On hitting times of bounded sets by random walks

Vlad Vysotsky

Imperial College London,
Arizona State University,
St. Petersburg Division of Steklov Institute

Warwick, May 19, 2015

1. The exit problem for random walks

Let $S_n := x + X_1 + \cdots + X_n$ be a random walk with i.i.d. increments $X_1, X_2, \ldots$.
Use $\mathbb{P}_x(\cdot)$ for the law of walk starting at x and $\mathbb{E}_x f := \int f d\mathbb{P}_x$.
Denote $\tau_B := \inf\{n \geq 1 : S_n \in B\}$ the hitting time of a set B.
A huge number of works is devoted to the asymptotic of $\mathbb{P}_x(\tau_B > n)$ under different assumptions of S_n and B.

1. The exit problem for random walks

Let $S_n := x + X_1 + \cdots + X_n$ be a random walk with i.i.d. increments $X_1, X_2, \ldots$.
Use $\mathbb{P}_x(\cdot)$ for the law of walk starting at x and $\mathbb{E}_x f := \int f d\mathbb{P}_x$.
Denote $\tau_B := \inf\{n \geq 1 : S_n \in B\}$ the hitting time of a set B.
A huge number of works is devoted to the asymptotic of $\mathbb{P}_x(\tau_B > n)$ under different assumptions of S_n and B.

- Unbounded B: a rather complete theory have been developed for $B = (-\infty, 0) \subset \mathbb{R}$ (from Sparre-Andersen '50s to Rogozin '72). In higher dimensions, there are many result on exit times from cones (resent most by Denisov and Wachtel '14).
- Bounded B: much less was known (Kesten and Spitzer '63, Port and Stone '67).

Kesten-Spitzer: For *any* aperiodic RW in $\mathbb{Z}^{1,2}$ and any *finite* $B \subset \mathbb{Z}^{1,2}$, there exists

$$\lim_{n\to\infty} \frac{\mathbb{P}_x(\tau_B > n)}{\mathbb{P}_0(\tau_{\{0\}} > n)} := g_B(x), \quad x \notin B.$$

The hard case is that of recurrent random walks.

- For $\mathbb{Z}^1$, if S_n is centred and asymptotically α-stable with $1 < \alpha \leq 2$, then $\mathbb{P}_0(\tau_{\{0\}} > n) \sim cn^{1/\alpha - 1} L(n)$.

Moreover, if $Var(X_1) < \infty$, then $\alpha = 2$ and $L(n) = const$.

Kesten-Spitzer: For *any* aperiodic RW in $\mathbb{Z}^{1,2}$ and any *finite* $B \subset \mathbb{Z}^{1,2}$, there exists

$$\lim_{n\to\infty} \frac{\mathbb{P}_x(\tau_B > n)}{\mathbb{P}_0(\tau_{\{0\}} > n)} := g_B(x), \quad x \notin B.$$

The hard case is that of recurrent random walks.

• For $\mathbb{Z}^1$, if S_n is centred and asymptotically α-stable with $1 < \alpha \le 2$, then $\mathbb{P}_0(\tau_{\{0\}} > n) \sim cn^{1/\alpha-1}L(n)$.
Moreover, if $Var(X_1) < \infty$, then $\alpha = 2$ and $L(n) = const$.

• $g_B(x)$ is harmonic for the walk killed at hitting B, that is

$$g_B(x) = \mathbb{E}_x g_B(S_1) \text{ for } x \in B^c \text{ and } g(x) := 0 \text{ on } B.$$

Why:

$$\begin{aligned} \mathbb{P}_x(\tau_B > n+1) &= \int_B \mathbb{P}_y(\tau_B > n)\mathbb{P}_x(S_1 \in dy) \\ &\sim \mathbb{P}_0(\tau_{\{0\}} > n)\mathbb{E}_x g_B(S_1)\mathbb{1}_{\{\tau_B>1\}}. \end{aligned}$$

Physical interpretation: $g_B(x)$ is the potential energy of the field due to the unit equilibrium charge on B.
Spitzer made this rigorous: for any aperiodic recurrent walk in $\mathbb{Z}^{1,2}$, the potential kernel

$$a(x) := \lim_{n\to\infty} \sum_{k=0}^{n} \big(\mathbb{P}_0(S_k = 0) - \mathbb{P}_x(S_k = 0)\big)$$

exists and solves $\Delta a = \delta_0$, where $\Delta = P - I$. For any finite $B \subset \mathbb{Z}^{1,2}$, the equilibrium charge on B is

$$\mu^*(y) = \begin{cases} \lim_{|x|\to\infty} \mathbb{E}_z(S_{T_{-B}} = -y), & d = 2 \text{ or } d = 1, \sigma^2 = \infty, \\ \frac{1}{2} \lim\limits_{x\to+\infty} \mathbb{E}_x(S_{T_{-B}} = -y) + \frac{1}{2} \lim\limits_{x\to-\infty} \mathbb{E}_x(S_{T_{-B}} = -y), & o/w. \end{cases}$$

Physical interpretation: $g_B(x)$ is the potential energy of the field due to the unit equilibrium charge on B.
Spitzer made this rigorous: for any aperiodic recurrent walk in $\mathbb{Z}^{1,2}$, the potential kernel

$$a(x) := \lim_{n\to\infty} \sum_{k=0}^{n} \big(\mathbb{P}_0(S_k = 0) - \mathbb{P}_x(S_k = 0)\big)$$

exists and solves $\Delta a = \delta_0$, where $\Delta = P - I$. For any finite $B \subset \mathbb{Z}^{1,2}$, the equilibrium charge on B is

$$\mu^*(y) = \begin{cases} \lim_{|x|\to\infty} \mathbb{E}_z(S_{T_{-B}} = -y), & d = 2 \text{ or } d = 1, \sigma^2 = \infty, \\ \frac{1}{2} \lim\limits_{x\to+\infty} \mathbb{E}_x(S_{T_{-B}} = -y) + \frac{1}{2} \lim\limits_{x\to-\infty} \mathbb{E}_x(S_{T_{-B}} = -y), & o/w. \end{cases}$$

The potential $h_B(x) := \sum_{y\in B} a(x-y)\mu^*(y)$ solves $\Delta h_B = \mu^*$ and is constant on B, called the capacity. Then

$$g_B(x) = h_B(x) - Cap_B.$$

This is a very implicit representation.

2. Our assumptions and a lower bound

Assume that the walk is in $\mathbb{R}$, $\mathbb{E}X_1 = 0$, $Var(X_1) := \sigma^2 \in (0, \infty)$. Let M be the state space of the random walk, that is $M := \lambda\mathbb{Z}$ if the walk is λ-arithmetic for some $\lambda > 0$ and $M := \mathbb{R}$ if otherwise. Consider the basic case that $B = (-d, d)$ for some $d > 0$. Put

$$p_n(x) := \mathbb{P}_x(\tau_{(-d,d)} > n), \quad x \notin B, x \in M.$$

2. Our assumptions and a lower bound

Assume that the walk is in $\mathbb{R}$, $\mathbb{E}X_1 = 0$, $Var(X_1) := \sigma^2 \in (0, \infty)$. Let M be the state space of the random walk, that is $M := \lambda\mathbb{Z}$ if the walk is λ-arithmetic for some $\lambda > 0$ and $M := \mathbb{R}$ if otherwise. Consider the basic case that $B = (-d, d)$ for some $d > 0$. Put

$$p_n(x) := \mathbb{P}_x(\tau_{(-d,d)} > n), \quad x \notin B, x \in M.$$

Hitting times for half-lines: for *any* $x \geq 0$,

$$\mathbb{P}_x(\tau_{(-\infty,0)} > n) \sim \sqrt{\frac{2}{\pi}} \frac{U_{\geqslant}(x)}{\sigma\sqrt{n}},$$

where $U_{\geqslant}(x)$ is the renewal function. It is harmonic for the walk killed as it enters $(-\infty, 0)$ and satisfies $U_{\geqslant}(x) = \mathbb{E}_x(x - S_{\tau_{(-\infty,0)}})$.

2. Our assumptions and a lower bound

Assume that the walk is in $\mathbb{R}$, $\mathbb{E}X_1 = 0$, $Var(X_1) := \sigma^2 \in (0, \infty)$. Let M be the state space of the random walk, that is $M := \lambda\mathbb{Z}$ if the walk is λ-arithmetic for some $\lambda > 0$ and $M := \mathbb{R}$ if otherwise. Consider the basic case that $B = (-d, d)$ for some $d > 0$. Put

$$p_n(x) := \mathbb{P}_x(\tau_{(-d,d)} > n), \quad x \notin B, x \in M.$$

Hitting times for half-lines: for *any* $x \geq 0$,

$$\mathbb{P}_x(\tau_{(-\infty,0)} > n) \sim \sqrt{\frac{2}{\pi}} \frac{U_{\geqslant}(x)}{\sigma\sqrt{n}},$$

where $U_{\geqslant}(x)$ is the renewal function. It is harmonic for the walk killed as it enters $(-\infty, 0)$ and satisfies $U_{\geqslant}(x) = \mathbb{E}_x(x - S_{\tau_{(-\infty,0)}})$.

Lower bound: for $|x| \geq d$, staying to one side of B gives

$$p_n(x) \geq \mathbb{P}_x(T_1 > n) \sim \sqrt{\frac{2}{\pi}} \frac{U_d(x)}{\sigma\sqrt{n}}, \quad U_d(x) := \mathbb{E}_x|x - S_{T_1}|,$$

where T_1 is the first moment of jump over either $-d$ or d.

3. Results for the basic case

Let T_k be the moment of the kth jump over $\{-d, d\}$ *from the outside*; let $H_k := S_{T_k}, k \geq 0$ be the overshoots; denote the # of jumps over $(-d, d)$ before it is hit as $\kappa := \min(k \geq 1 : |H_k| < d)$.

Theorem 1

Let S_n be a random walk with $\mathbb{E}X_1 = 0$, $\mathbb{E}X_1^2 := \sigma^2 \in (0, \infty)$. Then for any $d > 0$ and any x from the state space M,

$$p_n(x) \sim \sqrt{\frac{2}{\pi}} \frac{V_d(x)}{\sigma\sqrt{n}}, \quad V_d(x) := \mathbb{E}_x\left[\sum_{i=1}^{\kappa} |H_i - H_{i-1}|\right].$$

Moreover, this holds uniformly for $x = o(\sqrt{n})$. Further,

- *$V_d(x)$ is harmonic for the walk killed as it enters $(-d, d)$;*
- *$0 < U_d(x) \leq V_d(x) < \infty$ for $|x| \geq d$;*
- *$V_d(x) \sim |x|$ as $x \to \infty$.*

4. Ideas of the proof

1. It costs to jump over:

There exists a $\gamma \in (0, 1)$ such that

$$\mathbb{P}_x(|H_1| \geq d) \leq \gamma.$$

This follows since H_1 converge weakly as $x \to \pm\infty$ to the overshoots over "infinitely remote" levels.

4. Ideas of the proof

1. It costs to jump over:
There exists a $\gamma \in (0,1)$ such that

$$\mathbb{P}_x(|H_1| \geq d) \leq \gamma.$$

This follows since H_1 converge weakly as $x \to \pm\infty$ to the overshoots over "infinitely remote" levels.

2. *Regularity of $p_n(x)$* in both x and n is needed.
Lemma: For any $x \in \mathbb{R}$ and $n \geq 1$, $p_n(x) \leq C|x|n^{-1/2}$.
Roughly, $\mathbb{E}_x p_{n-T_1}(H_1)\mathbb{1}_{\{|H_1|\geq d, T_1 \leq n\}}$ is controlled by $\mathbb{E}_x|H_1|$.

4. Ideas of the proof

1. It costs to jump over:

There exists a $\gamma \in (0,1)$ such that

$$\mathbb{P}_x(|H_1| \geq d) \leq \gamma.$$

This follows since H_1 converge weakly as $x \to \pm\infty$ to the overshoots over "infinitely remote" levels.

2. *Regularity of $p_n(x)$* in both x and n is needed.

Lemma: For any $x \in \mathbb{R}$ and $n \geq 1$, $p_n(x) \leq C|x|n^{-1/2}$.

Roughly, $\mathbb{E}_x p_{n-T_1}(H_1)\mathbb{1}_{\{|H_1|\geq d, T_1 \leq n\}}$ is controlled by $\mathbb{E}_x|H_1|$.

3. The mechanism of stabilisation:

For any $\alpha \in (0,1)$ it holds that

$$\mathbb{E}_x|H_1| \leq \alpha|x| + K(\alpha), \quad |x| \geq d.$$

This follows from the known $\mathbb{E}_x|H_1| = o(|x|)$ as $|x| \to \infty$.

5. General sets

Denote T'_k the moments of jumps over $\{\inf B, \sup B\}$; $H'_k := S'_{T_k}$ the overshoots; and put $\kappa' := \min\{k \geq 1 : T'_k \geq \tau_B\}$.

5. General sets

Denote T'_k the moments of jumps over $\{\inf B, \sup B\}$; $H'_k := S'_{T_k}$ the overshoots; and put $\kappa' := \min\{k \geq 1 : T'_k \geq \tau_B\}$.

Theorem 2

Assume that $\mathbb{E}X_1 = 0$, $\mathbb{E}X_1^2 := \sigma^2 \in (0, \infty)$, and B is a bounded Borel set with the non-empty $Int_M(B)$. Then for any $x \in M$,

$$p'_n(x) \sim \frac{\sqrt{2} V_B(x)}{\sigma\sqrt{\pi n}}, \quad V_B(x) := \mathbb{E}_x \left[\sum_{i=1}^{\kappa'} \left| H'_i - H'_{i-1} \right| \mathbb{1}_{\{H'_{i-1} \notin Conv(B)\}} \right].$$

Moreover, this holds uniformly for $x = o(\sqrt{n})$. It is true that $0 < V_B(x) < \infty$ for $x \notin Conv(B)$ and clearly, $V_{(-d,d)}(x) = V_d(x)$.

5. General sets

Denote T'_k the moments of jumps over $\{\inf B, \sup B\}$; $H'_k := S'_{T_k}$ the overshoots; and put $\kappa' := \min\{k \geq 1 : T'_k \geq \tau_B\}$.

Theorem 2

Assume that $\mathbb{E}X_1 = 0$, $\mathbb{E}X_1^2 := \sigma^2 \in (0, \infty)$, and B is a bounded Borel set with the non-empty $Int_M(B)$. Then for any $x \in M$,

$$p'_n(x) \sim \frac{\sqrt{2} V_B(x)}{\sigma\sqrt{\pi n}}, \quad V_B(x) := \mathbb{E}_x\left[\sum_{i=1}^{\kappa'} \left|H'_i - H'_{i-1}\right| \mathbb{1}_{\{H'_{i-1} \notin Conv(B)\}}\right].$$

Moreover, this holds uniformly for $x = o(\sqrt{n})$. It is true that $0 < V_B(x) < \infty$ for $x \notin Conv(B)$ and clearly, $V_{(-d,d)}(x) = V_d(x)$.

Heuristics

1. It costs exponentially in time to stay within $Conv(B) \setminus B$.
2. Each return from B^c to $Conv(B) \setminus B$ costs multiplicatively.
3. The rest is as in the basic case.

6. Conditional functional limit theorem

Define $\hat{S}_n(t)$: for $t = k/n$ with a $k \in \mathbb{N}$ put $\hat{S}_n(k/n) := S_k/(\sigma\sqrt{n})$, and define the other values by linear interpolation.

Theorem 3

Under assumptions of Thm 2, for any $x \in M$ such that $V_B(x) > 0$,

$$Law_x(\hat{S}_n(\cdot)|\tau_B > n) \stackrel{\mathcal{D}}{\to} Law(\rho W_+) \qquad in\ C[0,1],$$

where W_+ is a Brownian meander, ρ is a r.v. independent of W_+ with the distribution given by $\mathbb{P}(\rho = \pm 1) = \frac{1}{2} \pm \frac{x - \mathbb{E}_x S_{\tau_B}}{2V_B(x)}$.

For integer-valued asymptotically α-stable walks ($1 \le \alpha \le 2$) the weak convergence was proved by Belkin '72.

7. Applications to the largest problem

Define the largest gap (maximal spacing) within the range of S_n:

$$Gap(\{S_k\}_{k\geq 1}^{n}) := G_n := \max_{1\leq k\leq n-1}\big(S_{(k+1,n)} - S_{(k,n)}\big),$$

where $m_n := S_{(1,n)} \leq S_{(2,n)} \leq \cdots \leq S_{(n,n)} =: M_n$ denote the elements of $S_1, \ldots, S_n$ arranged in the weakly ascending order.

Theorem 4

If $\mathbb{E}X_1 = 0$, $Var(X_1) < \infty$, *then*

$$G_n \xrightarrow{\mathcal{D}} G,$$

where G *is a non-degenerate proper random variable.*