

Multiple Random Walks: Cover Times, Hitting Times and Applications

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(joint work with Robert Elsässer)

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Outline

Introduction

Upper Bounds on the Multiple Cover Time

Lower Bounds on the Multiple Cover Time

Concrete Networks

Conclusion



(Single) Random Walks

- Random walk on an undirected, connected and finite graph $G = (V, E)$:

$$X_0 = u, X_1, X_2, \dots$$

- Let P be the transition matrix of a lazy random walk on G

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Theorem [Feige]

For any graph G with $n = |V|$ vertices, $n \ln n \lesssim t_{cov} \lesssim n^3$.



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Motivation and Challenges of Multiple Random Walks

Applications

- many randomised algorithms employ multiple random walks
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Mathematical Challenges

- various configurations for start vertices
- deal with short random walks
- already for $k = 2$, hard(er) to apply linearity of expectations
- (dependencies and interactions)



- modified transition probabilities
 - locally computable transition probabilities with $t_{cov} \lesssim n^2 \log n$
[Ikeda, Kubo, Okumoto, Yamashita]
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- Neighborhood exploration
 - avoid visited vertices (or edges)
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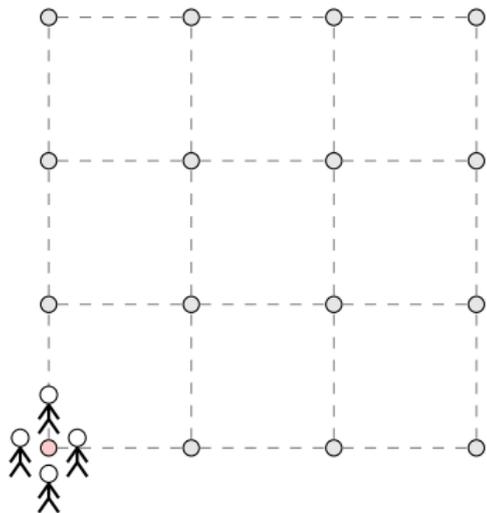


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- Multiple Walks (this talk)

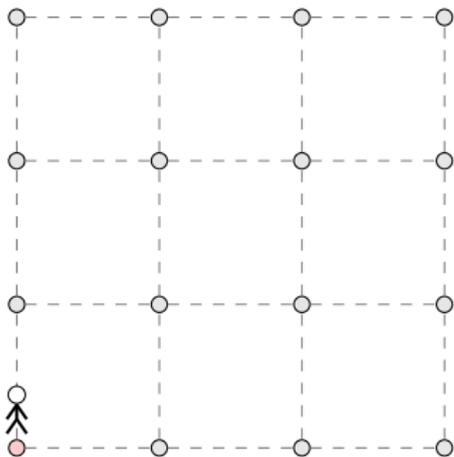


Example

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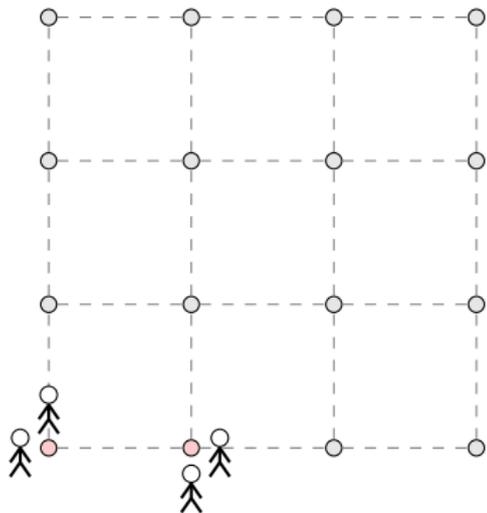


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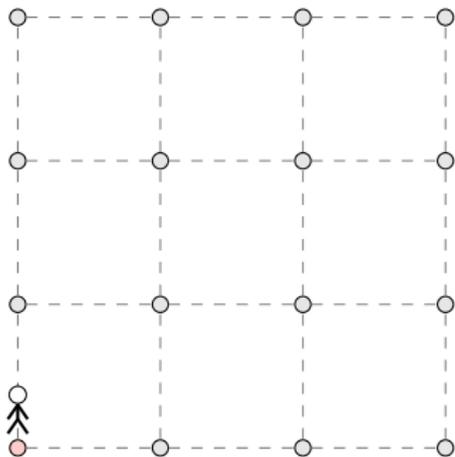


Example

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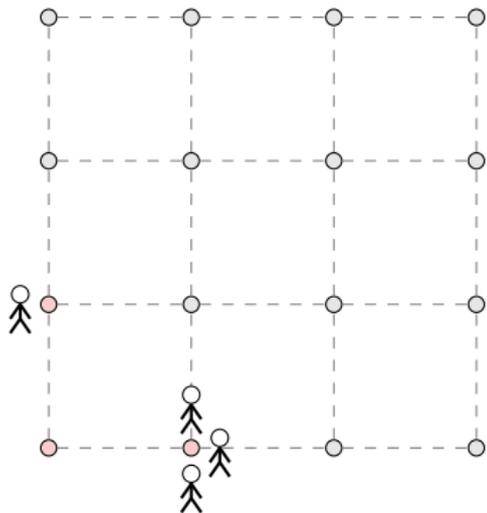


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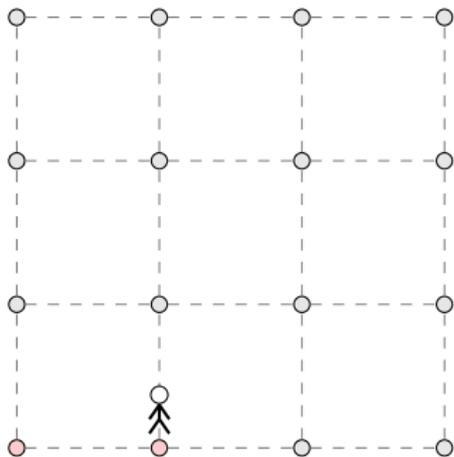


Example

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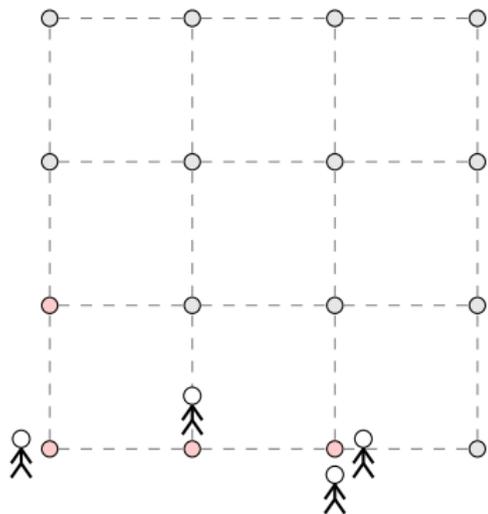


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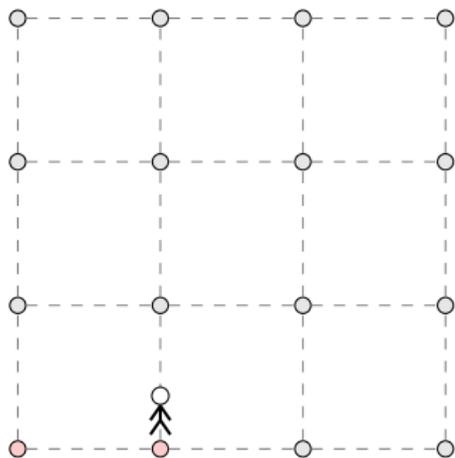


Example

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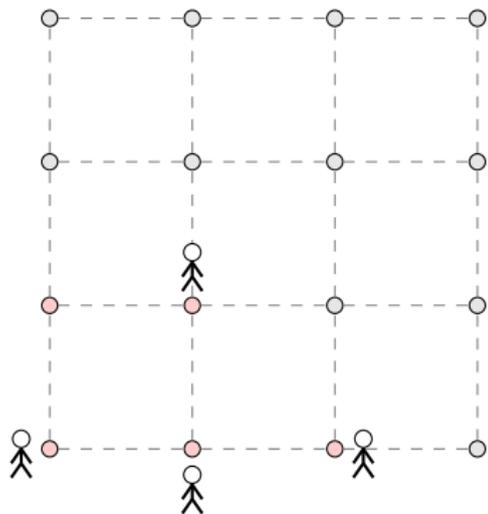


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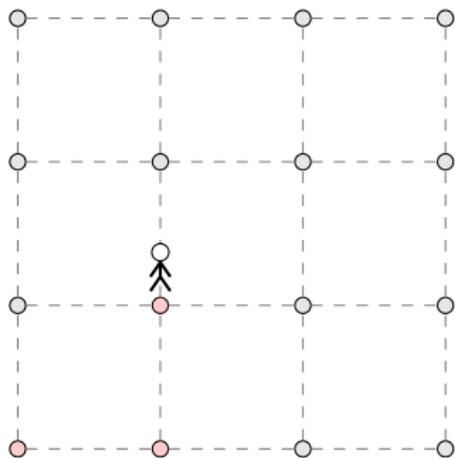


Example

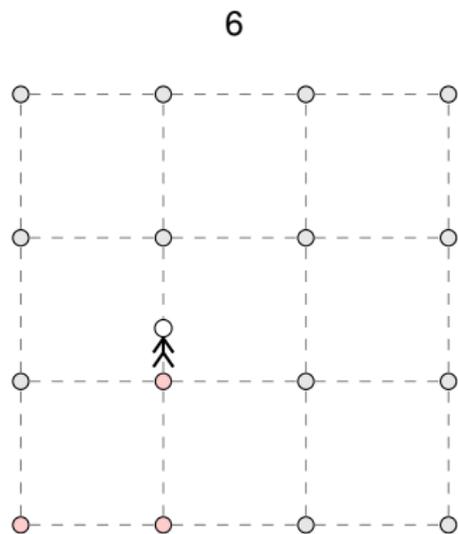
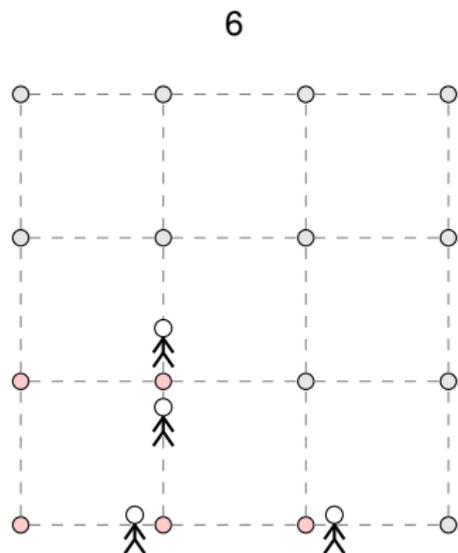
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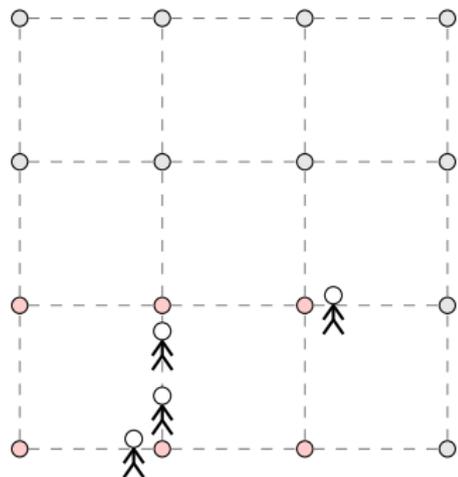


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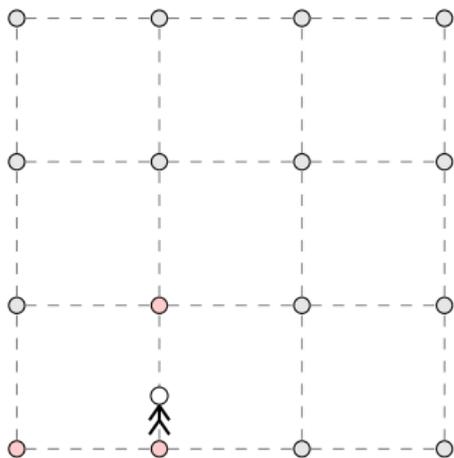


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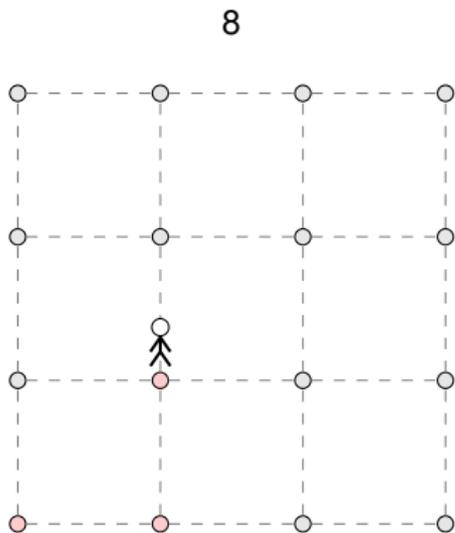
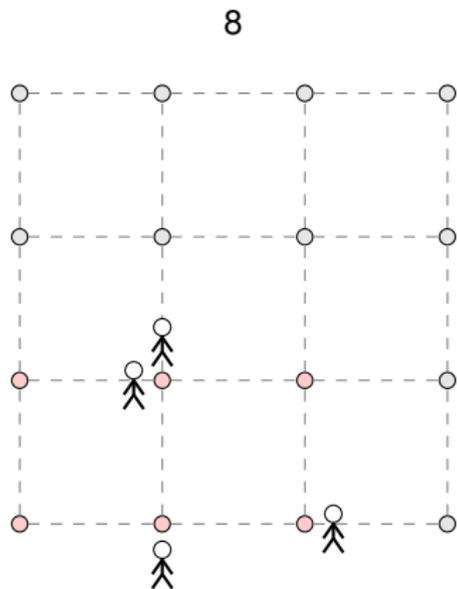
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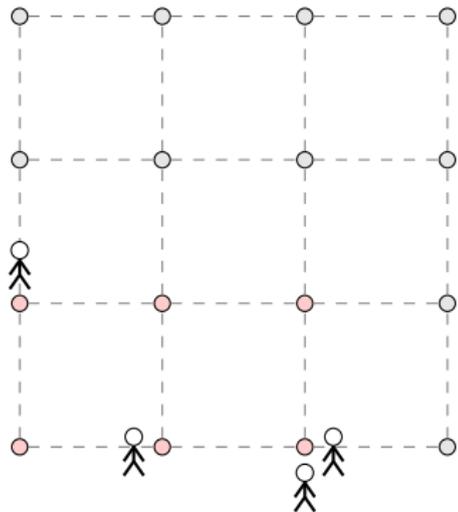


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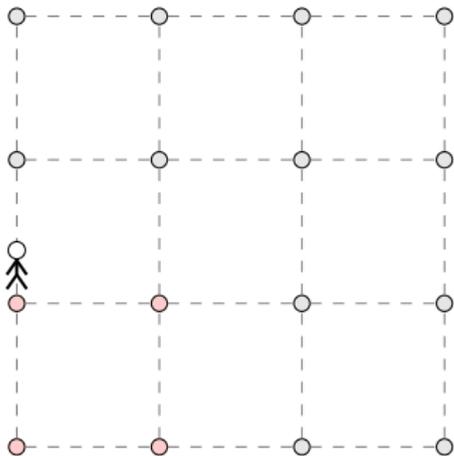


Example

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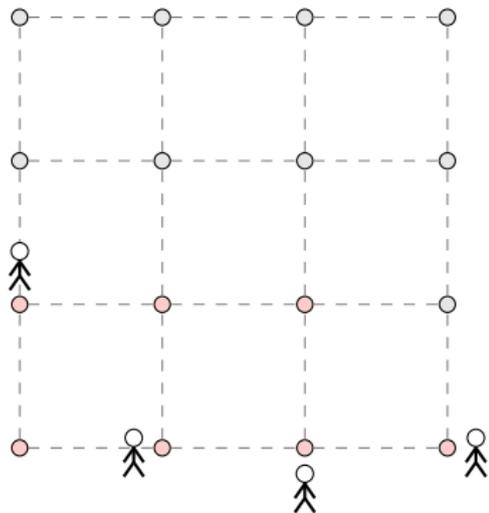


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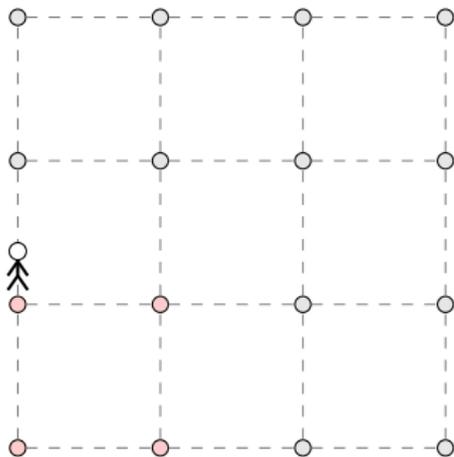


Example

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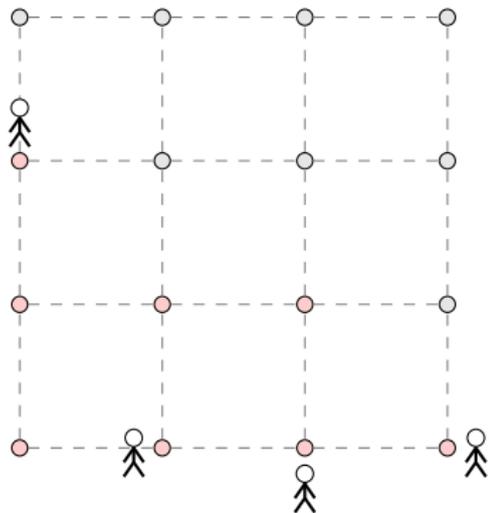


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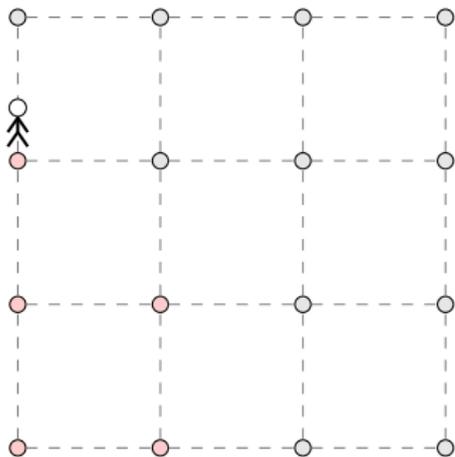


Example

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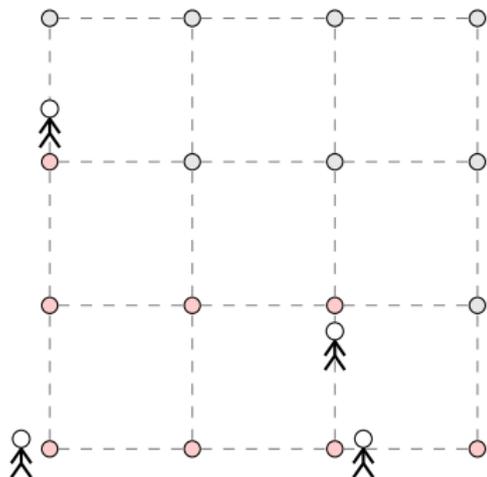


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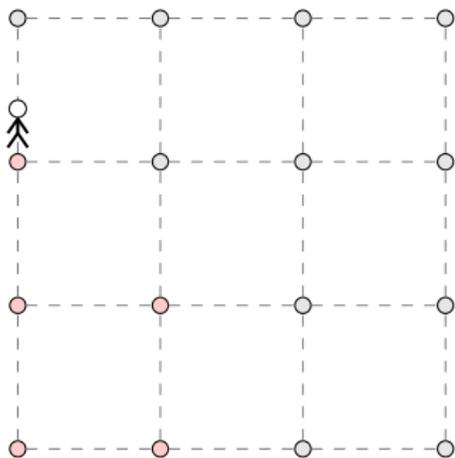


Example

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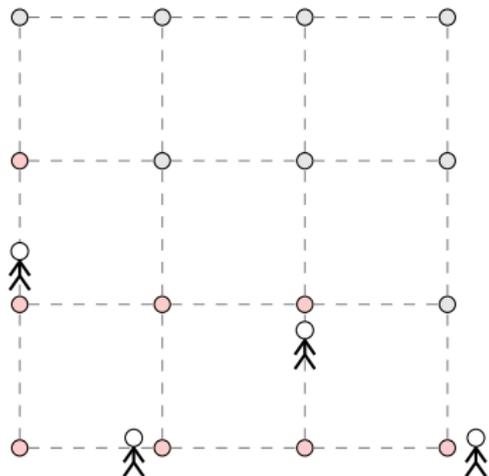


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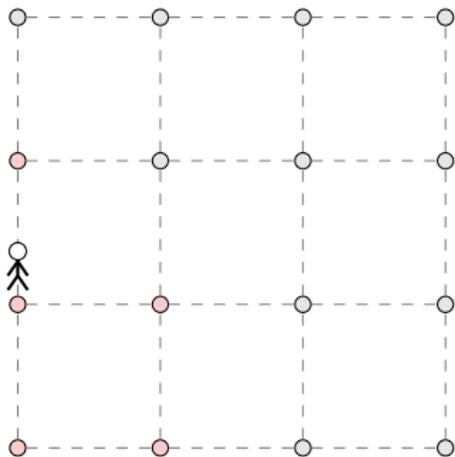


Example

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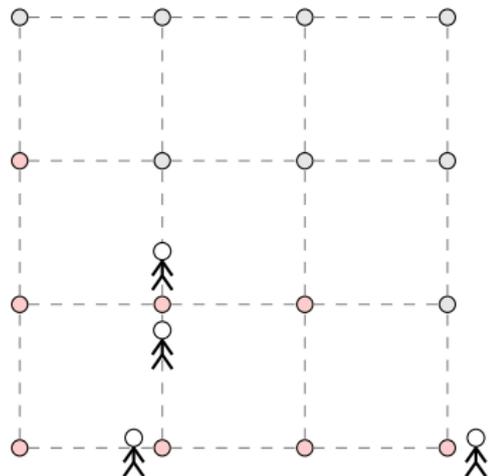


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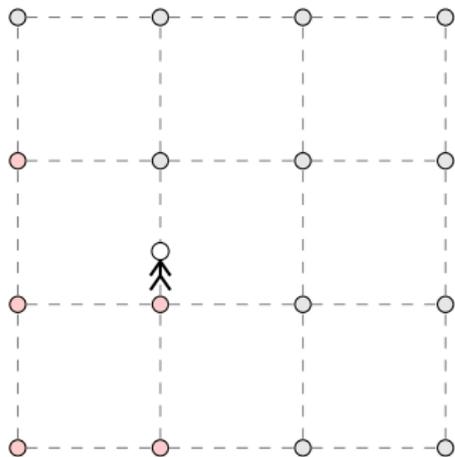


Example

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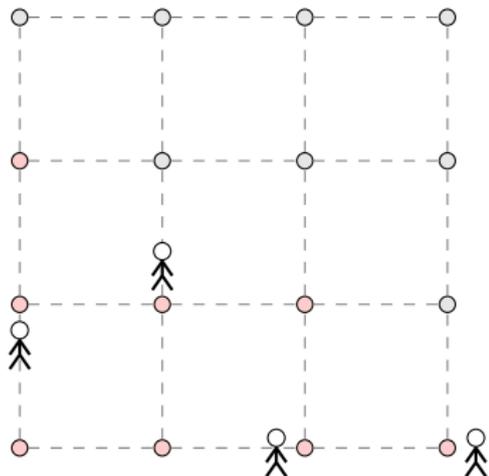


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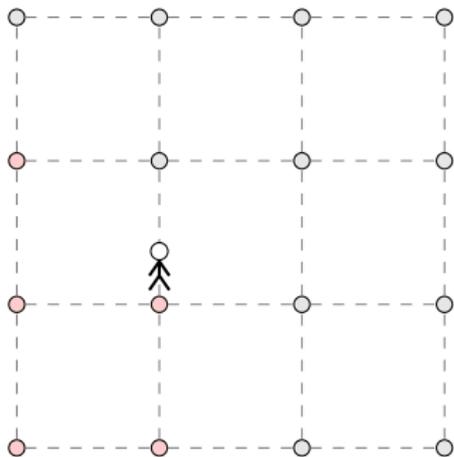


Example

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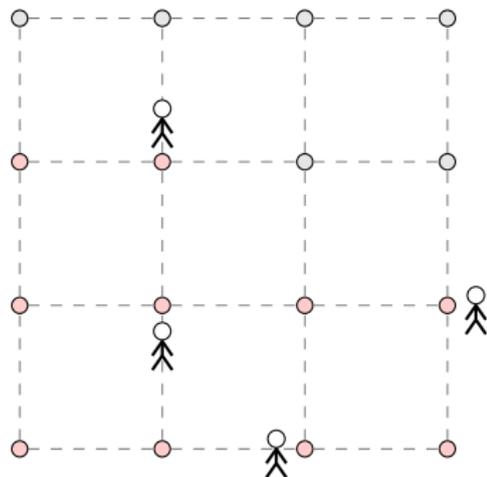


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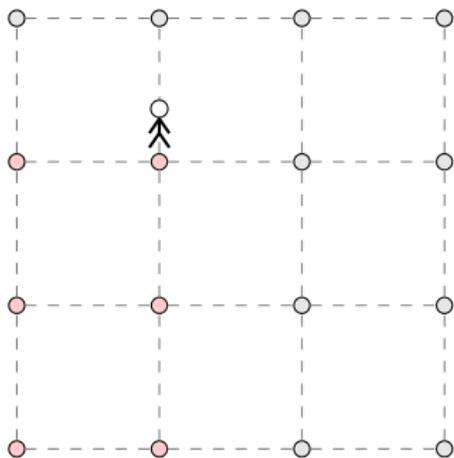


Example

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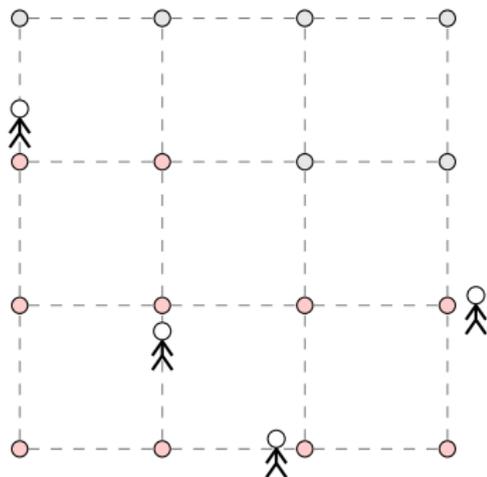


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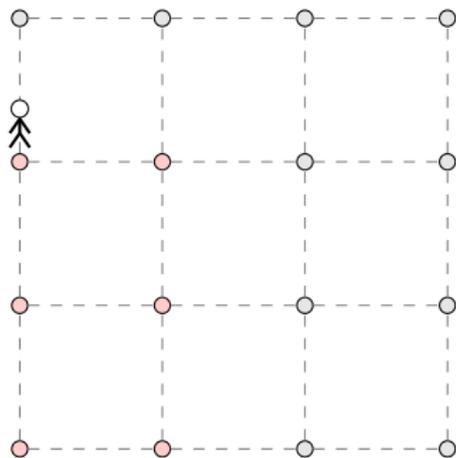


Example

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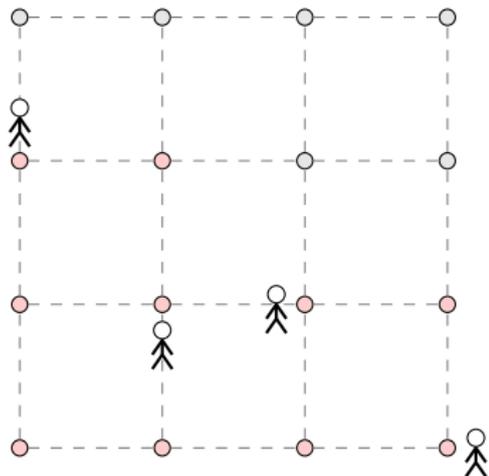


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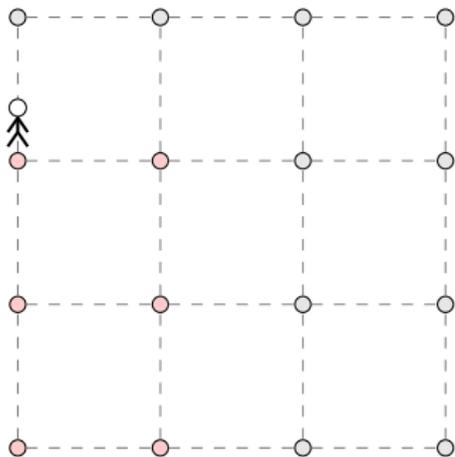


Example

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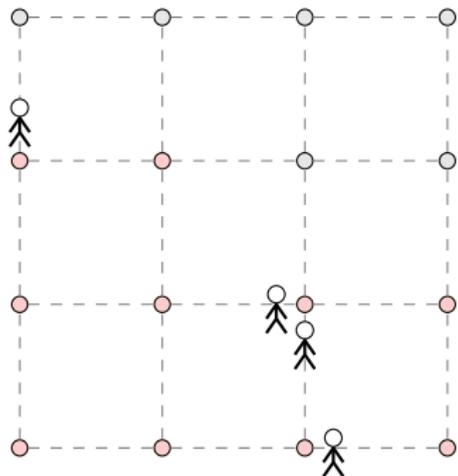


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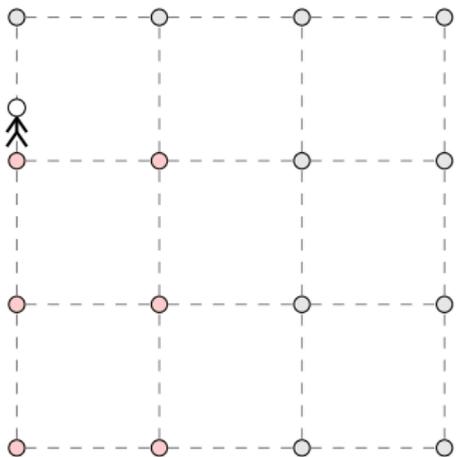


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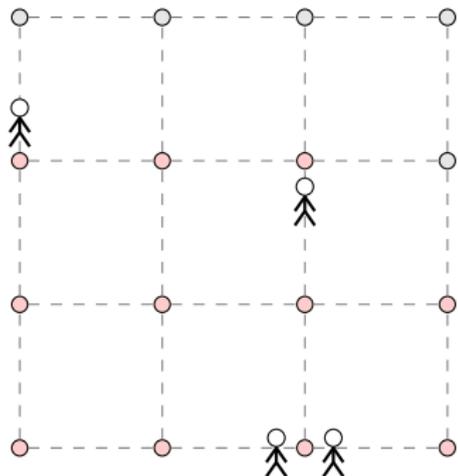


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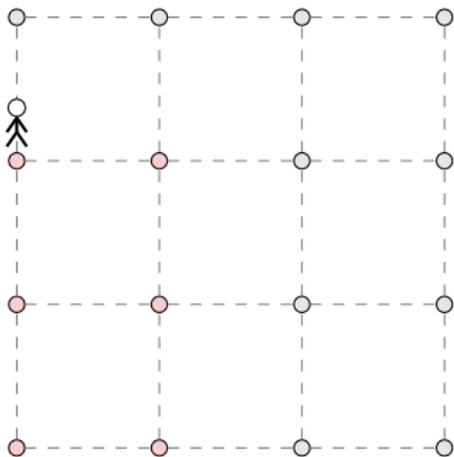


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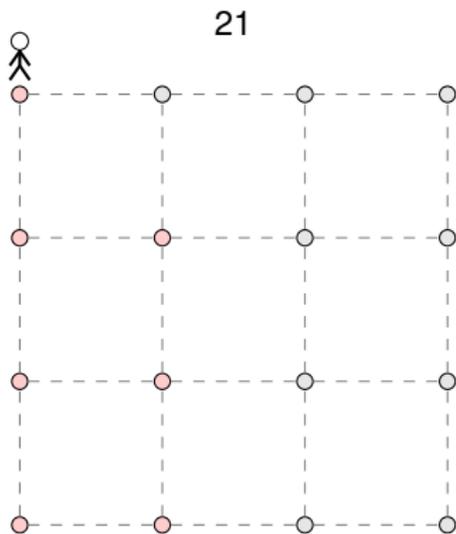
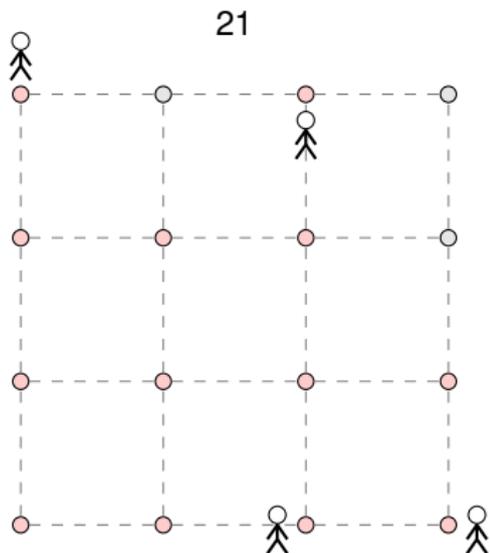
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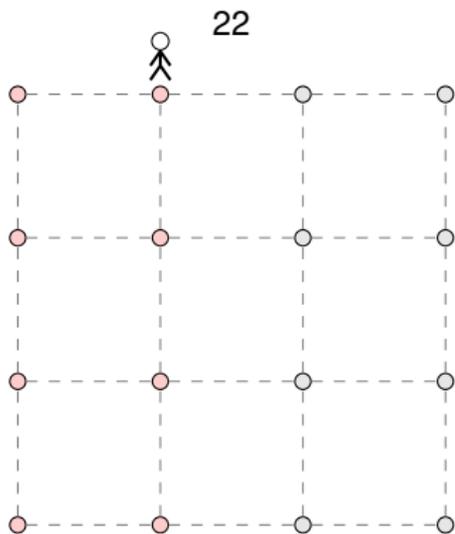
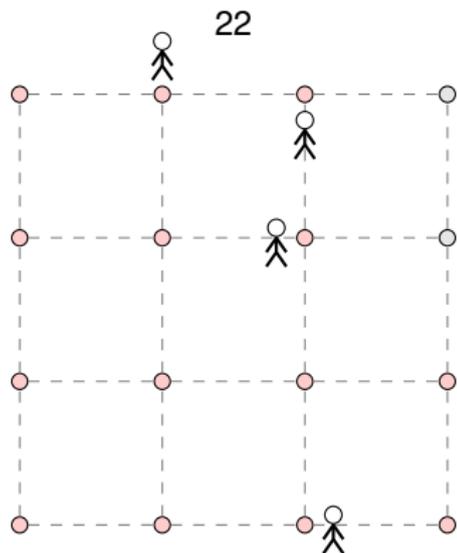
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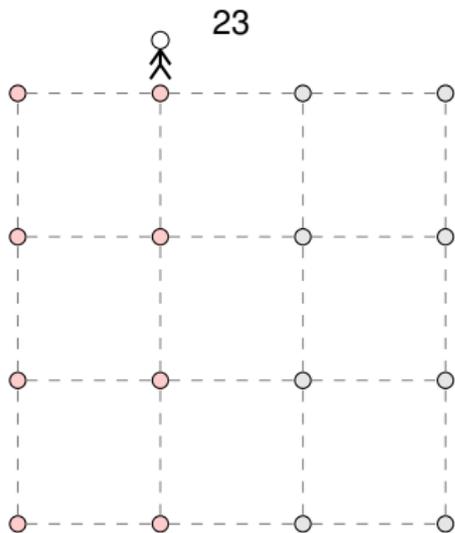
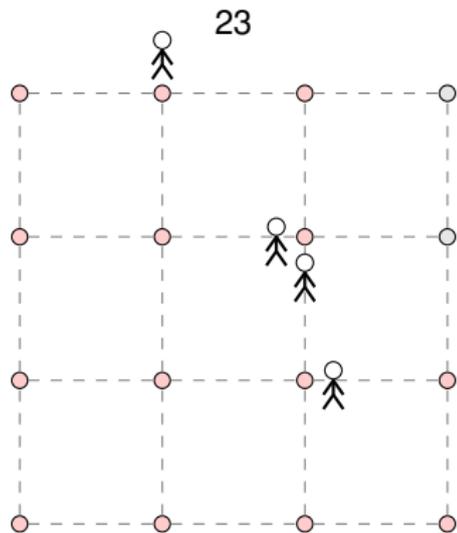
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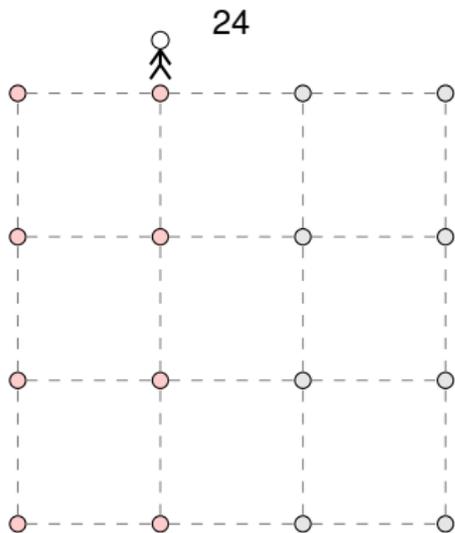
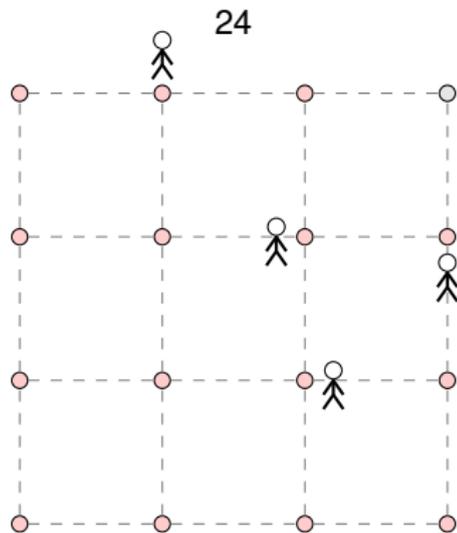
Example



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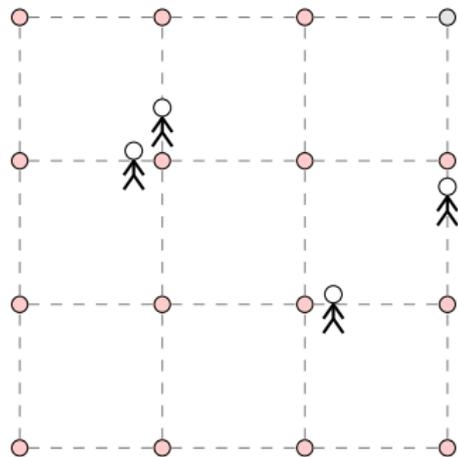


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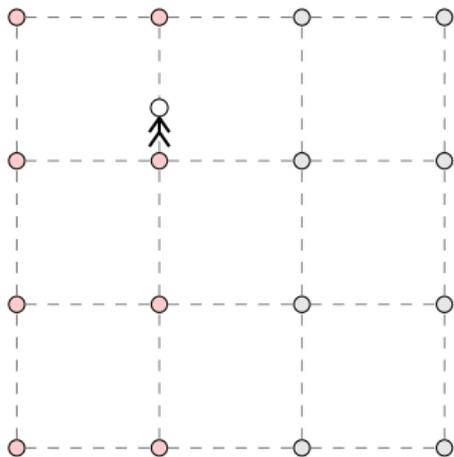


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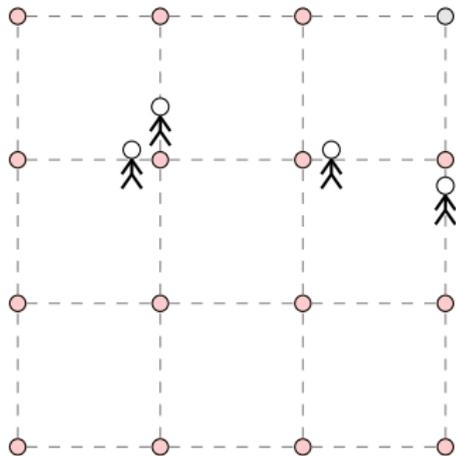


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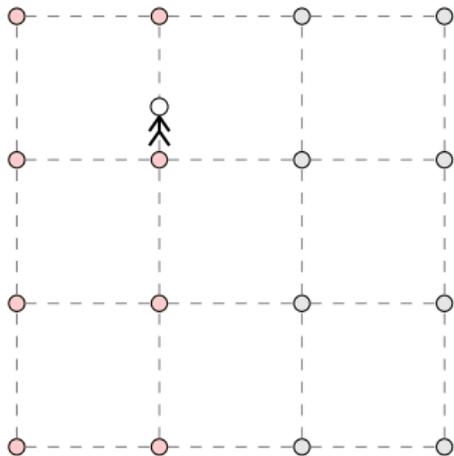


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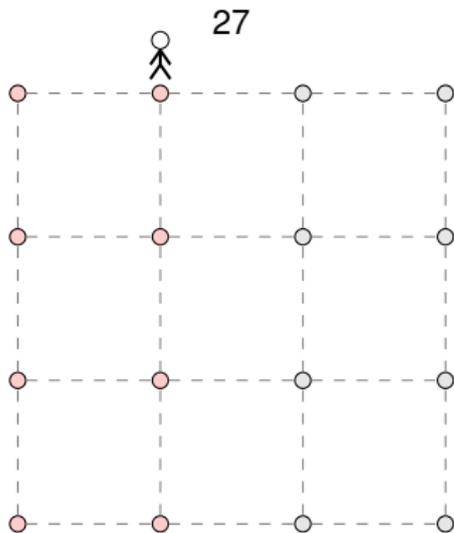
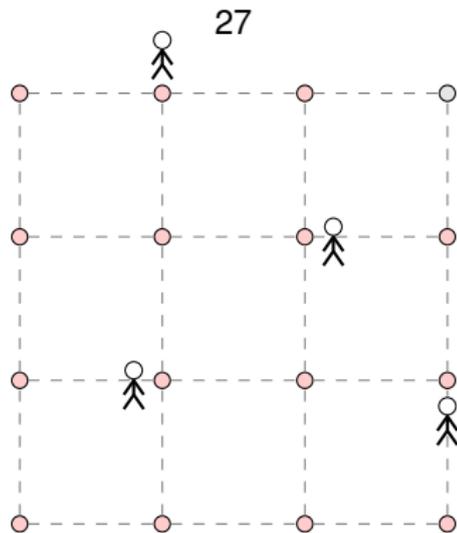
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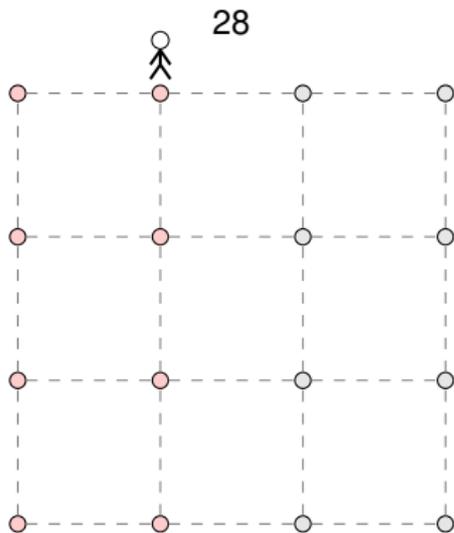
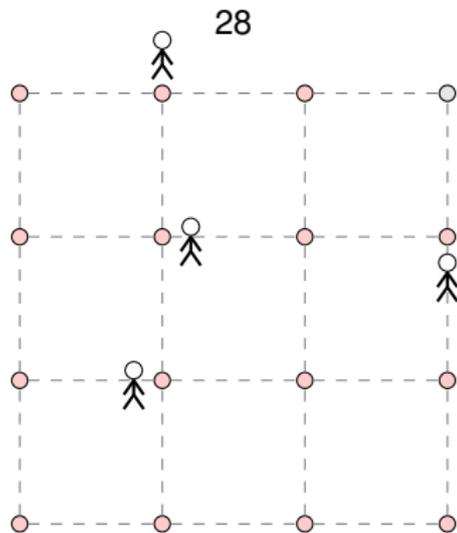
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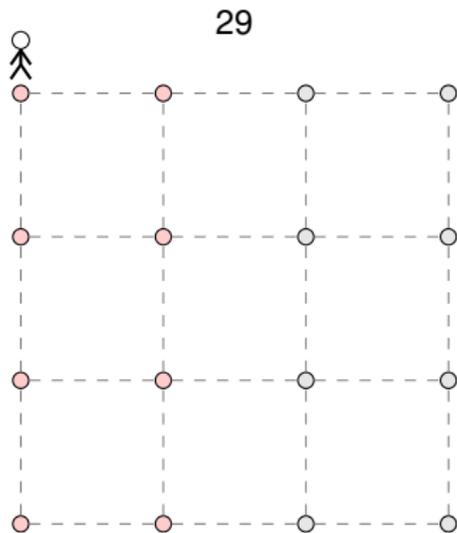
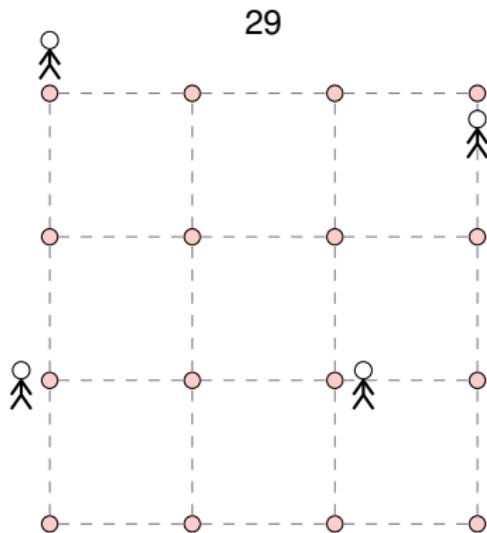
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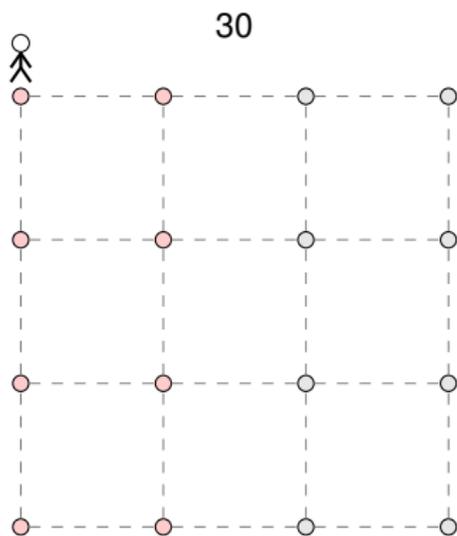
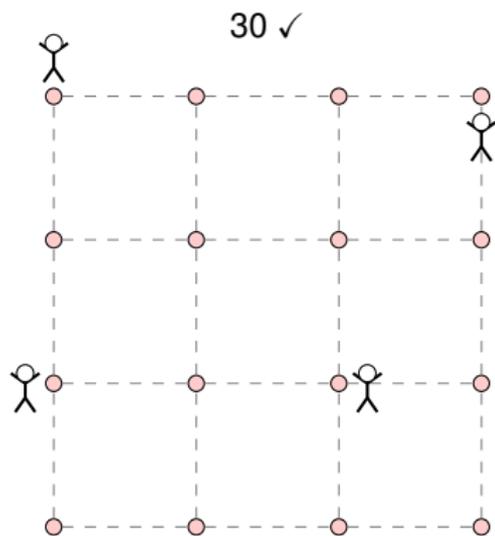
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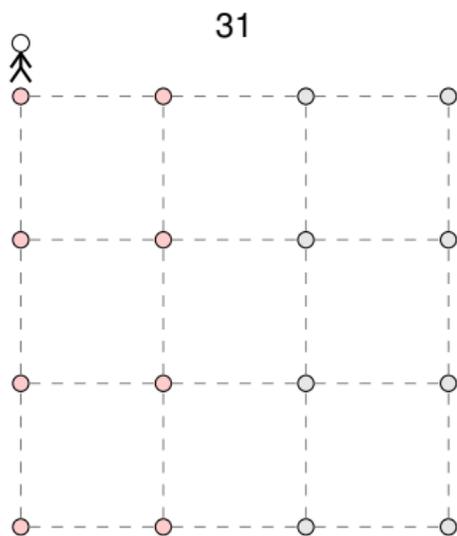
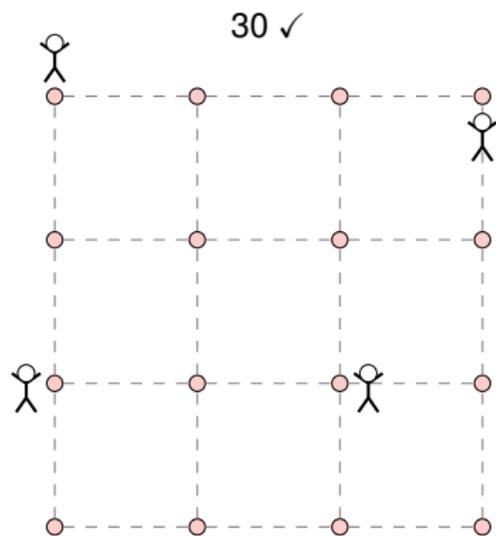
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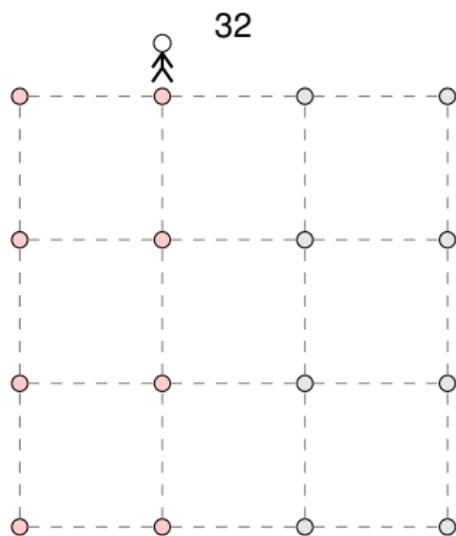
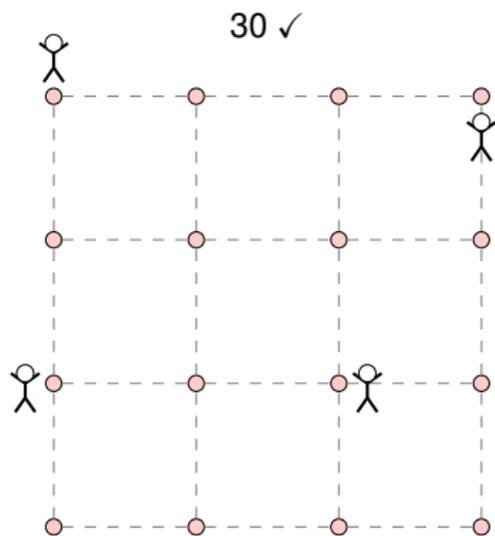
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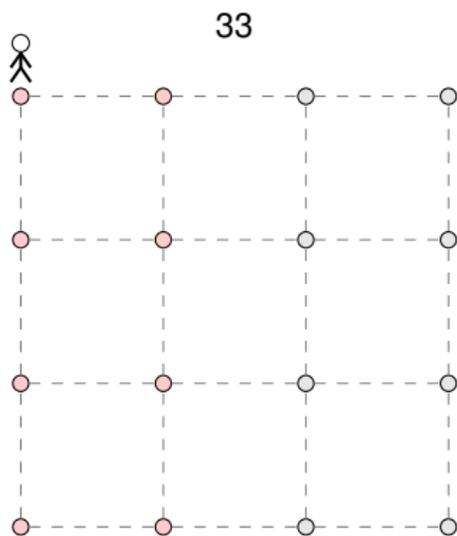
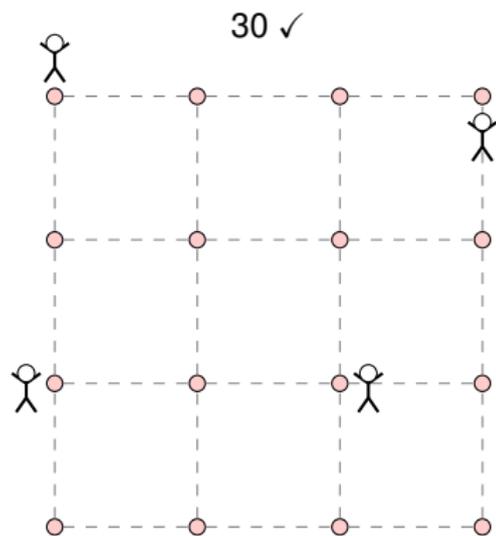
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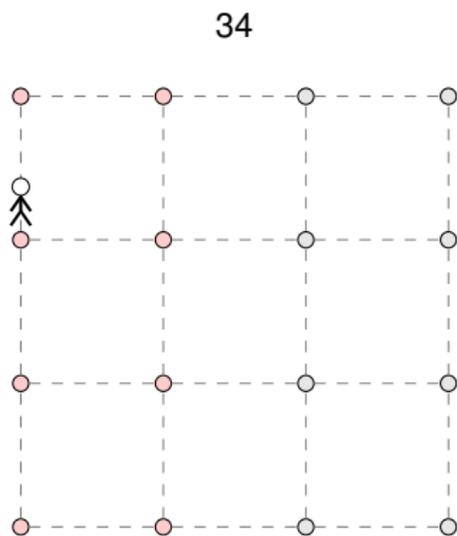
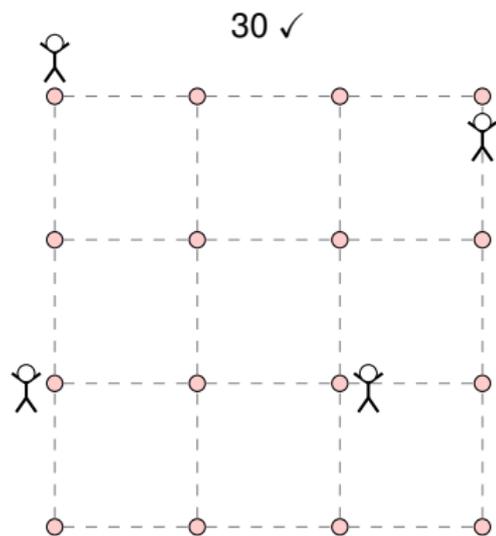
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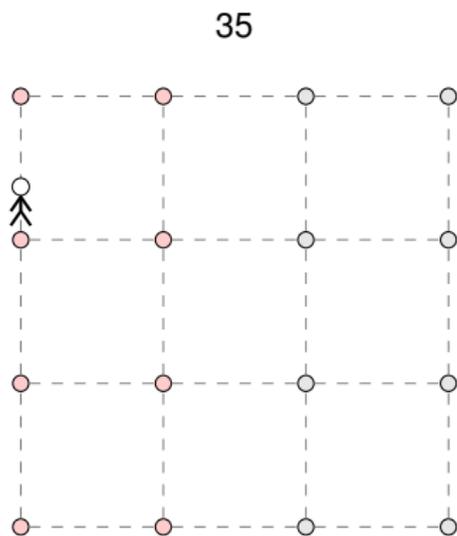
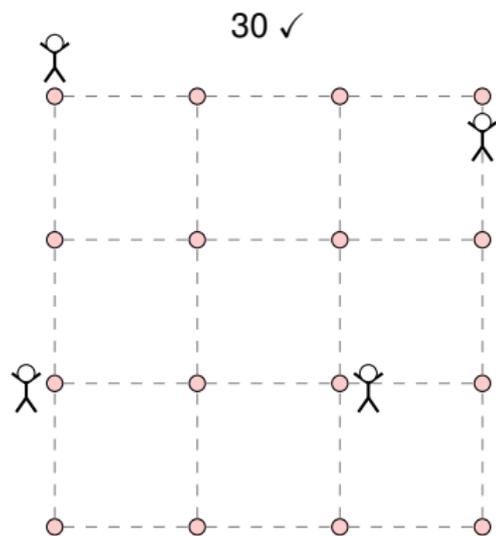
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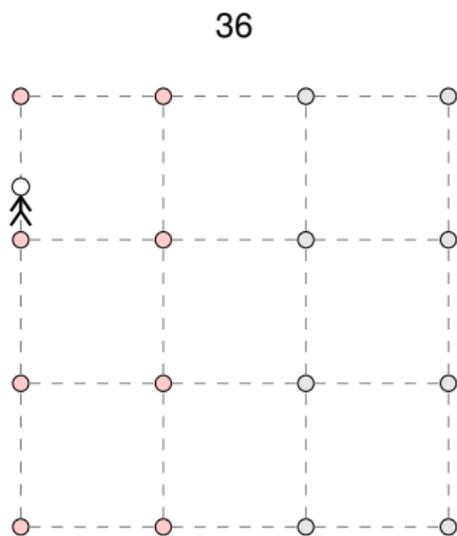
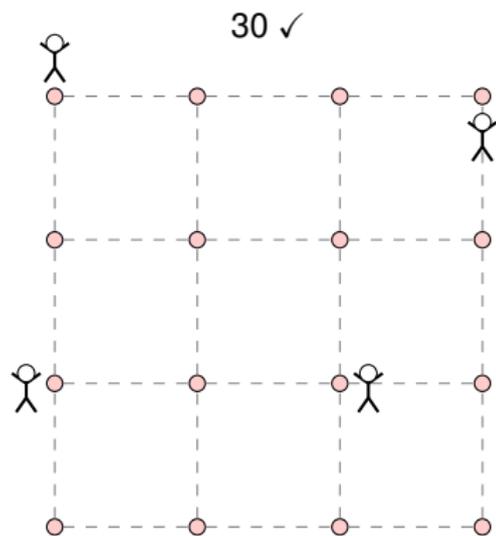
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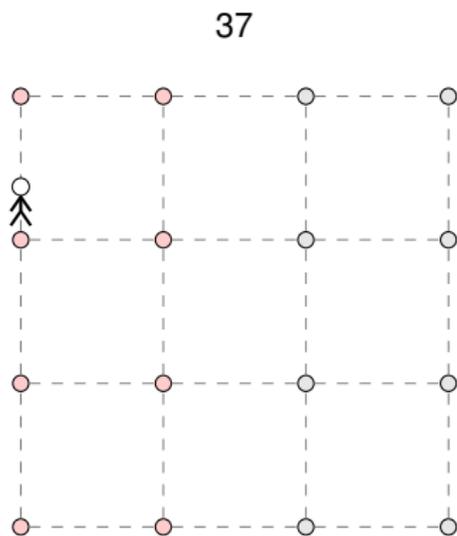
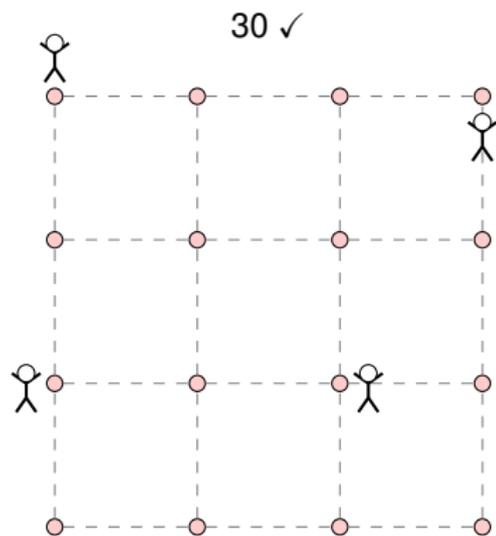
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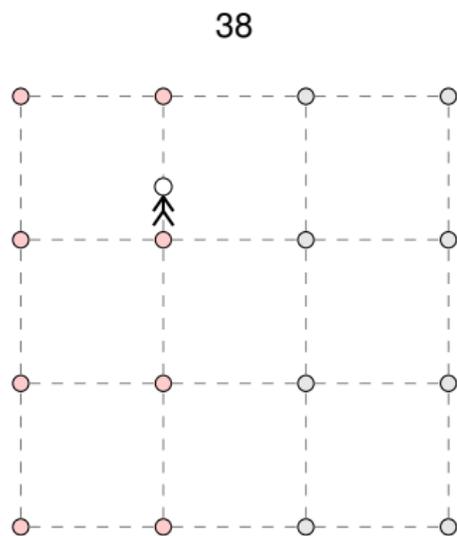
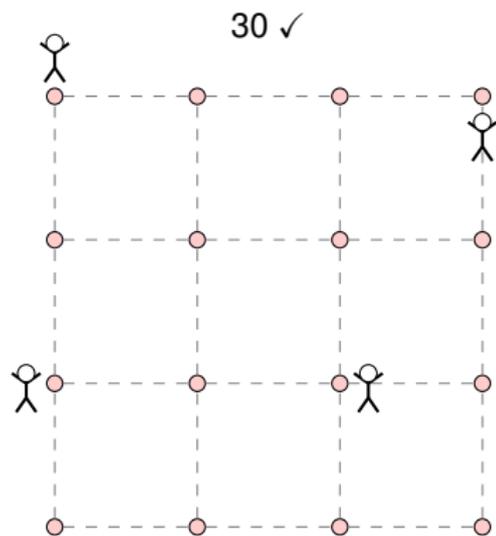
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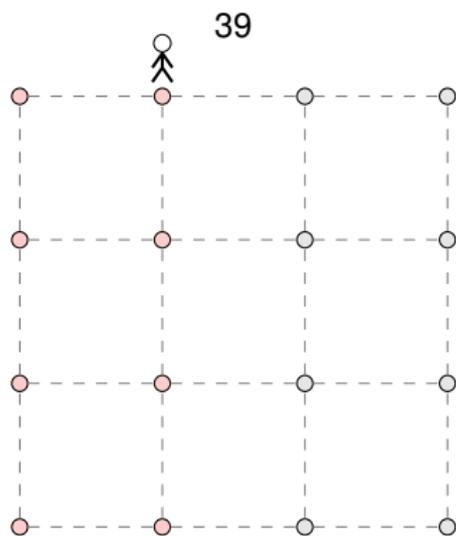
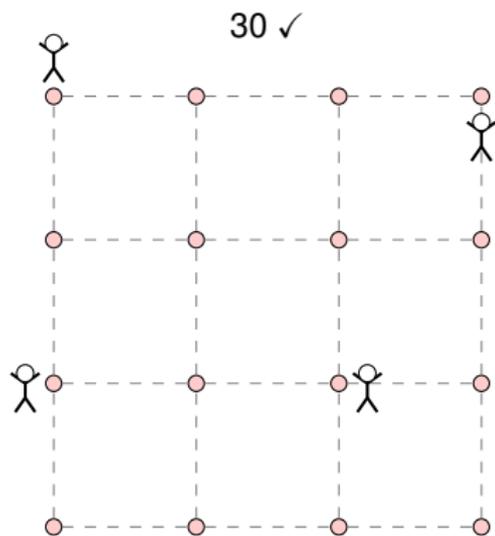
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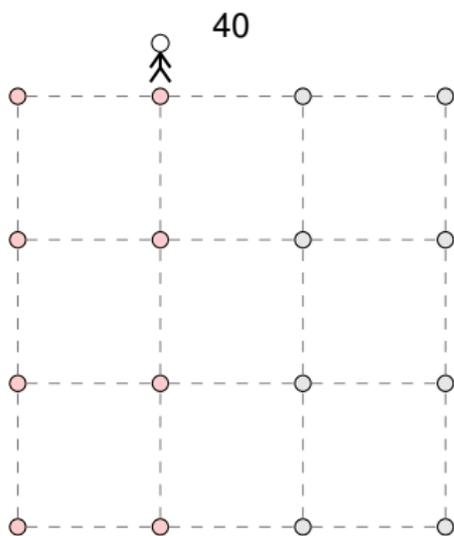
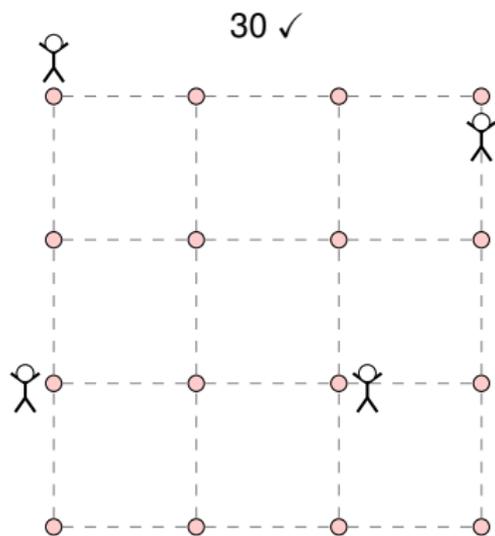
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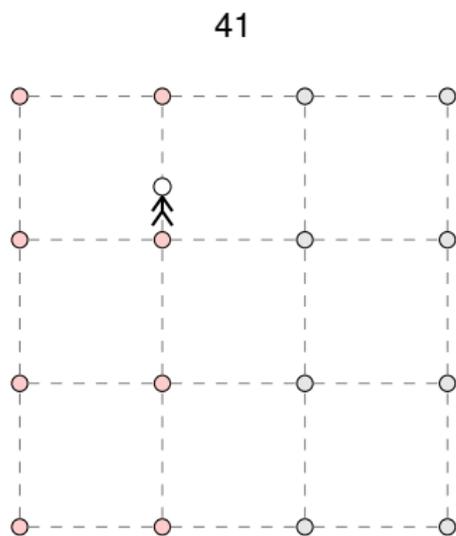
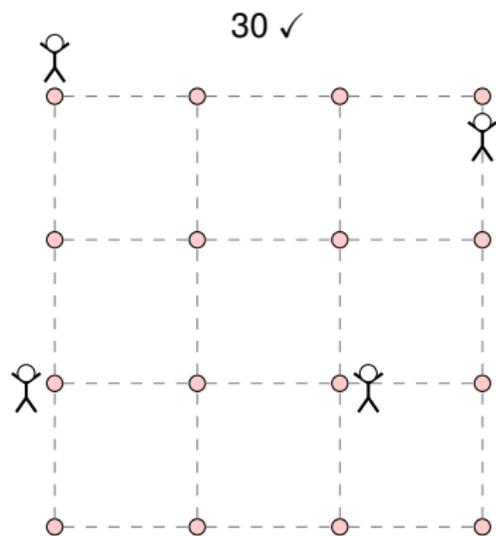
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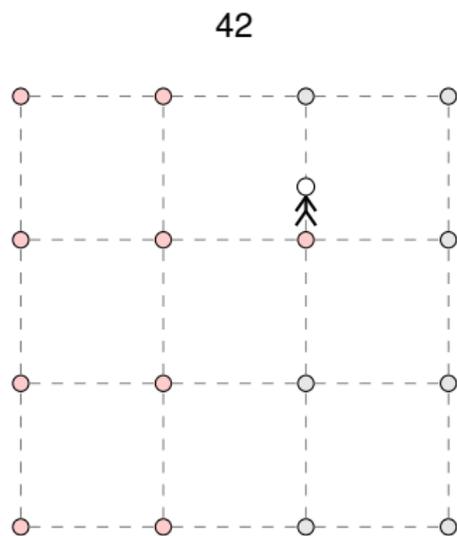
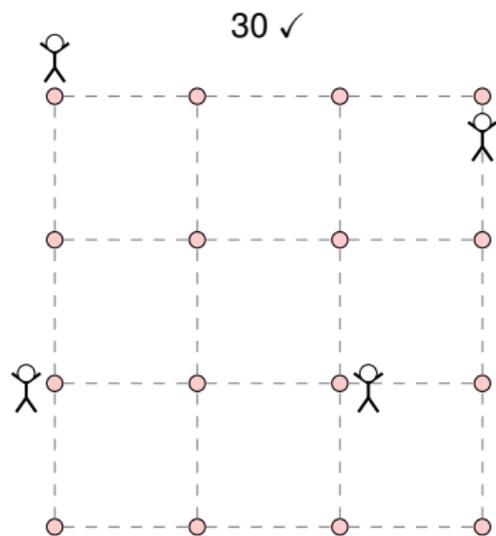
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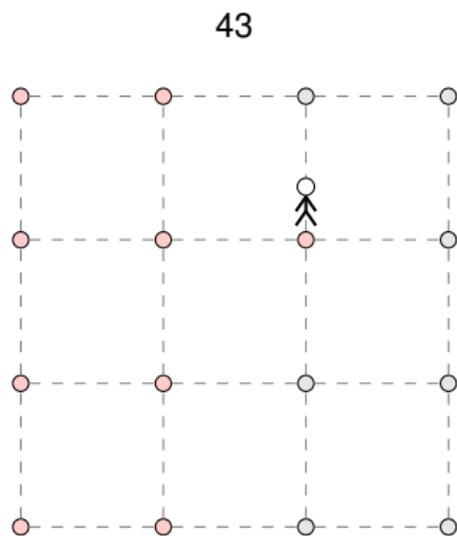
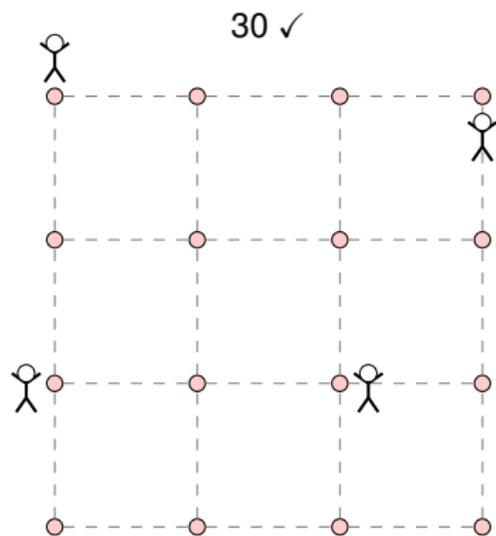
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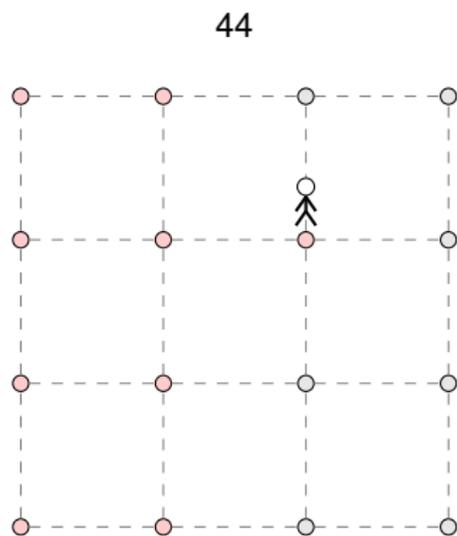
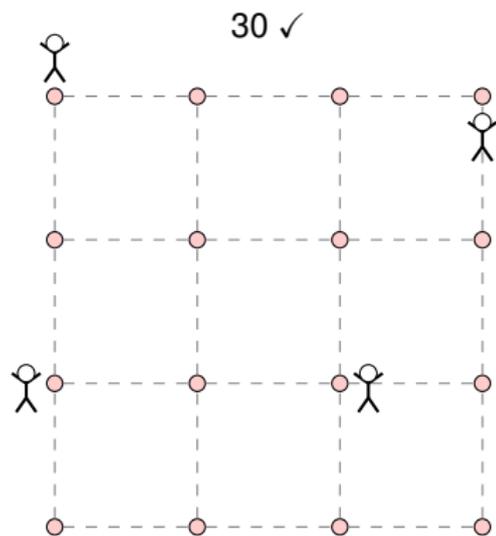
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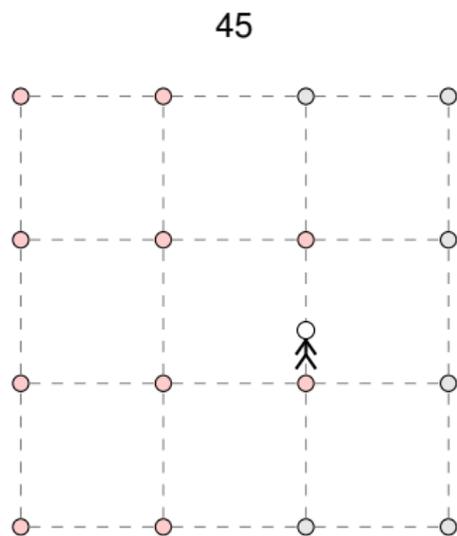
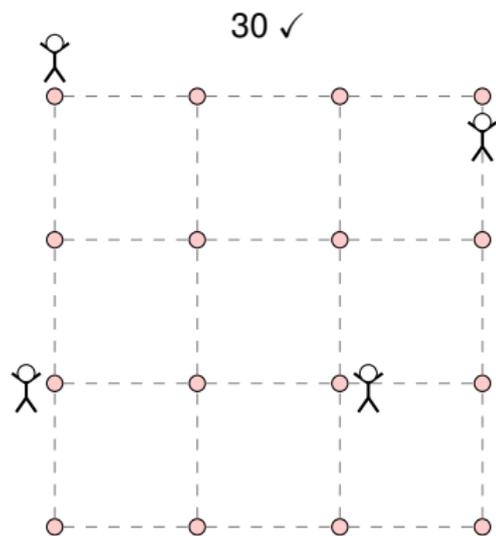
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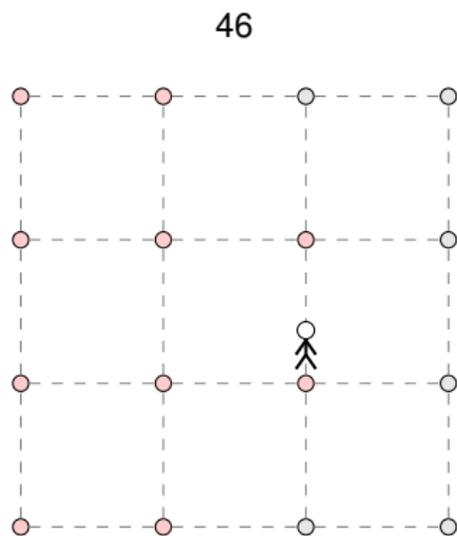
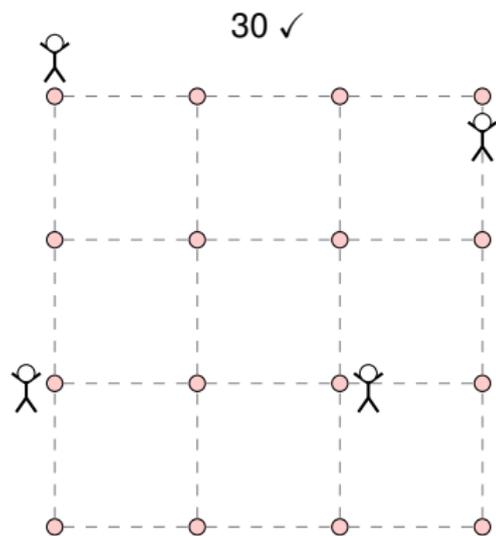
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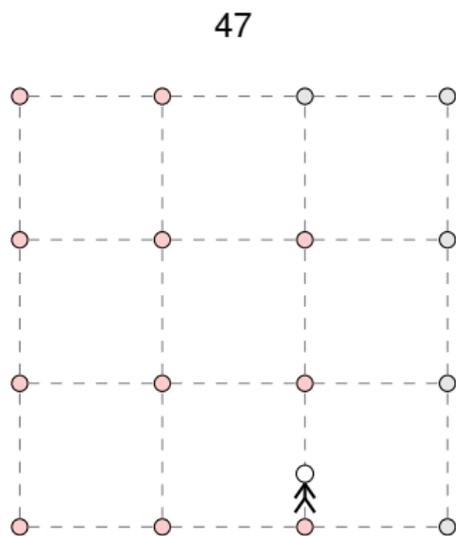
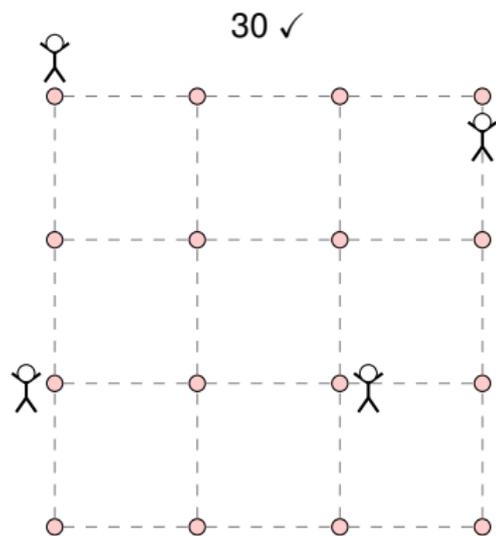
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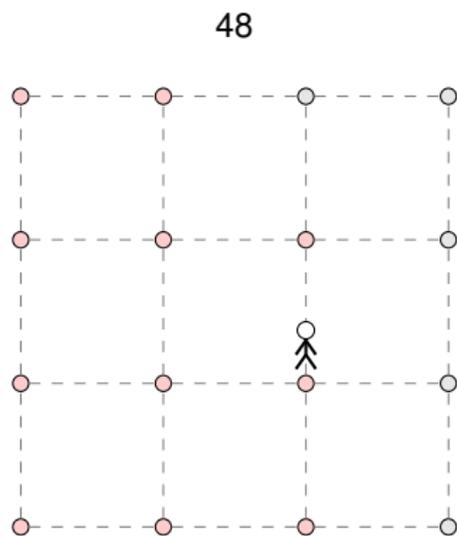
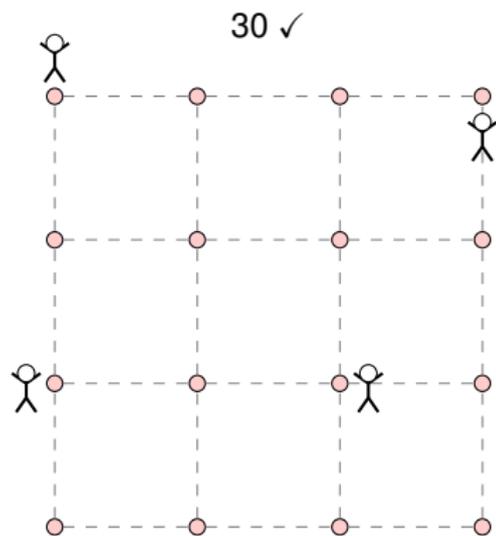
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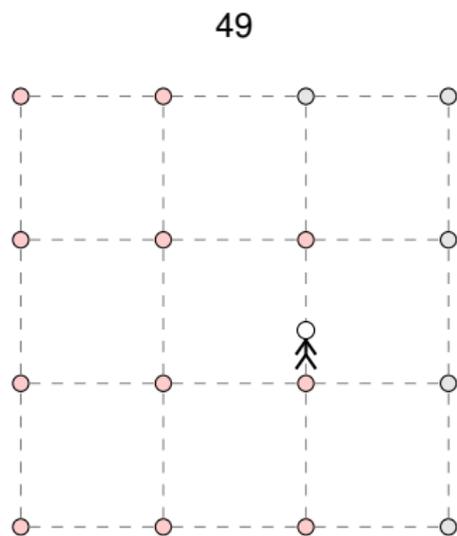
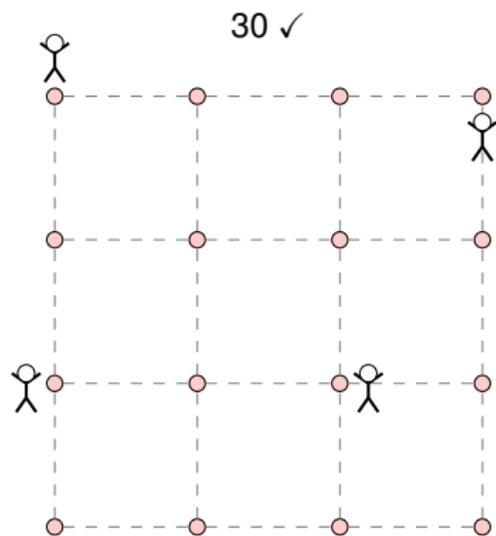
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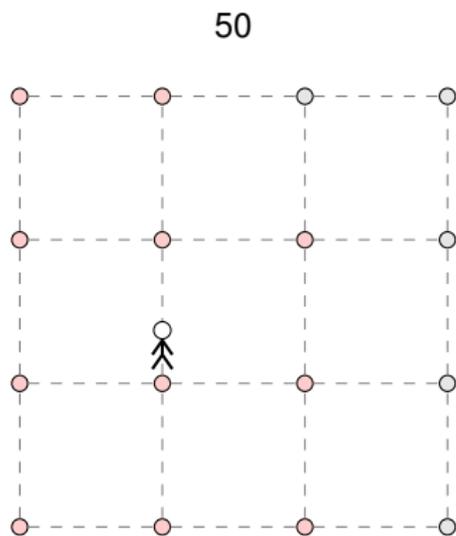
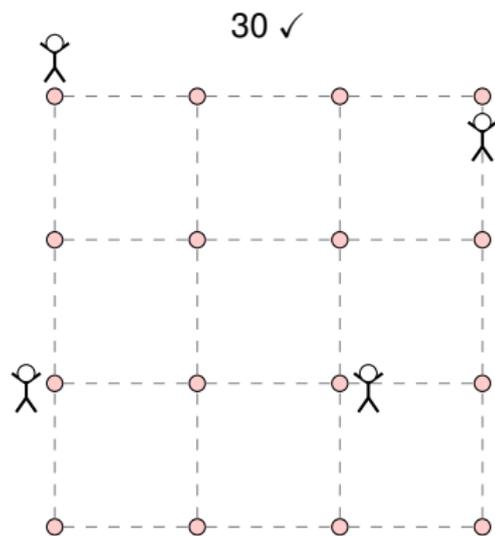
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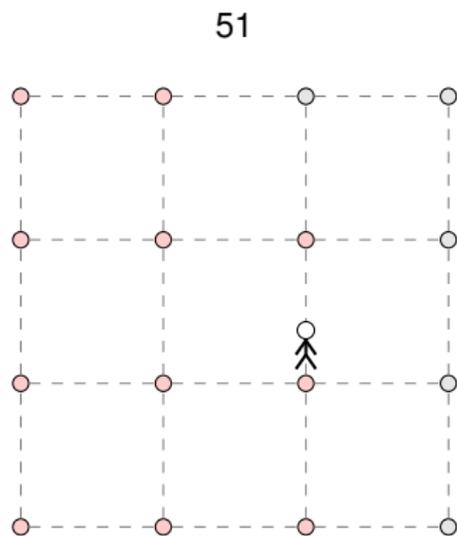
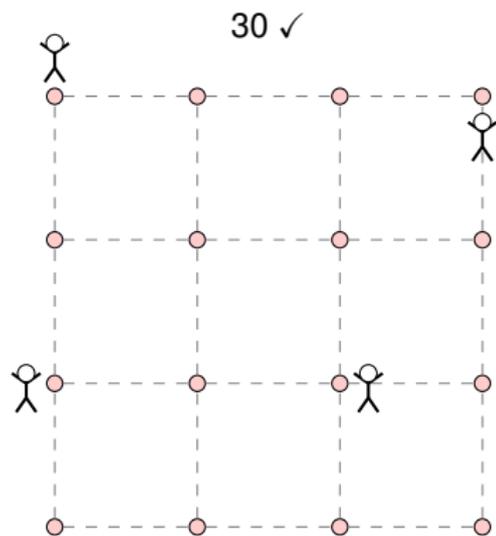
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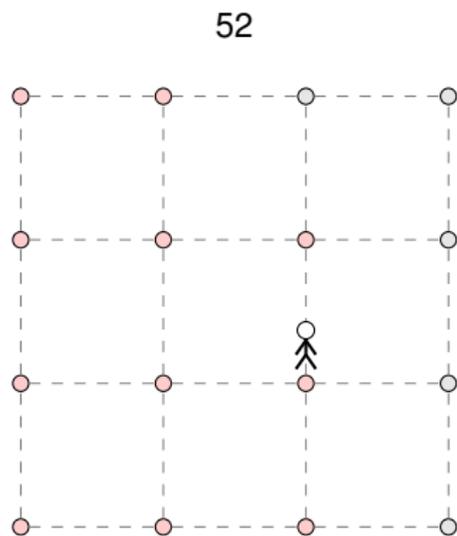
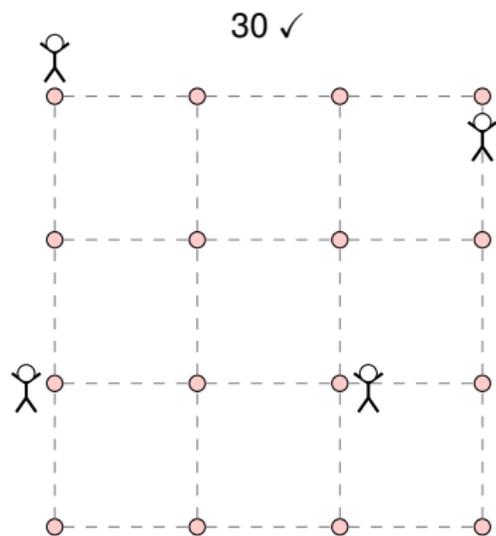
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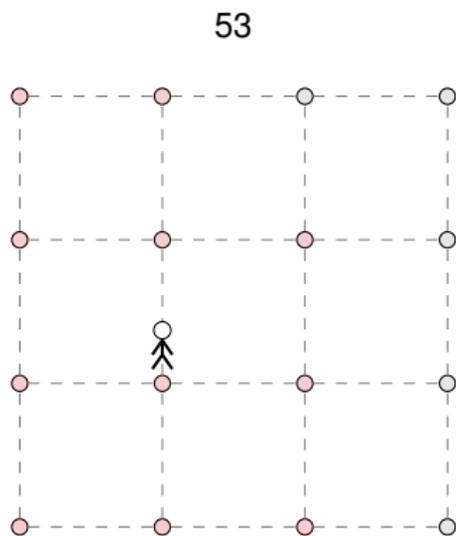
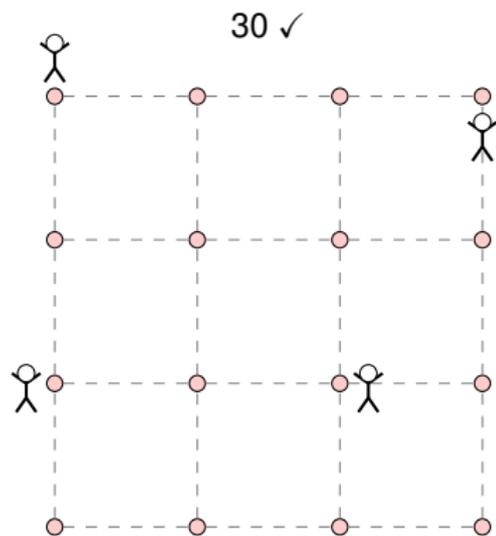
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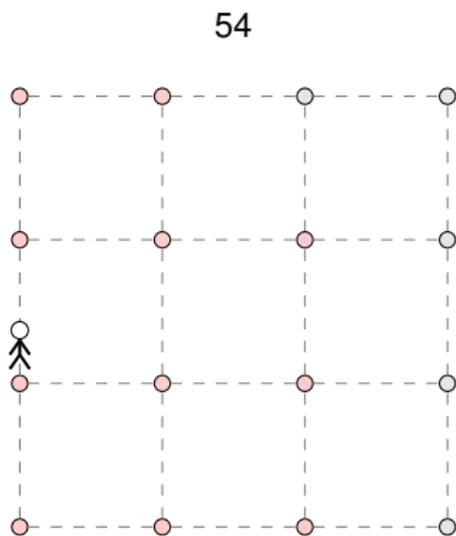
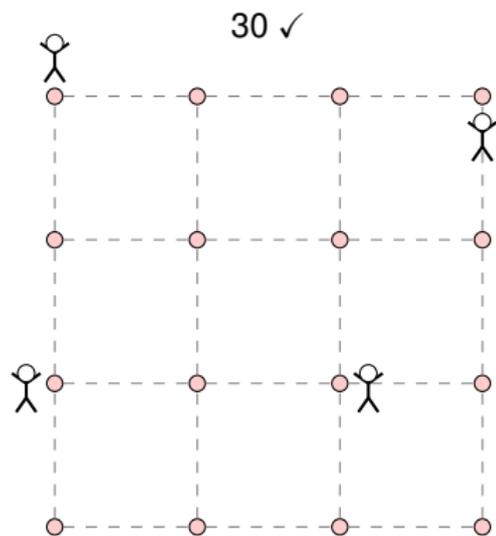
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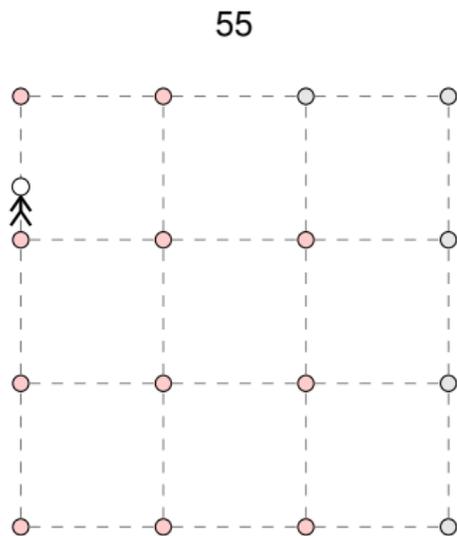
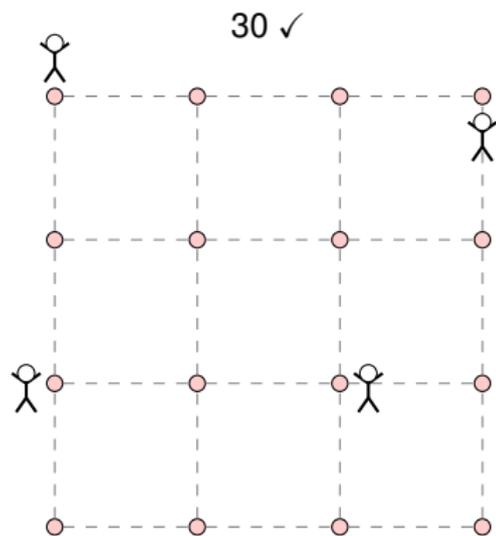
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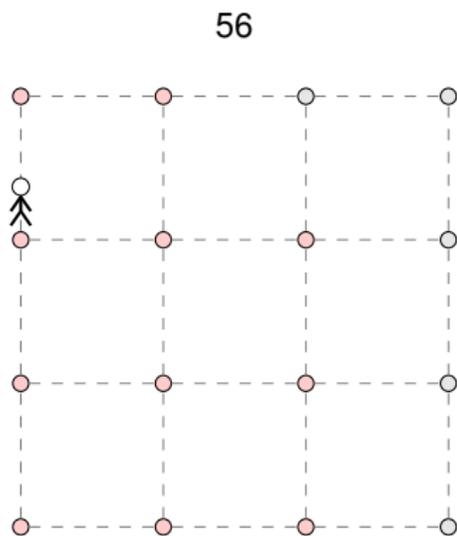
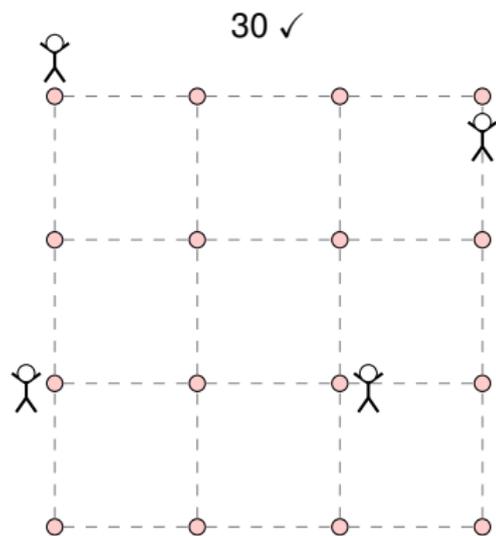
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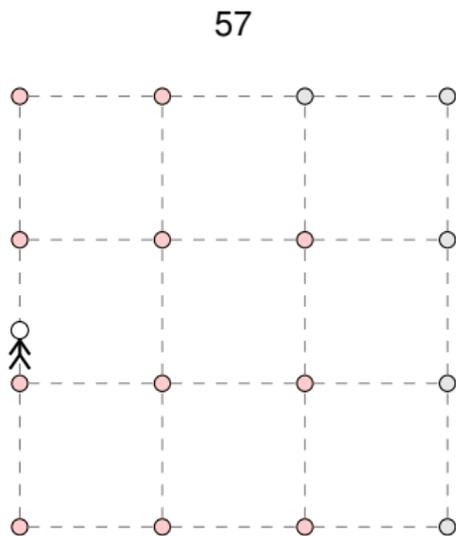
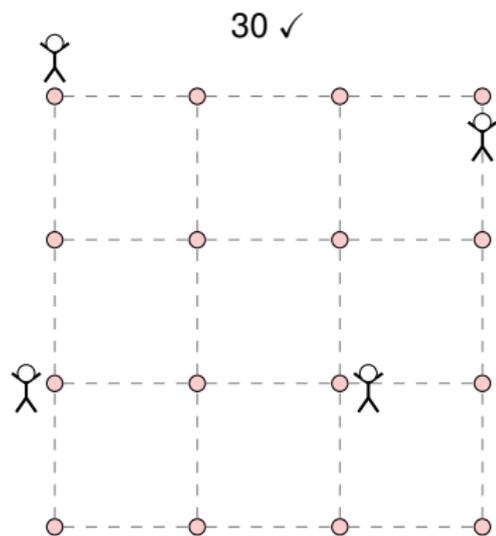
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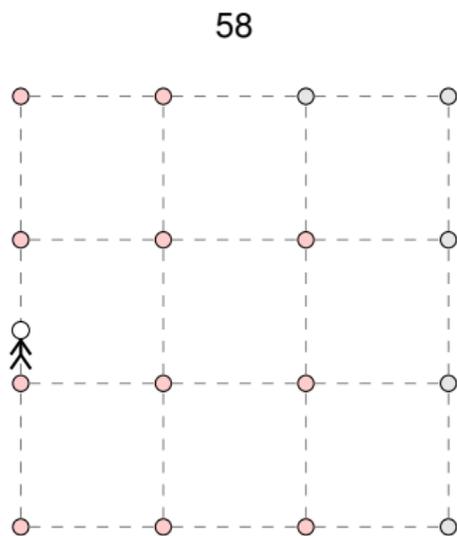
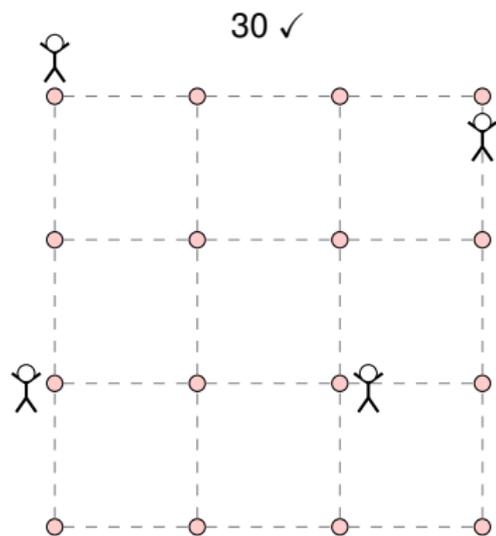
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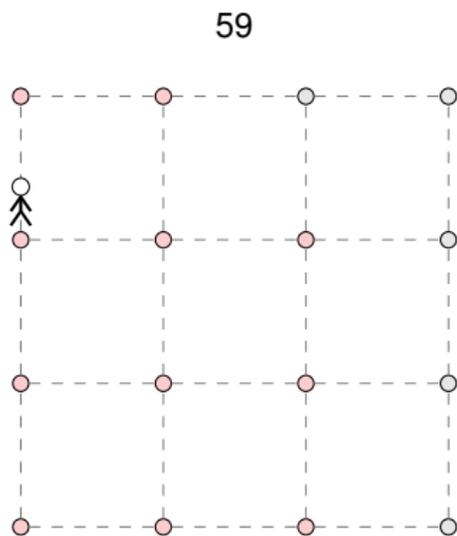
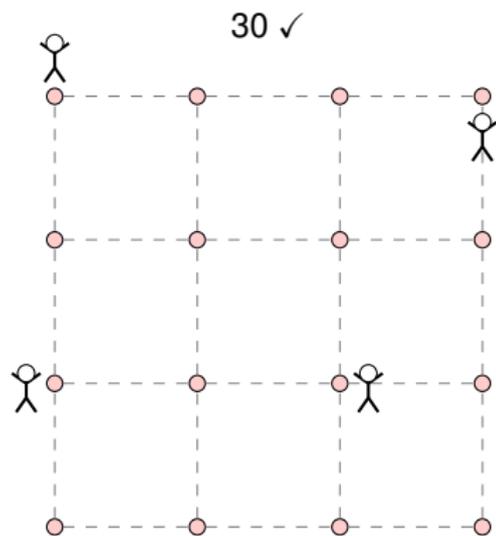
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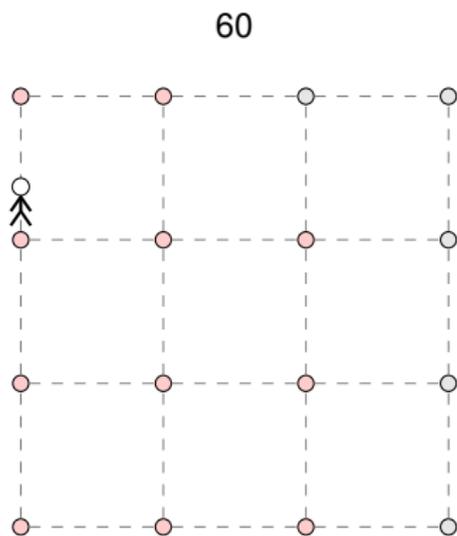
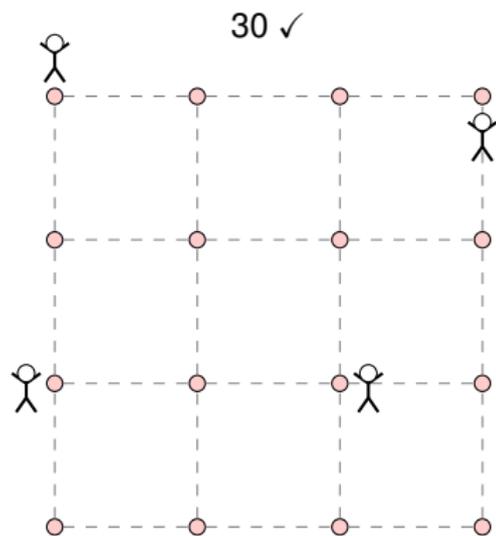
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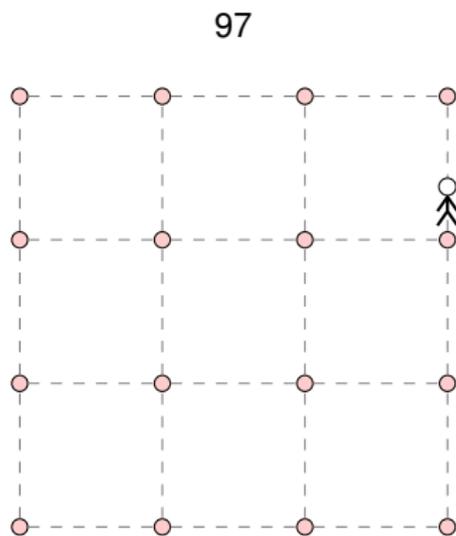
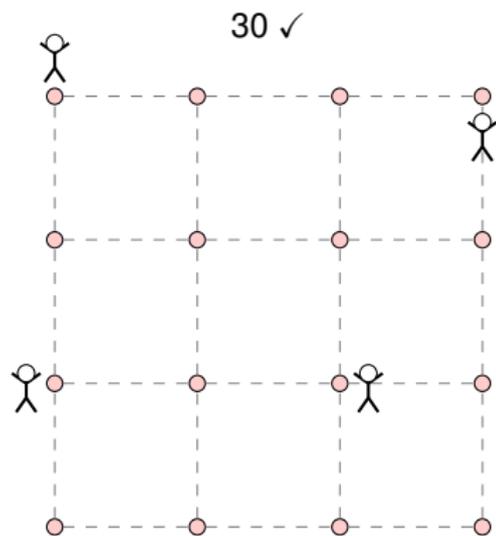
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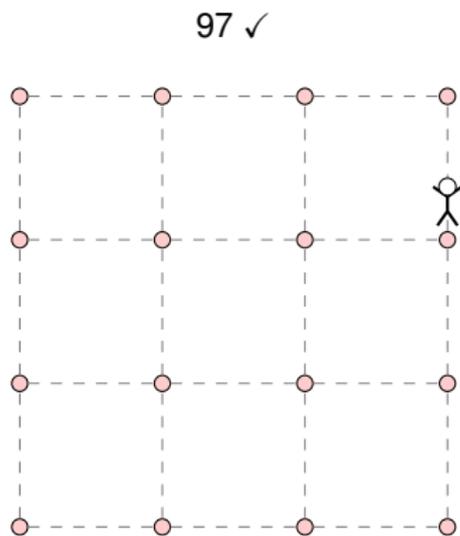
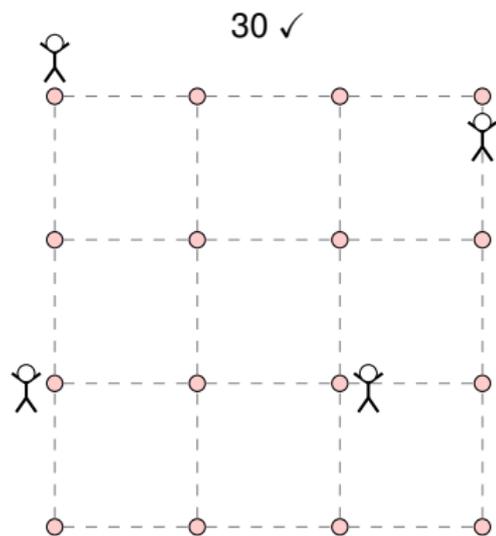
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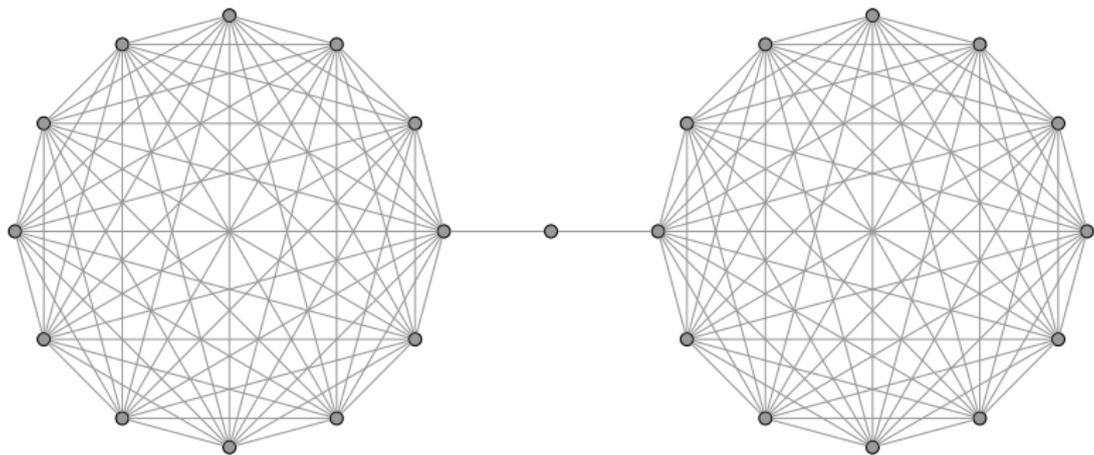
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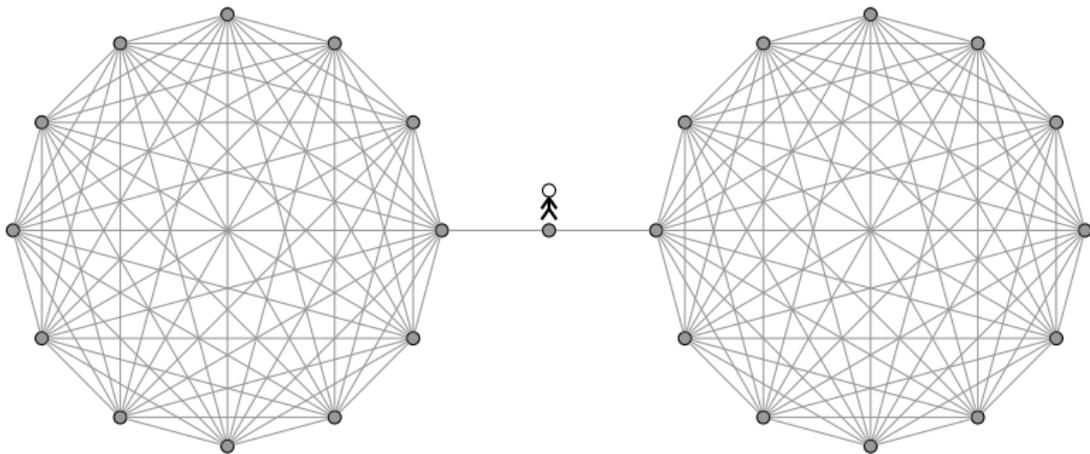
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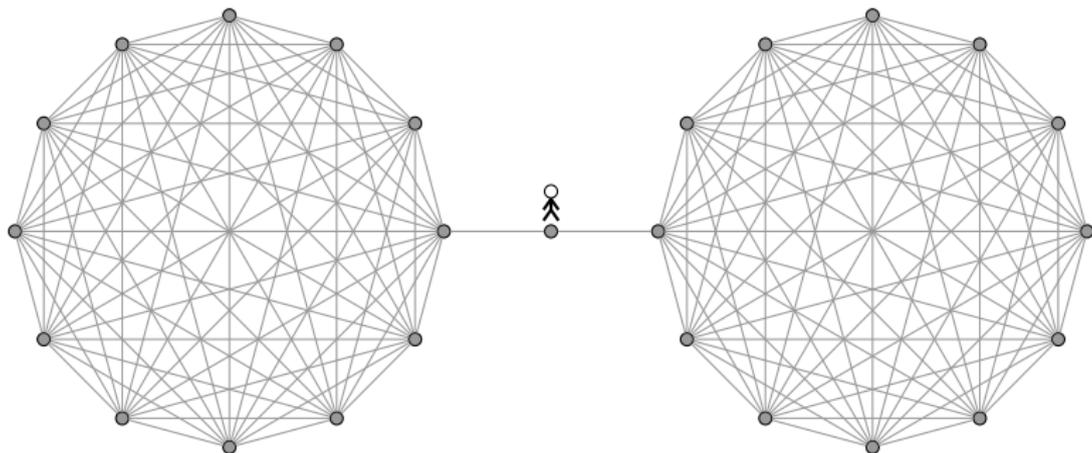
Speed-up for Best-Case Start Vertices



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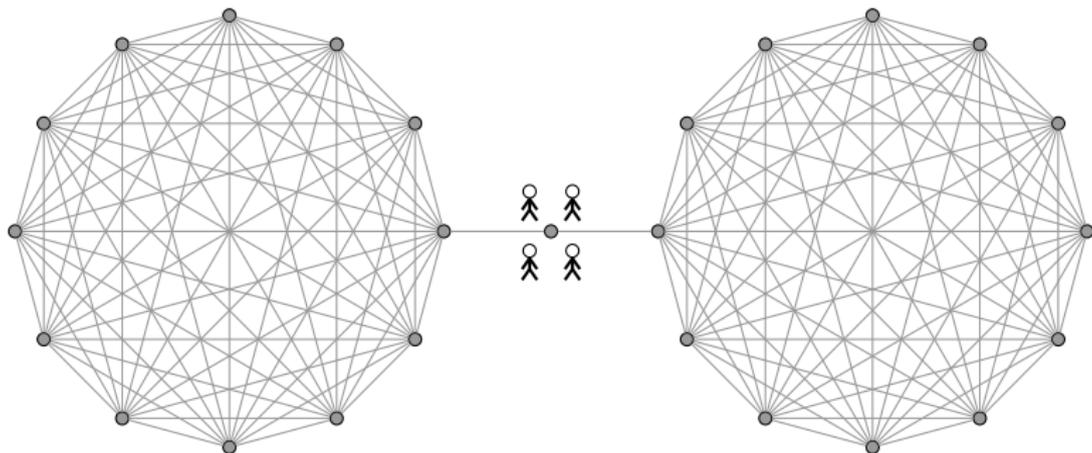
Speed-up for Best-Case Start Vertices



- $k = 1$: $\min_{u \in V} \mathbf{E}_u [\min \{ t : \cup_{s=0}^t X_s = V \}] \asymp n^2$

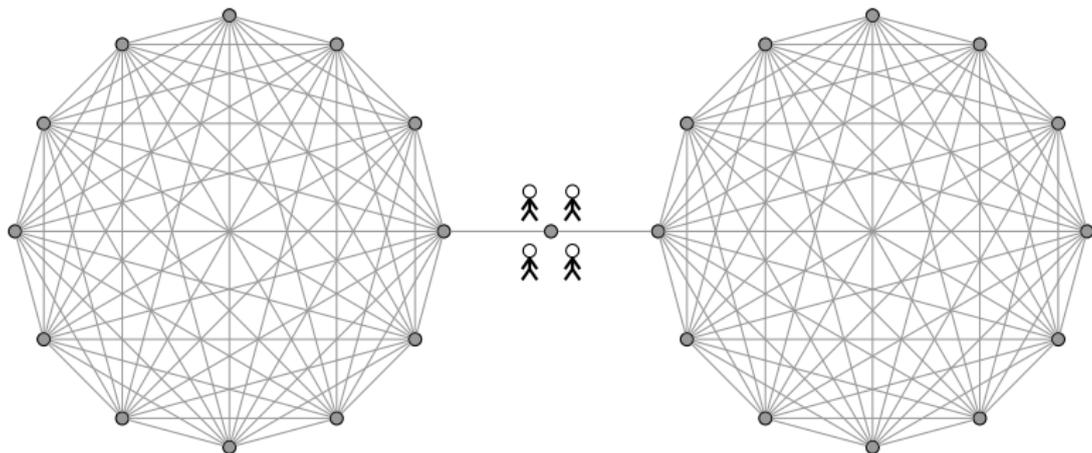


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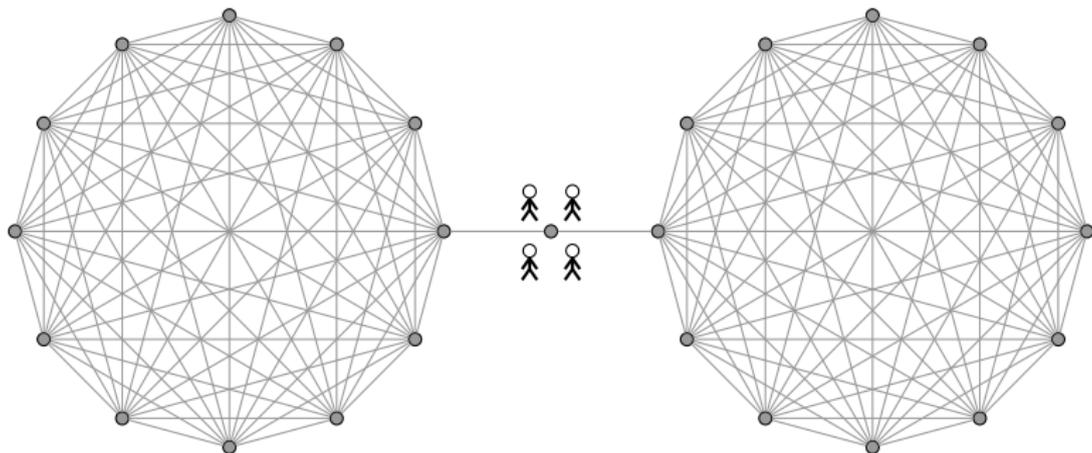
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- $k = 1$: $\min_{u \in V} \mathbf{E}_u \left[\min \{ t : \cup_{s=0}^t X_s = V \} \right] \asymp n^2$
- $k = \log n$: $\min_{u \in V} \mathbf{E}_{(u, \dots, u)} \left[\min \{ t : \cup_{i=1}^{\log n} \cup_{s=0}^t X_s^i = V \} \right] \asymp n$



Speed-up for Best-Case Start Vertices

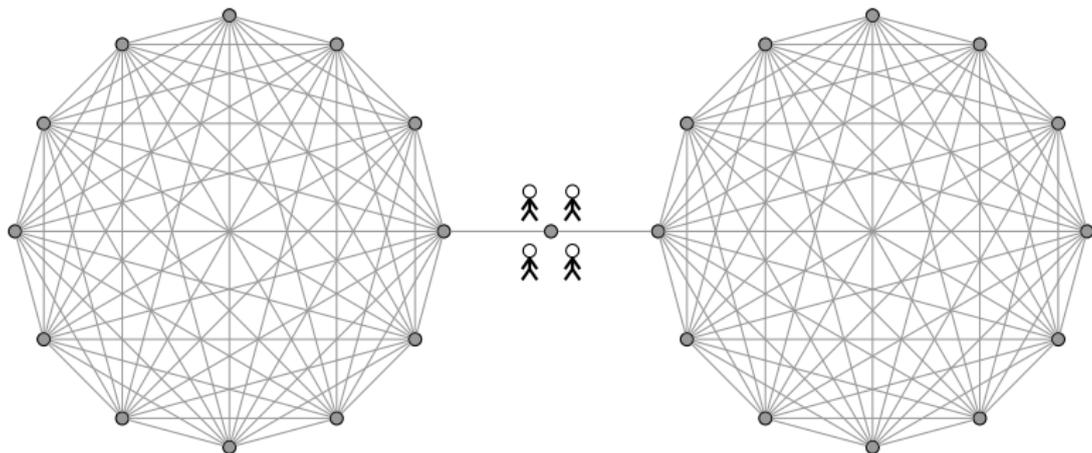


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$$\Rightarrow S^{(\log n)}$$



Speed-up for Best-Case Start Vertices

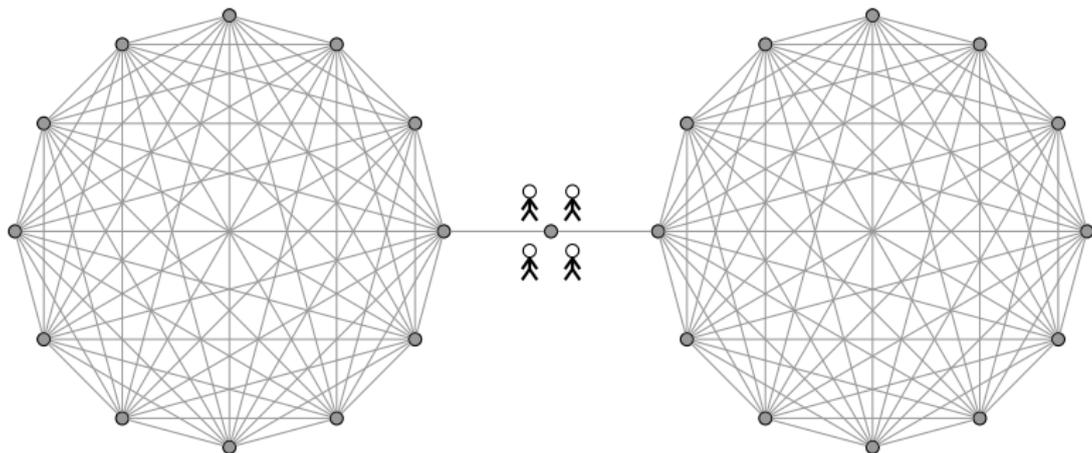


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$$\Rightarrow S^{(\log n)} = \frac{\min_{u \in V} \mathbf{E}_u [\min \{ t : \cup_{s=0}^t X_s = V \}]}{\min_{u \in V} \mathbf{E}_{(u, \dots, u)} [\min \{ t : \cup_{i=1}^{\log n} \cup_{s=0}^t X_s^i = V \}]}$$



Speed-up for Best-Case Start Vertices



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Speed-up for Stationary Start Vertices

Theorem [*Broder, Karlin, Raghavan and Upfal*]

For any graph G , the cover time for k stationary random walks satisfies

$$\mathbf{E}_{(\pi, \dots, \pi)} \left[\min \{ t : \cup_{i=1}^k \cup_{s=0}^t X_s^i = V \} \right] \lesssim \left(\frac{|E|}{k} \right)^2 \cdot \log^3 n.$$

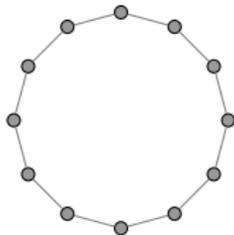


Speed-up for Stationary Start Vertices

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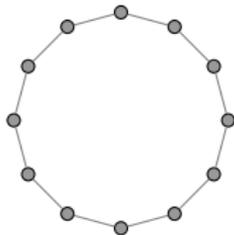


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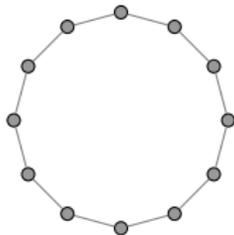


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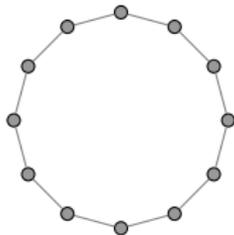


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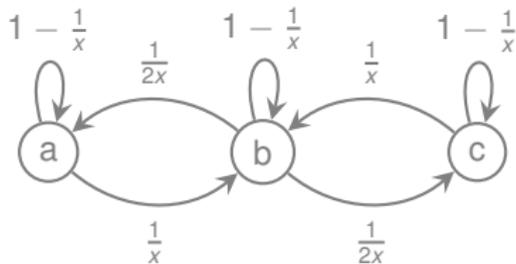
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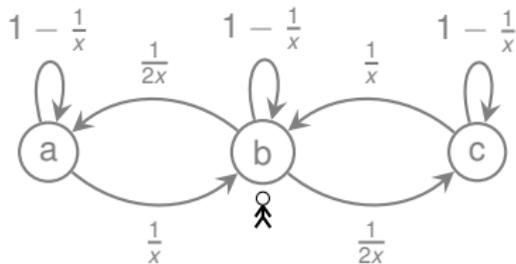
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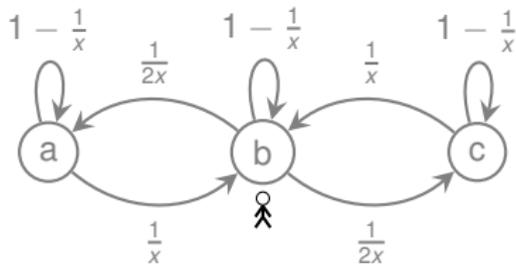


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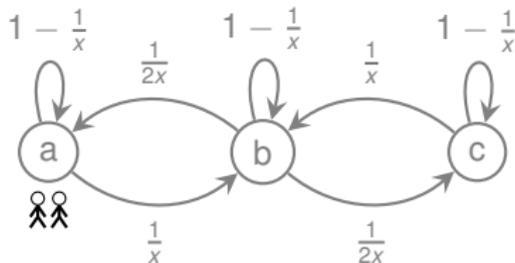


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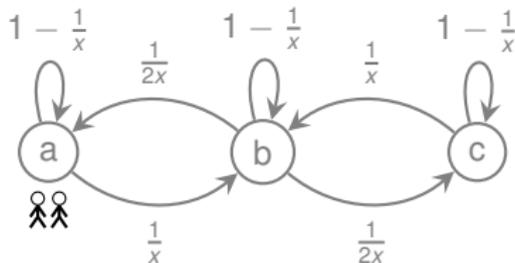


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- $S^{(k)} > k$ possible!



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Since $t_{\text{cov}} \lesssim n \log n \cdot t_{\text{mix}}$, the lower bound on $S^{(k)}$ will be always $o(k)$!



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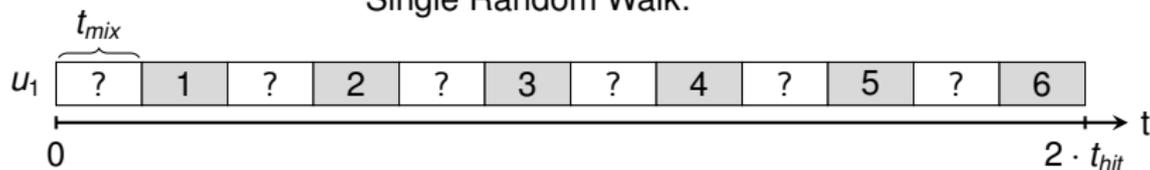
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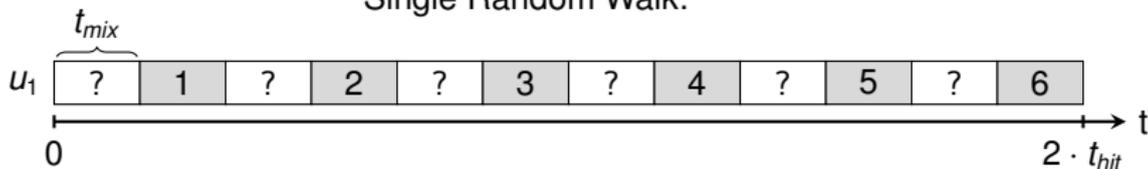
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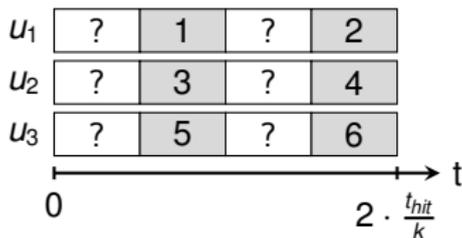
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Unnatural condition

Question: Does $t_{\text{cov}}^{(k)} \gtrsim \frac{n}{k} \cdot \log n$ hold for every k and every graph G ?



Conjecture: Does $S^{(k)} \lesssim k$ hold for every graph G ?



General Bound

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Bottlenecks

Conductance

$$\Phi(G) := \min_{\substack{\emptyset \subsetneq S \subsetneq V: \\ \text{vol}(S) \leq |E|}} \left\{ \frac{|E(S, V \setminus S)|}{\text{vol}(S)} \right\}$$



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Conductance

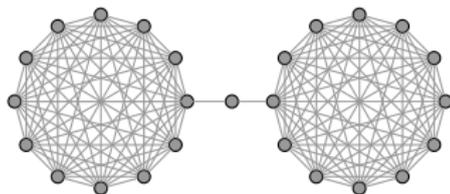
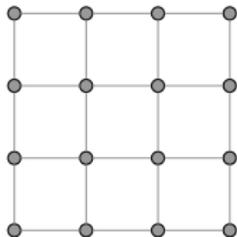
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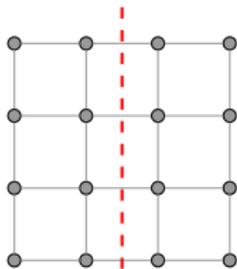
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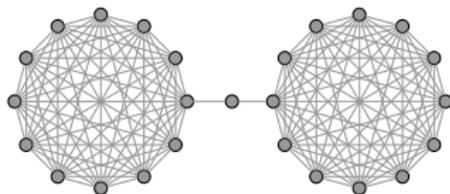
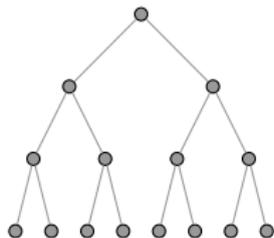
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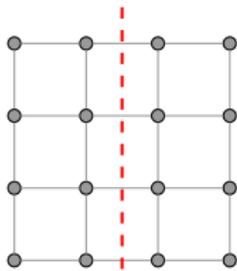
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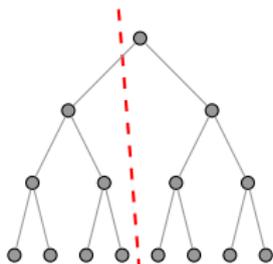
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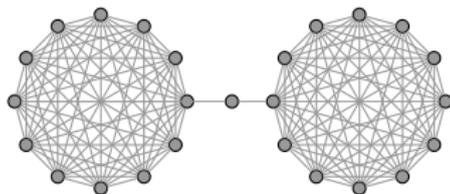
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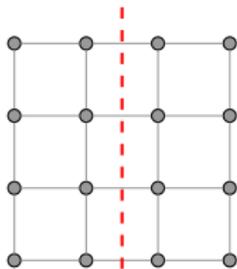
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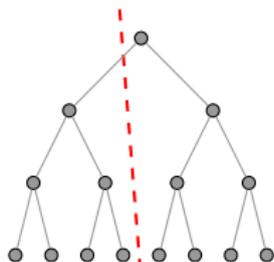
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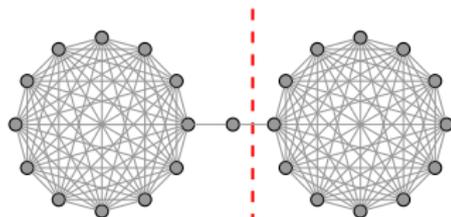
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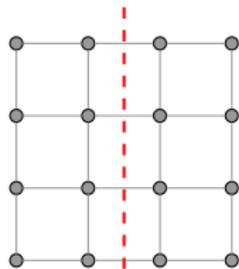
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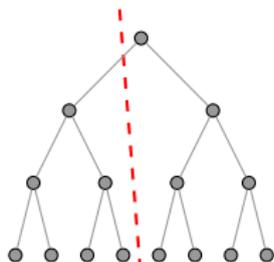
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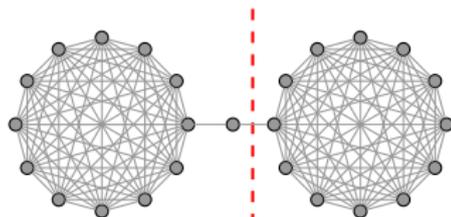
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Some bottlenecks slow down single and multiple random walks, others only affect multiple random walks.



Relating Conductance to Multiple Cover Time

Proposition

Let $G = (V, E)$ be any graph and $1 \leq k \leq n$. Then,

$$t_{\text{cov}}^{(k)} \gtrsim \sqrt{\frac{n}{k} \cdot \frac{1}{\Phi}}.$$



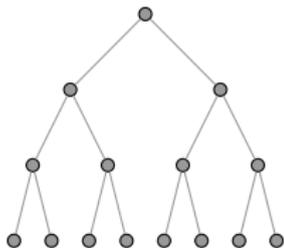
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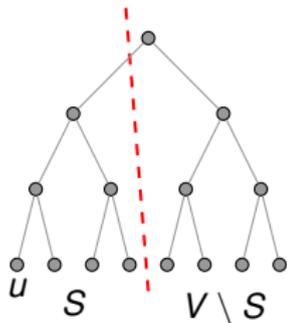
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Relating Conductance to Multiple Cover Time

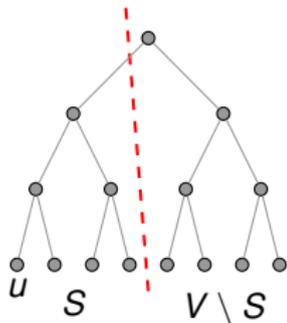
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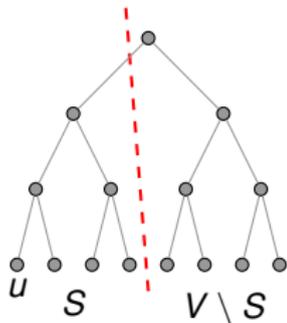
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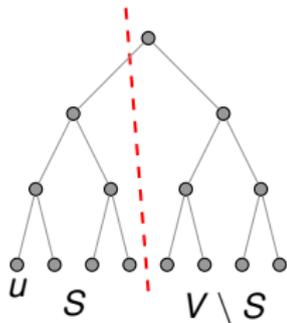
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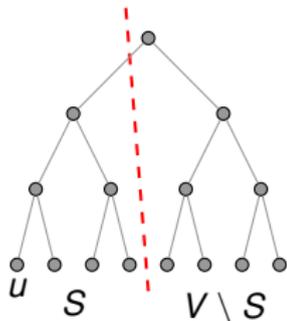
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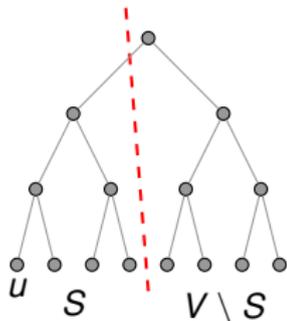
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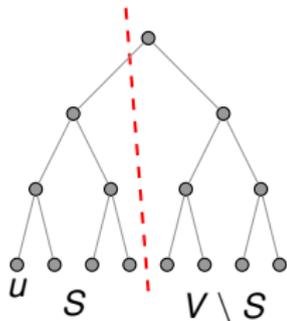
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Let $G = (V, E)$ be any graph and $1 \leq k \leq n$. Then,

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Corollary

$$\sqrt{\frac{1}{\Phi}} \lesssim t_{\text{cov}}^{(n)} \lesssim \frac{\log^2 n}{\Phi^2}$$



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Corollary

Cover time of n multiple random walks captures rapid mixing

$$\sqrt{\frac{1}{\Phi}} \lesssim t_{\text{cov}}^{(n)} \lesssim \frac{\log^2 n}{\Phi^2}$$



Outline

Introduction

Upper Bounds on the Multiple Cover Time

Lower Bounds on the Multiple Cover Time

Concrete Networks

Conclusion



1D Grid

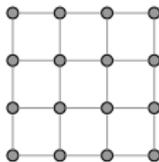


$$t_{mix} \asymp n^2$$

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2D Grid

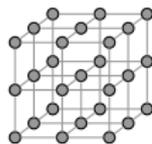


$$t_{mix} \asymp n$$

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3D Grid



$$t_{mix} \asymp n^{2/3}$$

$$t_{hit} \asymp n$$

$$t_{cov} \asymp n \log n$$



1D Grid

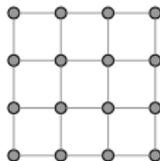


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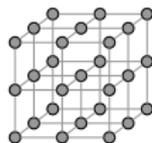


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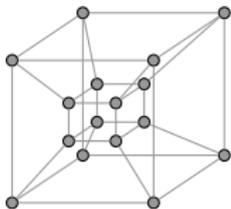


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Hypercube

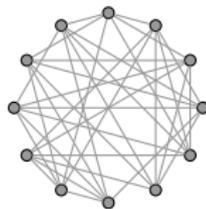


$$t_{mix} \asymp \log n \log \log n$$

$$t_{hit} \asymp n$$

$$t_{cov} \asymp n \log n$$

Expander Graph



$$t_{mix} \asymp \log n$$

$$t_{hit} \asymp n$$

$$t_{cov} \asymp n \log n$$

Binary Tree



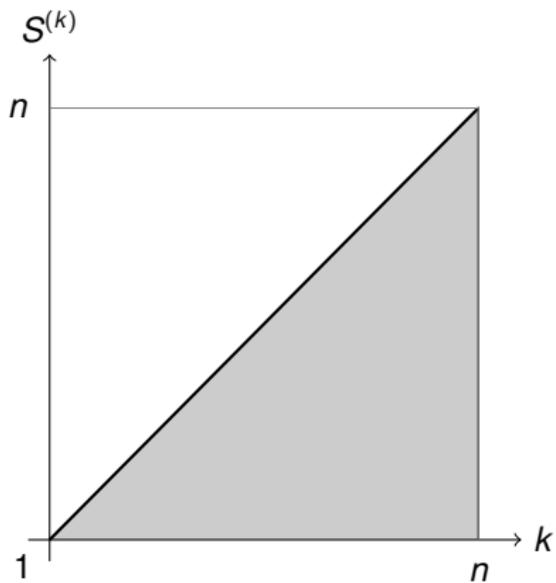
$$t_{mix} \asymp n$$

$$t_{hit} \asymp n \log n$$

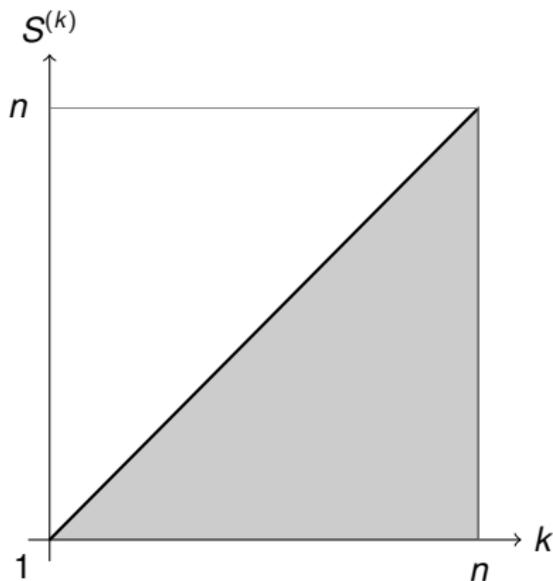
$$t_{cov} \asymp n \log^2 n$$



Speed-up for Expander Graphs



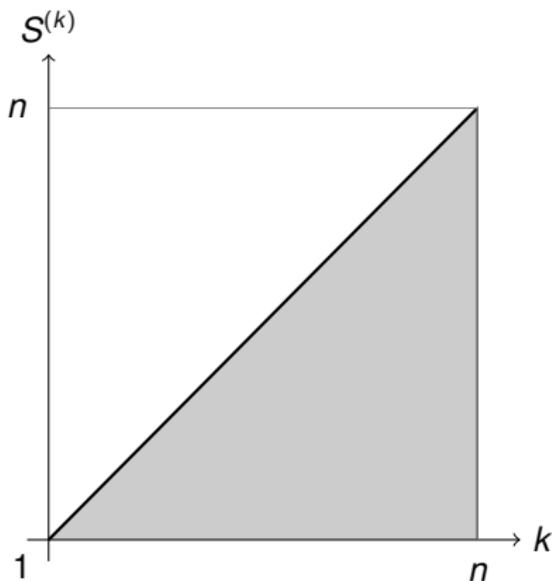
Speed-up for Expander Graphs



- $t_{cov}^{(k)} \lesssim \frac{t_{hit} \cdot \log n}{k} + t_{mix}$



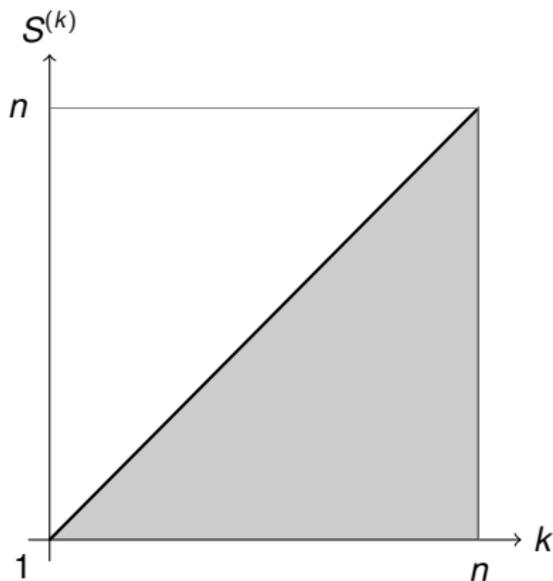
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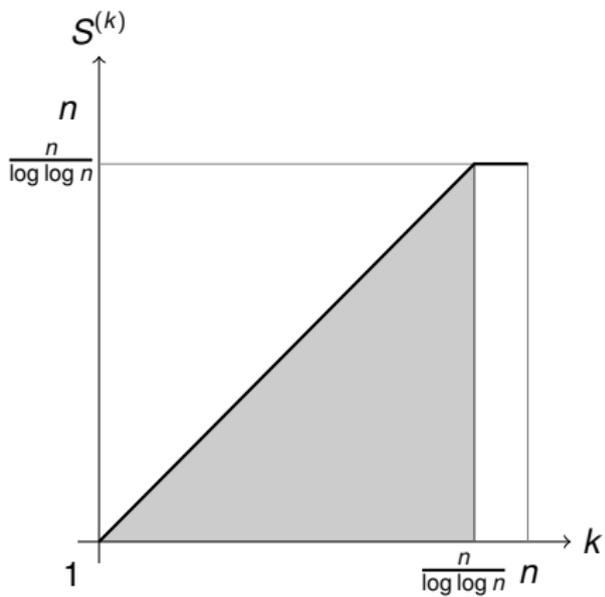
Speed-up for Expander Graphs



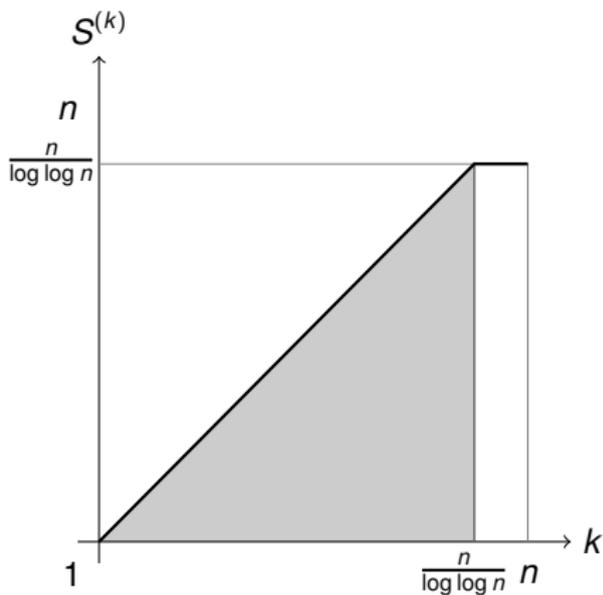
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Speed-up for Hypercubes



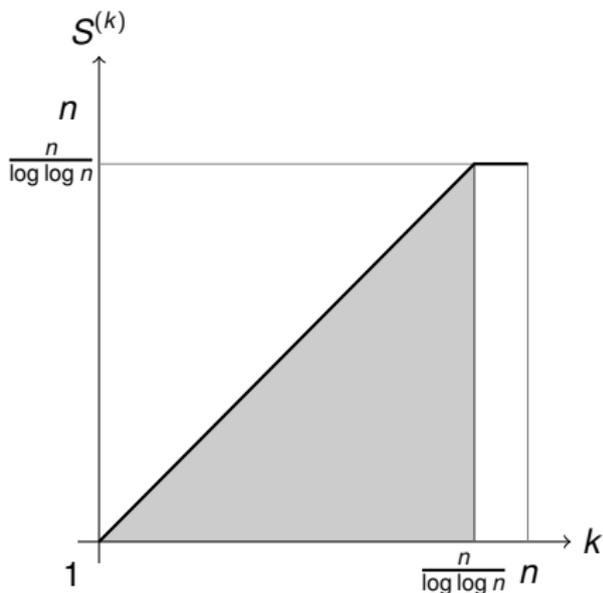
Speed-up for Hypercubes



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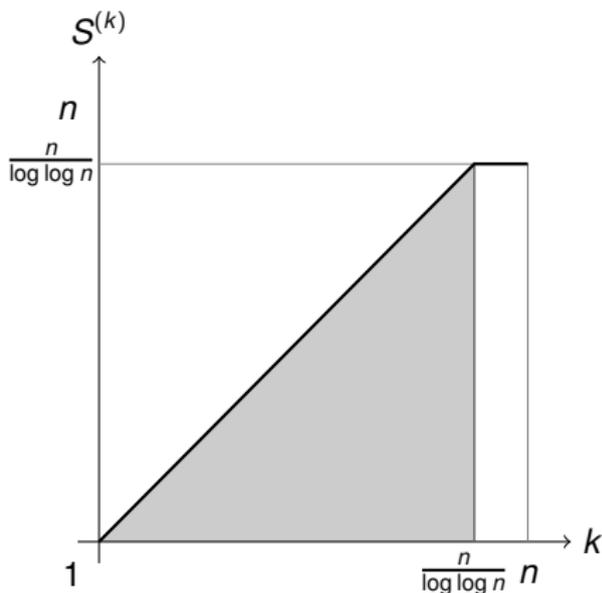
Speed-up for Hypercubes



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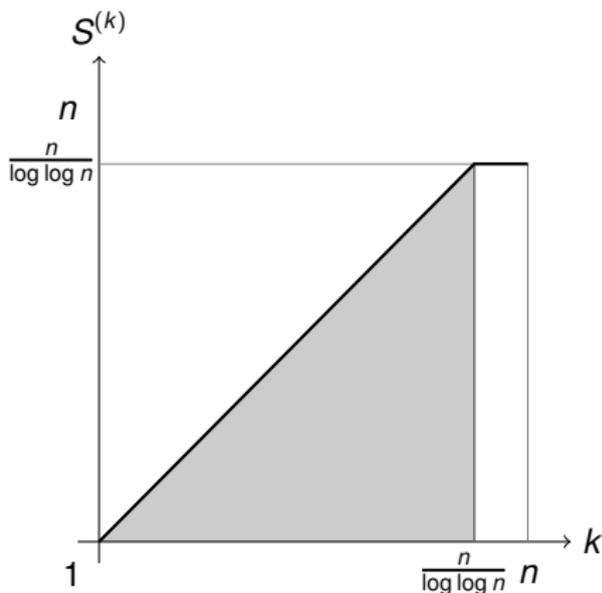
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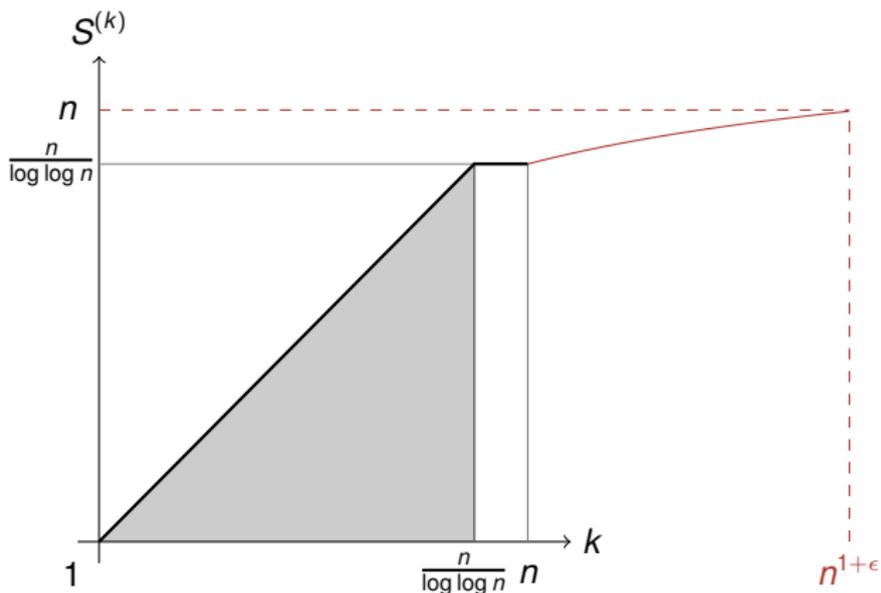
Speed-up for Hypercubes



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- $t_{cov}^{(n)} \gtrsim \log n \log \log n$ (due to coupon-collecting argument)



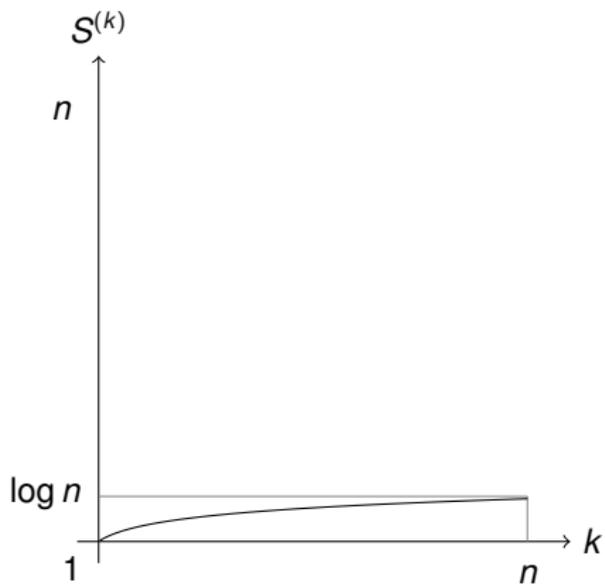
Speed-up for Hypercubes



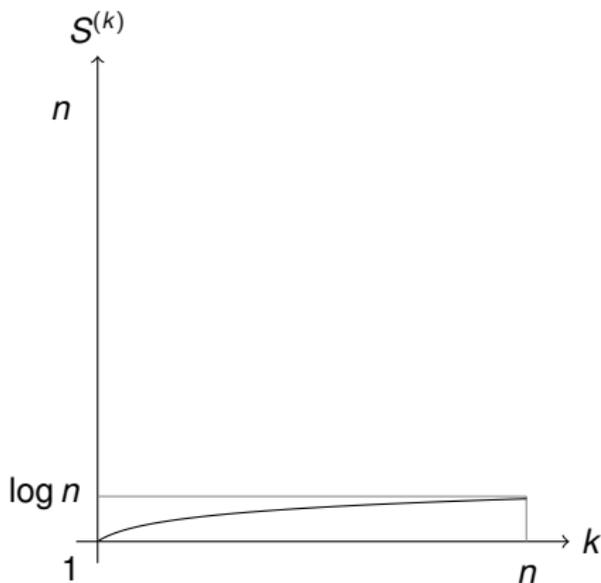
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Speed-up for Cycles



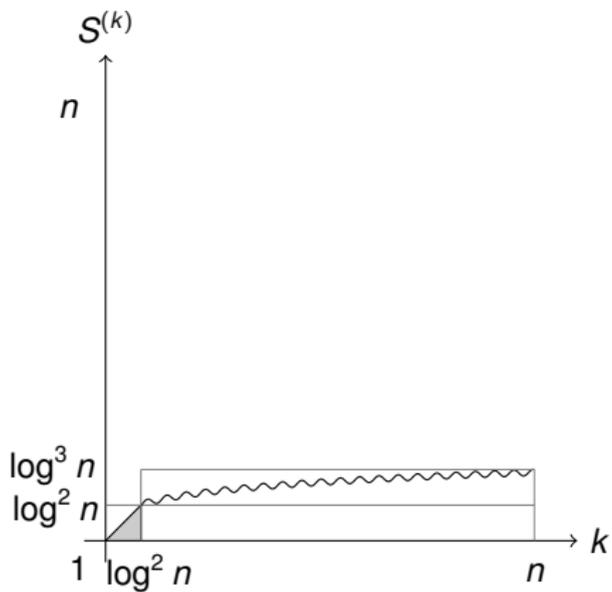
Speed-up for Cycles



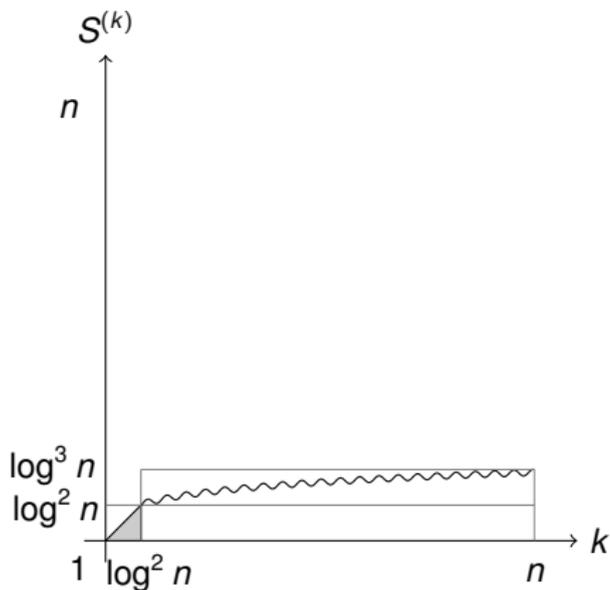
- Direct Analysis
[Alon, Avin, Koucký, Kozma, Lotker and Tuttle]



Speed-up for 2D Grids



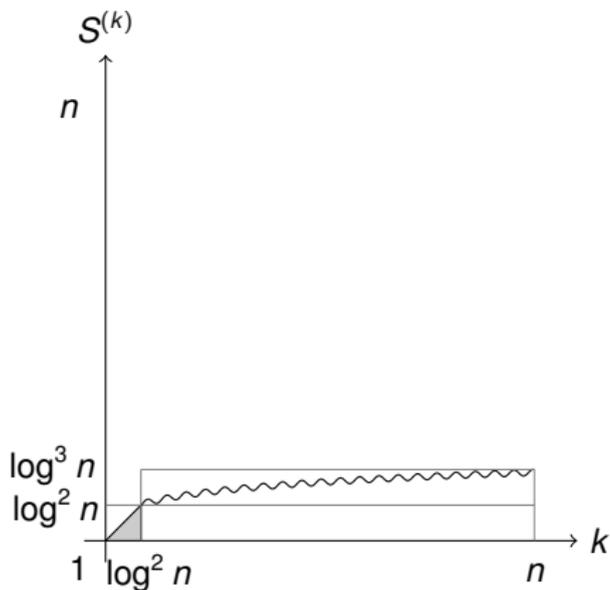
Speed-up for 2D Grids



- $t_{cov}^{(k)} \lesssim \frac{t_{hit} \cdot \log n}{k} + t_{mix}$



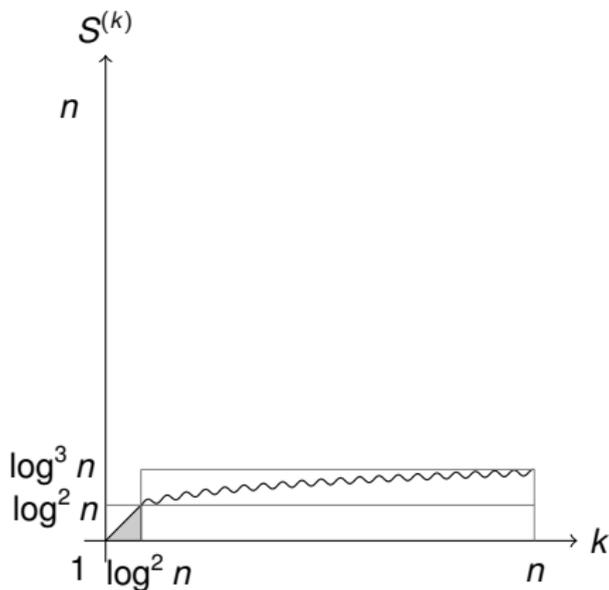
Speed-up for 2D Grids



- $t_{cov}^{(k)} \lesssim \frac{t_{hit} \cdot \log n}{k} + t_{mix} \lesssim \frac{n \cdot \log^2 n}{k} + n$



Speed-up for 2D Grids

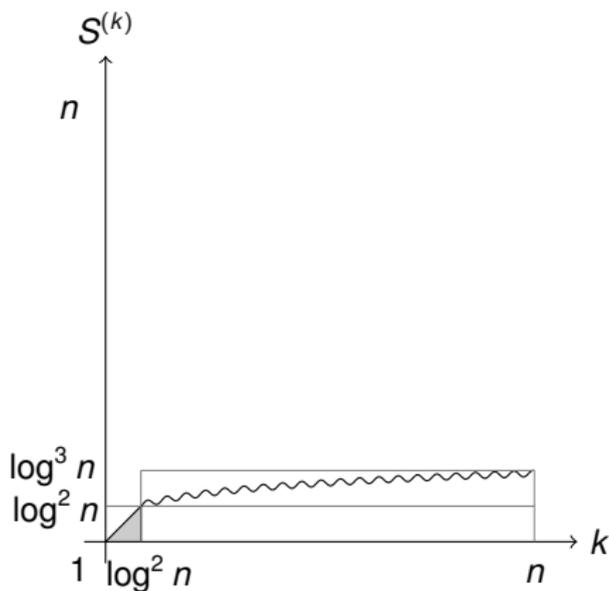


- $t_{cov}^{(k)} \lesssim \frac{t_{hit} \cdot \log n}{k} + t_{mix} \lesssim \frac{n \cdot \log^2 n}{k} + n$
- $t_{cov}^{(n)} \gtrsim \frac{\text{diam}(G)^2}{\log n}$

[Carne]



Speed-up for 2D Grids

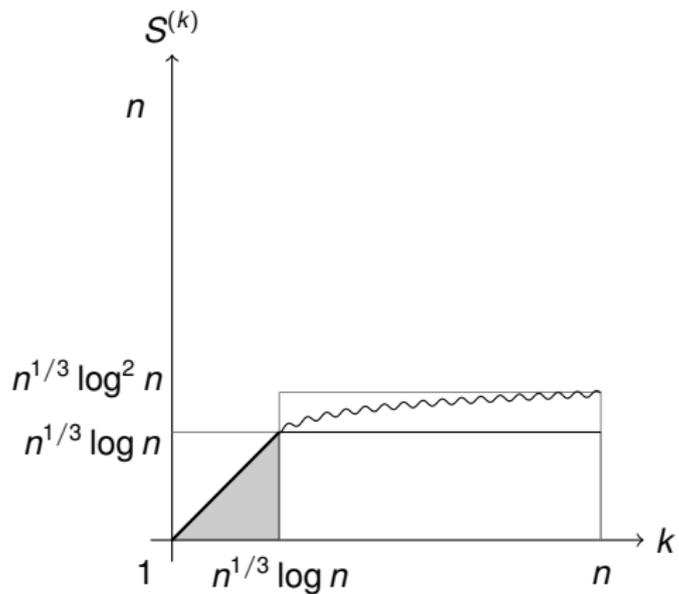


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- $t_{cov}^{(n)} \gtrsim \frac{\text{diam}(G)^2}{\log n} = \frac{n}{\log n}$

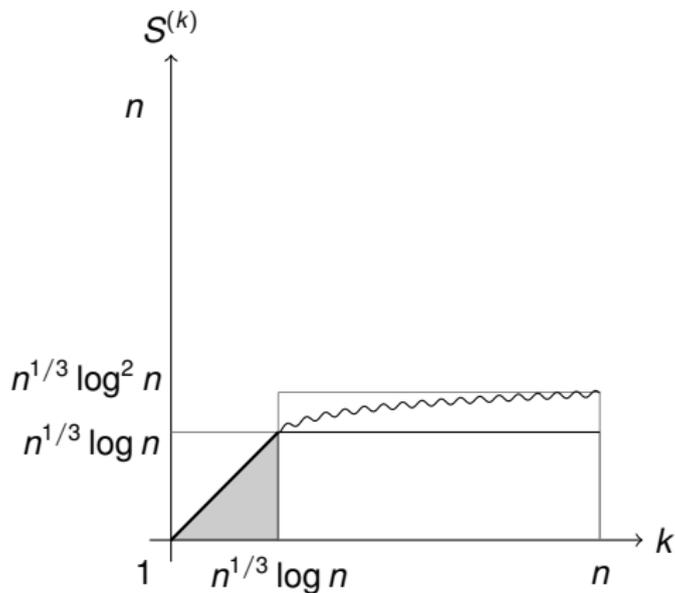
[Carne]



Speed-up for 3D Grids



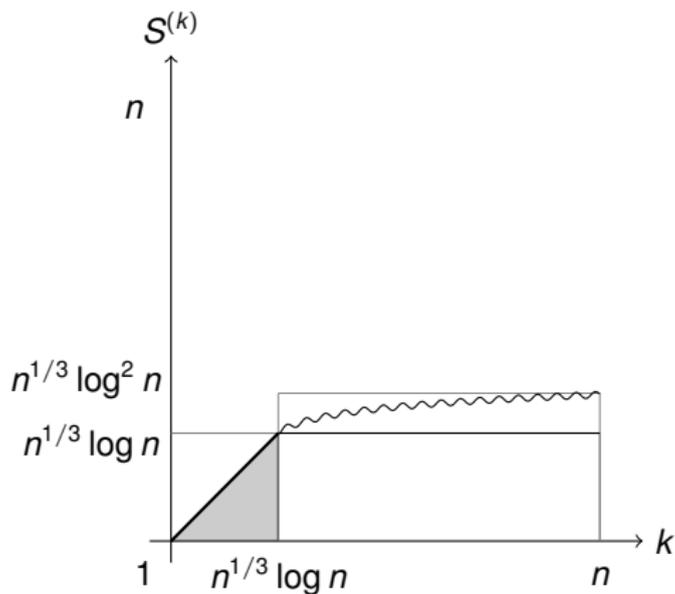
Speed-up for 3D Grids



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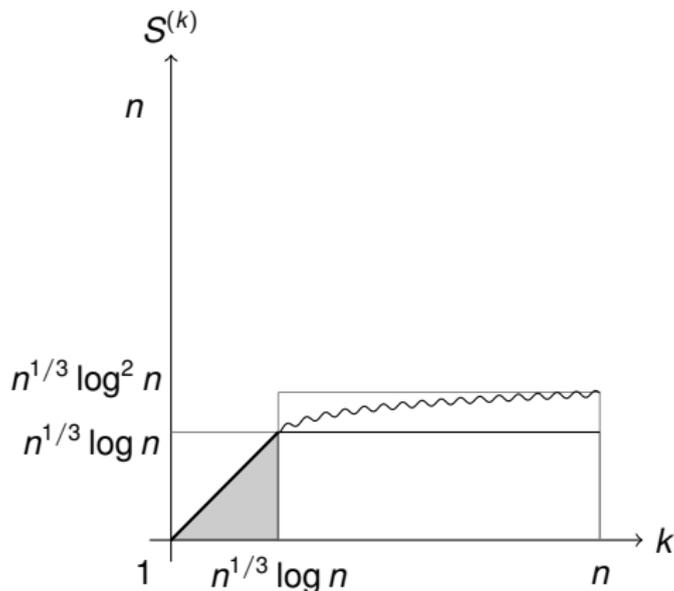
Speed-up for 3D Grids



- $t_{cov}^{(k)} \lesssim \frac{t_{hit} \cdot \log n}{k} + t_{mix} \lesssim \frac{n \cdot \log n}{k} + n^{2/3}$



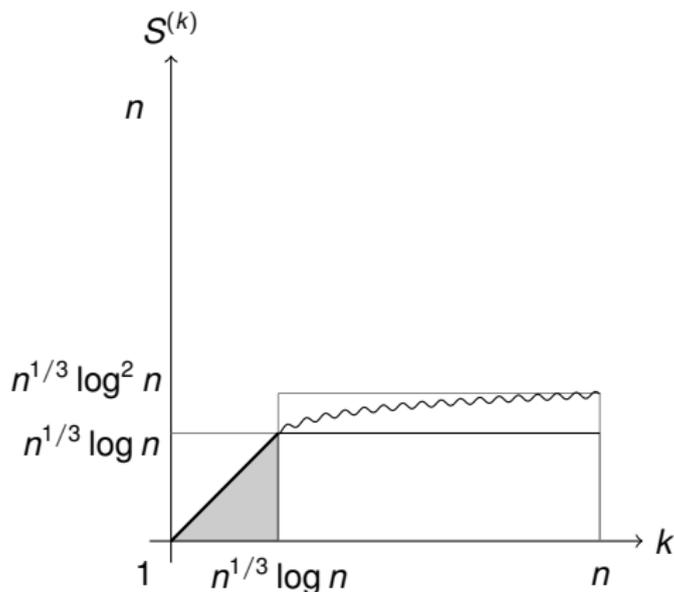
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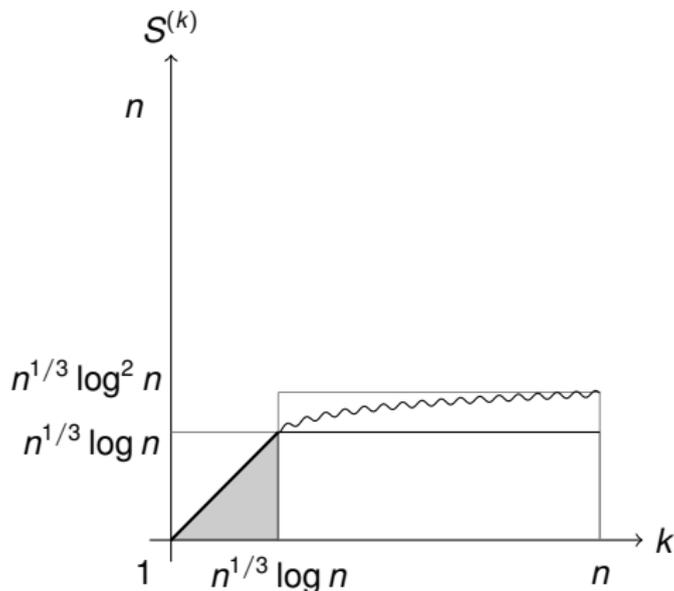
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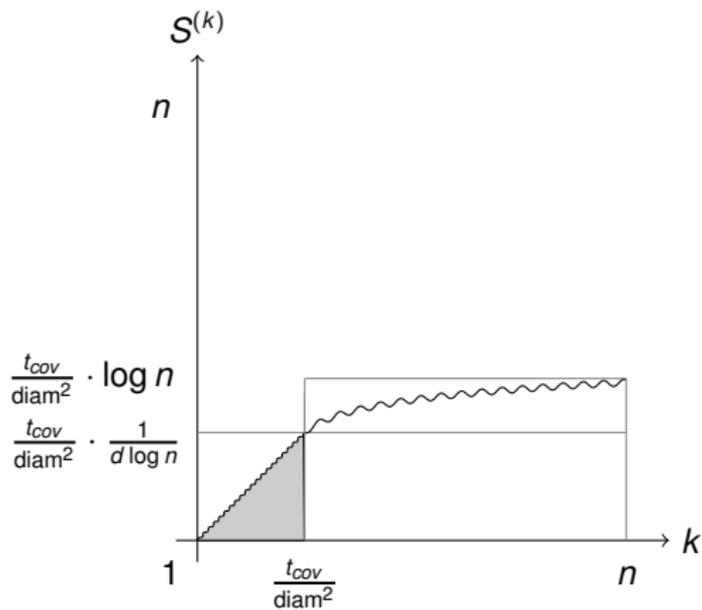
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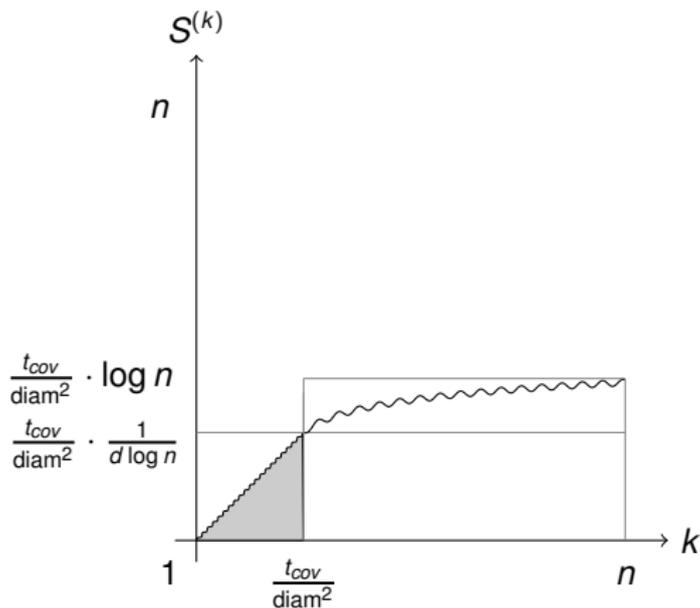
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Speed-up for Vertex Transitive Graphs



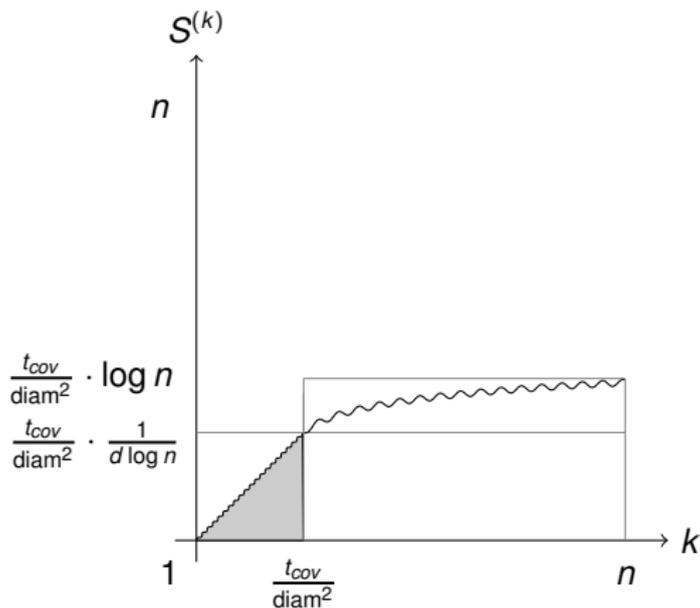
Speed-up for Vertex Transitive Graphs



- $t_{cov}^{(k)} \lesssim \frac{t_{hit} \cdot \log n}{k} + t_{mix}$



Speed-up for Vertex Transitive Graphs

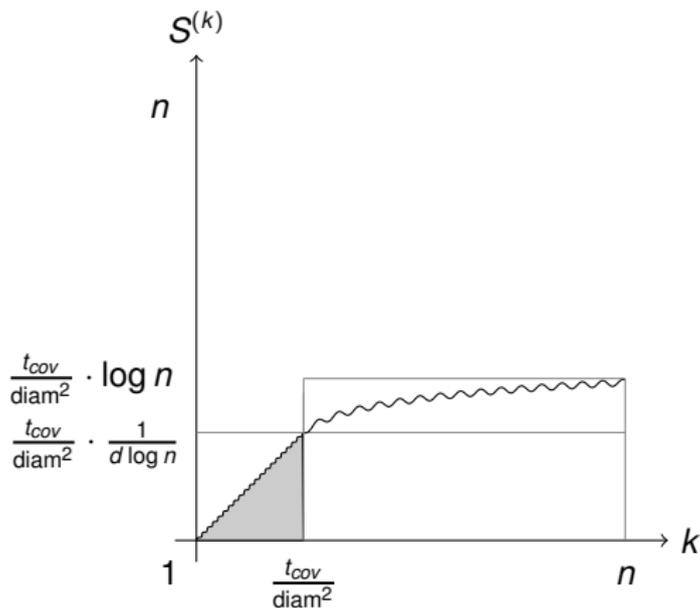


- $t_{cov}^{(k)} \lesssim \frac{t_{hit} \cdot \log n}{k} + t_{mix} \lesssim \frac{t_{cov} \cdot \log n}{k} + d \text{diam}^2 \log n$
- $S^{(k)} = O(k \log n)$

[Babai]



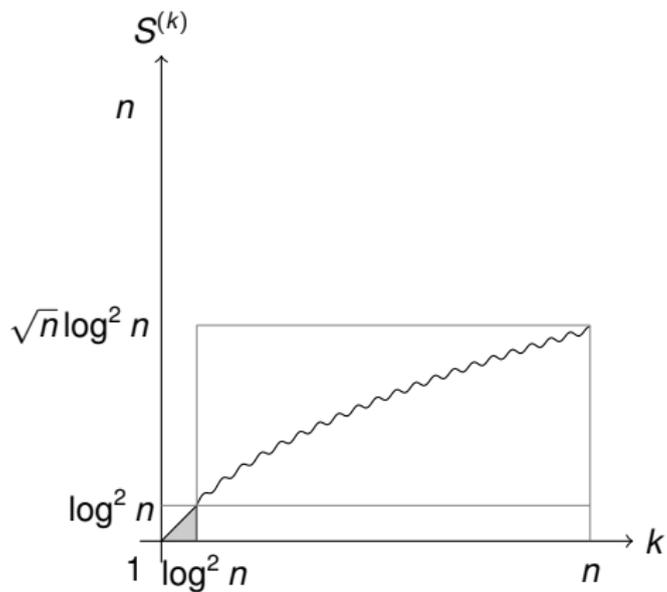
Speed-up for Vertex Transitive Graphs



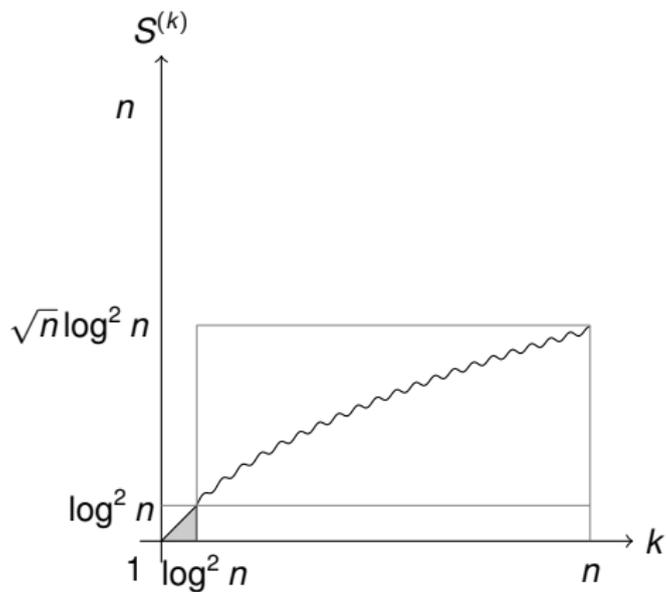
- $t_{cov}^{(k)} \lesssim \frac{t_{hit} \cdot \log n}{k} + t_{mix} \lesssim \frac{t_{cov} \cdot \log n}{k} + d \text{diam}^2 \log n$ [Babai]
- $S^{(k)} = O(k \log n)$
- $t_{cov}^{(n)} \gtrsim \frac{\text{diam}^2}{\log n}$ [Carne]



Binary Trees



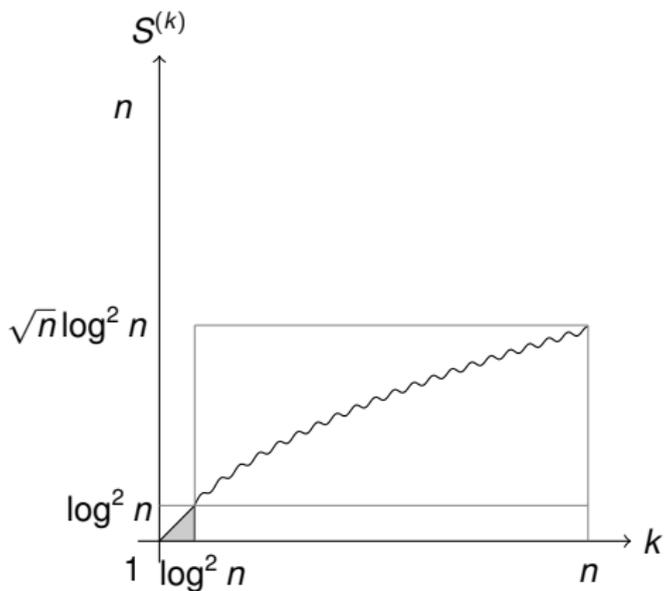
Binary Trees



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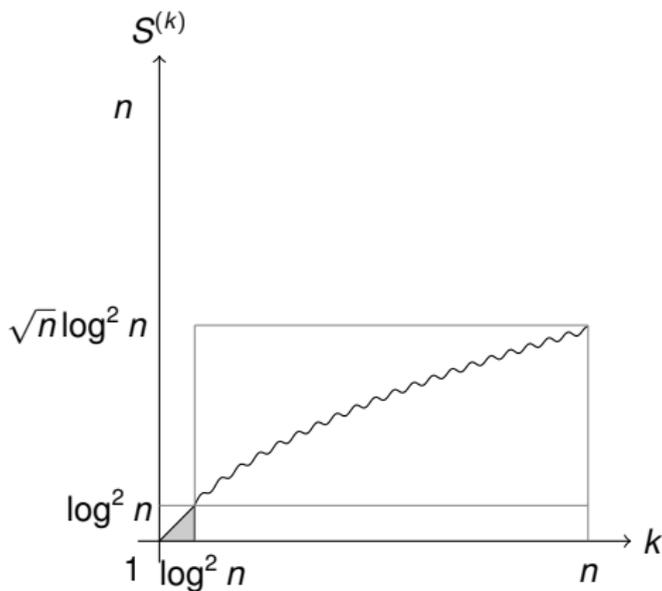


Binary Trees



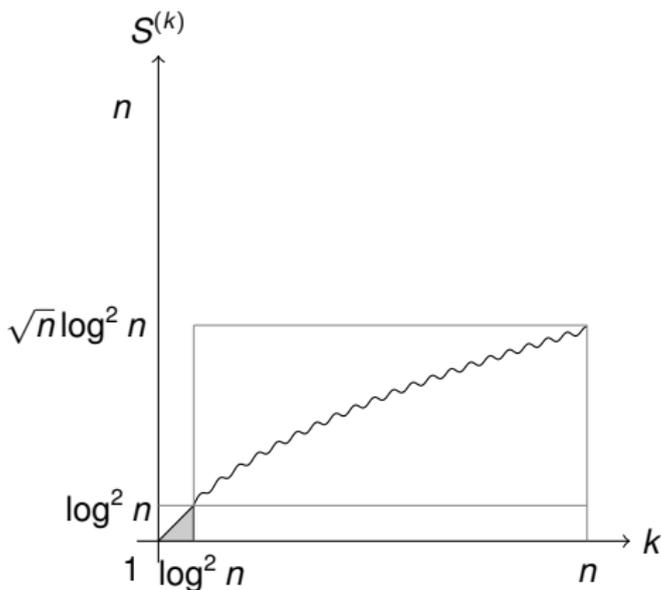
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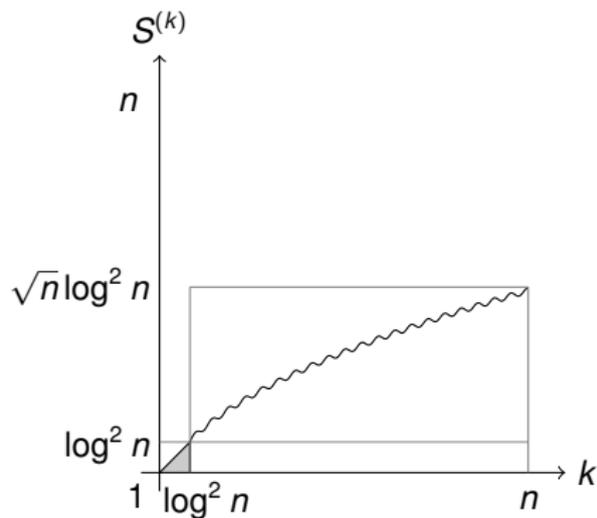
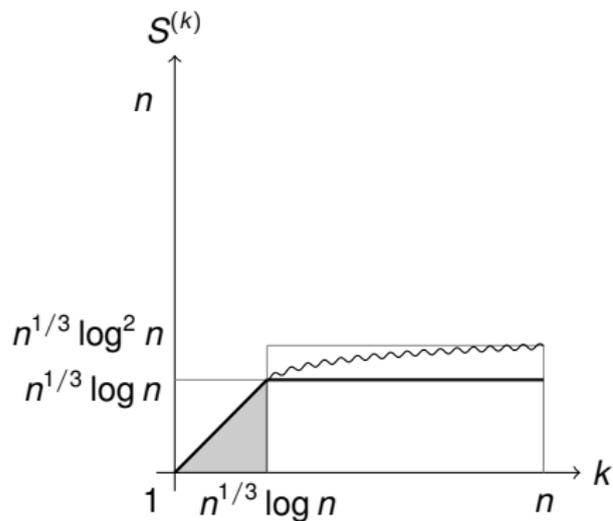




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- $t_{cov}^{(k)} \lesssim \frac{n}{\sqrt{k}} \cdot \log^5 n$ (specific analysis)



Binary Trees vs. 3D Grids



Outline

Introduction

Upper Bounds on the Multiple Cover Time

Lower Bounds on the Multiple Cover Time

Concrete Networks

Conclusion



Summary and Open Problems

General Conjectures [*Alon, Avin, Koucký, Kozma, Lotker and Tuttle*]

- $S^{(k)} \gtrsim \log k$
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Summary and Open Problems

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Both conjectures hold in the **router-router model** (a.k.a. Propp machine)! [*Dereniowski, Kosowski, Pajak, Uznanski*]



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Load Balancing

- tight analysis of a natural protocol based on multiple random walks
- load tokens \rightsquigarrow **negatively correlated** multiple random walks
- **qualitative** relationship based on hitting-set probabilities



The End

Further Directions

- Can we exploit **multiple cover time** for clustering applications?
- Can we extend our results to random walks of **varying lengths**?
- What happens on **dynamic** graphs?



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