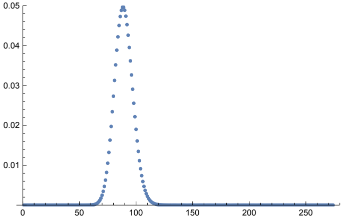
The number of loops

Theorem

Let $\overline{L}_n$ denote the random number of cycles in the random 2-factor model. The mean and variance of $\overline{L}_n$ are asymptotically given by

$$\overline{\mu}_n \sim 0.119986 \cdot 3^n, \qquad \overline{\sigma}_n^2 \sim 0.085573 \cdot 3^n.$$

The normalised random variable $\frac{\overline{L}_n - \overline{\mu}_n}{\overline{\sigma}_n}$ converges weakly to a standard normal distribution.



The distribution for $n = 6$.

Short loops

A similar result holds for the number of cycles of fixed length:

Theorem

For a fixed integer $k \geq 3$, let $L_{k,n}$ and $\overline{L}_{k,n}$ be the number of cycles of length k in the random edge partition model and the random 2-factor model on X_n respectively.
There exist positive constants α_k, β_k and $\overline{\alpha}_k, \overline{\beta}_k$ such that mean and variance of $L_{k,n}$ and $\overline{L}_{k,n}$ are asymptotically equal to

$$\mu_{k,n} \sim \alpha_k \cdot 3^n, \qquad \sigma_{k,n}^2 \sim \beta_k \cdot 3^n$$

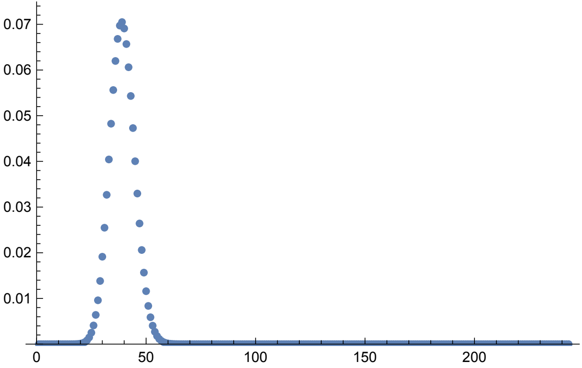
and

$$\overline{\mu}_{k,n} \sim \overline{\alpha}_k \cdot 3^n, \qquad \overline{\sigma}_{k,n}^2 \sim \overline{\beta}_k \cdot 3^n$$

respectively. Moreover, the normalised random variables $\frac{L_{k,n}-\mu_{k,n}}{\sigma_{k,n}}$ and $\frac{\overline{L}_{k,n}-\overline{\mu}_{k,n}}{\overline{\sigma}_{k,n}}$ converge weakly to a standard normal distribution.

An example of the distribution

Number of triangles in a random 2-factor of X_6:



Long loops

In both models, “almost all” the cycles are short, since we have the following trivial property:

Proposition

In both models, the number of cycles of length $> k$ is $O(3^n/k)$. Thus if $k \to \infty$ (arbitrarily slowly), the proportion of such cycles goes to 0.

Nonetheless, this raises the question how long “long” cycles typically are. The answers are vastly different.

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Long loops in the edge partition model

Consider the cycle containing a fixed corner of X_n in the edge partition model, and let J be the smallest index for which this cycle fits inside a copy of X_J.

Theorem

There exists a constant $C < 1$ such that

$$\mathbb{P}(J = j) = O(C^{2^j}).$$

Corollary

With high probability, the longest cycle in the edge partition model has length $O(n^{\log_2 3})$.

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We have seen that $a_{0,n} = a_{1,n} = a_{2,n} = a_{3,n} = a_n$ and $b_{0,n} = b_{1,n} = b_n$ for $n \geq 1$, and that $a_n / b_n \to \frac{1}{2}$.

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Long loops in the 2-factor model

In the recursion

$$a_{n+1} = 8a_n^3 + b_n^3,$$

the first term stands for the number of configurations without a cycle surrounding the central hole, the second term stands for configurations with such a cycle. Since they are asymptotically equal, the asymptotic probability for a cycle around the central hole is $\frac{1}{2}$.

If there is no cycle around the central hole, then in each of the three parts the probability for such a cycle is asymptotically $\frac{1}{2}$, etc.

In other words: fairly long cycles (at least in the order of 2^n) exist with high probability. How long are they actually, and what do they look like?

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In other words: fairly long cycles (at least in the order of 2^n) exist with high probability. How long are they actually, and what do they look like?

Long loops in the 2-factor model

In the recursion

$$a_{n+1} = 8a_n^3 + b_n^3,$$

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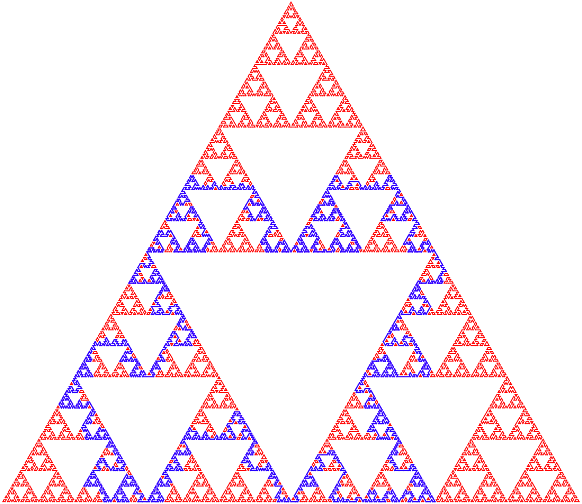
Long loops in the 2-factor model

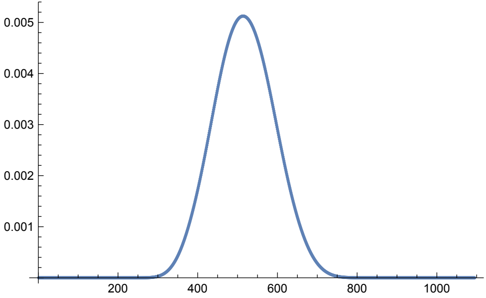
A random 2-factor of X_8 and its longest cycle:



Long loops in the 2-factor model

Theorem

Let M_n be the length of the longest cycle in a random 2-factor of X_n. The normalised random variable $n^{-1/10}\left(\frac{5}{2}\right)^{-n} M_n$ converges weakly to a limiting distribution.



The distribution for $n = 6$.

Long loops in the 2-factor model

Heuristic explanation:

Consider the case that there is a cycle around the central hole. When we decompose X_n into its three pieces, we obtain three configurations that are each counted by $b_{0,n-1} = b_{n-1}$.

Call configurations counted by a_n "type A configurations" and those counted by b_n "type B configurations".

The recursion

$$b_n = 4a_{n-1}b_{n-1}^2 + 2b_{n-1}^3$$

has two terms that are asymptotically equal. The first corresponds to a split into one type A and two type B configurations, the second to a split into three type B configurations.

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Long loops in the 2-factor model

Thus with asymptotic probability $\frac{1}{2}$, a piece of the long cycle in a type B copy of X_n decomposes into three similar pieces in copies of X_{n-1}, otherwise only two.

So we can regard this essentially as a Galton-Watson process, for which standard theorems would be available. The main issue is the fact that the probabilities only hold asymptotically (this is also the reason for the curious $n^{1/10}$).

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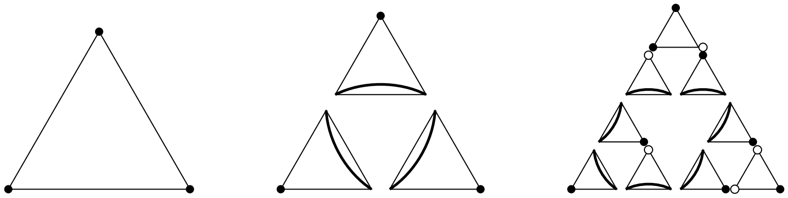
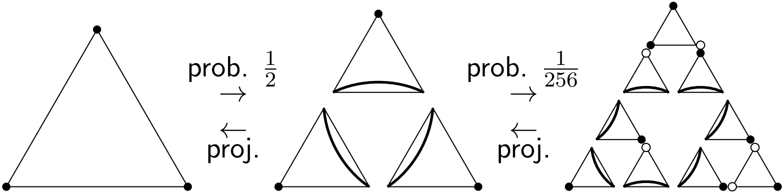
A scaling limit

What do random 2-factors look like on a macroscopic level? To formalise this question, we can use the different types of subgraphs induced on pieces that we also used for counting purposes. The following picture shows a random 2-factor at increasing resolutions:



A scaling limit

These "increasing resolutions" come with a natural projection map, turning this into a projective system with a projective (inverse) limit. The limiting probabilities of the various types equip this limit structure with a natural probability measure.



The geometry of long loops

Theorem

In the scaling limit of random 2-factors on the Sierpiński graph X_n, consider a cycle around a fixed hole of the Sierpiński gasket X (conditioned on the event that such a cycle exists). It is a random closed curve that is almost surely

- *continuous everywhere,*
- *non-differentiable everywhere,*
- *and self-avoiding.*

Its Hausdorff dimension is almost surely

$$\frac{\log \frac{5}{2}}{\log 2} \approx 1.32193.$$

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Spanning trees and the LERW

In earlier work (Teufl and W., 2014), it was shown that random spanning trees on X_n have a natural limit. This limit can be seen as a random metric on the Sierpiński gasket that is a *real tree*.

The connection between random spanning trees and the loop-erased random walk (LERW) is well established (Wilson's algorithm). In particular, the unique path between e.g. the two bottom corners of X_n in a spanning tree follows the same distribution as the LERW.

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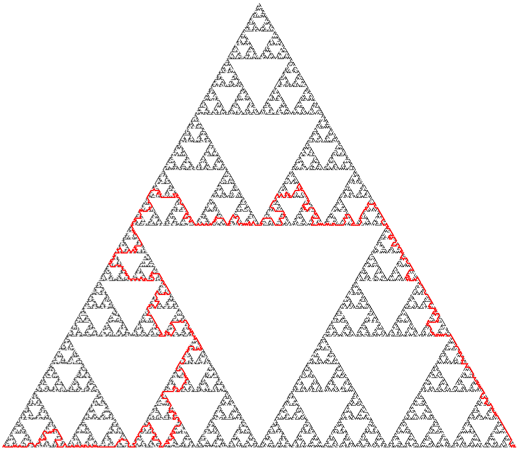
Spanning trees and the LERW

A randomly generated spanning tree on X_8 and the unique path from the bottom left corner to the bottom right corner:



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As a consequence of this correspondence, we also obtained some properties of the LERW on Sierpiński graphs that were also proven independently by K. Hattori and M. Mizuno (2014):

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As $n \to \infty$, the loop-erased random walk on the Sierpiński graphs X_n converges (suitably normalised) to a limit process. The limit curve is almost surely

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