On hitting times of bounded sets by random walks

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1. The exit problem for random walks
Let $S_n := x + X_1 + \cdots + X_n$ be a random walk with i.i.d. increments $X_1, X_2, \ldots$.
Use $\mathbb{P}_x(\cdot)$ for the law of walk starting at $x$ and $\mathbb{E}_x f := \int f d\mathbb{P}_x$.
Denote $\tau_B := \inf\{n \geq 1 : S_n \in B\}$ the hitting time of a set $B$.
A huge number of works is devoted to the asymptotic of $\mathbb{P}_x(\tau_B > n)$ under different assumptions of $S_n$ and $B$. 

• Unbounded $B$: a rather complete theory have been developed for $B = (-\infty, 0) \subset \mathbb{R}$ (from Sparre-Andersen '50s to Rogozin '72). In higher dimensions, there are many result on exit times from cones (resent most by Denisov and Wachtel '14).
• Bounded $B$: much less was known (Kesten and Spitzer '63, Port and Stone '67).
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  (resent most by Denisov and Wachtel ’14).
- Bounded $B$: much less was known (Kesten and Spitzer ’63, Port
  and Stone ’67).
**Kesten-Spitzer:** For *any* aperiodic RW in \( \mathbb{Z}^{1,2} \) and any *finite* \( B \subset \mathbb{Z}^{1,2} \), there exists

\[
\lim_{n \to \infty} \frac{\mathbb{P}_x(\tau_B > n)}{\mathbb{P}_0(\tau_{\{0\}} > n)} := g_B(x), \quad x \notin B.
\]

The hard case is that of recurrent random walks.

- For \( \mathbb{Z}^1 \), if \( S_n \) is centred and asymptotically \( \alpha \)-stable with \( 1 < \alpha \leq 2 \), then \( \mathbb{P}_0(\tau_{\{0\}} > n) \sim cn^{1/\alpha-1}L(n) \).

Moreover, if \( \text{Var}(X_1) < \infty \), then \( \alpha = 2 \) and \( L(n) = \text{const} \).
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Moreover, if $\text{Var}(X_1) < \infty$, then $\alpha = 2$ and $L(n) = \text{const}$.

- $g_B(x)$ is harmonic for the walk killed at hitting $B$, that is

$$g_B(x) = \mathbb{E}_x g_B(S_1) \text{ for } x \in B^c \text{ and } g(x) := 0 \text{ on } B.$$ 

Why:

$$\mathbb{P}_x(\tau_B > n + 1) = \int_B \mathbb{P}_y(\tau_B > n)\mathbb{P}_x(S_1 \in dy)$$

$$\sim \mathbb{P}_0(\tau\{0\} > n)\mathbb{E}_x g_B(S_1)1_{\{\tau_B > 1\}}.$$

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Physical interpretation: $g_B(x)$ is the potential energy of the field due to the unit equilibrium charge on $B$. Spitzer made this rigorous: for any aperiodic recurrent walk in $\mathbb{Z}^{1,2}$, the potential kernel

$$a(x) := \lim_{n \to \infty} \sum_{k=0}^{n} (\mathbb{P}_0(S_k = 0) - \mathbb{P}_x(S_k = 0))$$

exists and solves $\Delta a = \delta_0$, where $\Delta = P - I$. For any finite $B \subset \mathbb{Z}^{1,2}$, the equilibrium charge on $B$ is

$$\mu^*(y) = \begin{cases} 
\lim_{|x| \to \infty} \mathbb{E}_z(S_{T_{-B}} = -y), & d = 2 \text{ or } d = 1, \sigma^2 = \infty, \\
\frac{1}{2} \lim_{x \to +\infty} \mathbb{E}_x(S_{T_{-B}} = -y) + \frac{1}{2} \lim_{x \to -\infty} \mathbb{E}_x(S_{T_{-B}} = -y), & o/w.
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The potential $h_B(x) := \sum_{y \in B} a(x - y)\mu^*(y)$ solves $\Delta h_B = \mu^*$ and is constant on $B$, called the capacity. Then

$$g_B(x) = h_B(x) - \text{Cap}_B.$$

This is a very implicit representation.
2. Our assumptions and a lower bound
Assume that the walk is in $\mathbb{R}$, $\mathbb{E}X_1 = 0$, $\text{Var}(X_1) := \sigma^2 \in (0, \infty)$. Let $M$ be the state space of the random walk, that is $M := \lambda\mathbb{Z}$ if the walk is $\lambda$-arithmetic for some $\lambda > 0$ and $M := \mathbb{R}$ if otherwise. Consider the basic case that $B = (-d, d)$ for some $d > 0$. Put

$$p_n(x) := \mathbb{P}_x(\tau_{(-d,d)} > n), \quad x \notin B, x \in M.$$
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Hitting times for half-lines: for any $x \geq 0$,

$$\mathbb{P}_x(\tau_{(-\infty, 0)} > n) \sim \sqrt{\frac{2}{\pi}} \frac{U_\geq(x)}{\sigma \sqrt{n}},$$

where $U_\geq(x)$ is the renewal function. It is harmonic for the walk killed as it enters $(-\infty, 0)$ and satisfies $U_\geq(x) = \mathbb{E}_x(x - S_{\tau_{(-\infty, 0)}})$. 


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Lower bound: for $|x| \geq d$, staying to one side of $B$ gives

$$p_n(x) \geq \mathbb{P}_x(T_1 > n) \sim \sqrt{\frac{2}{\pi}} \frac{U_d(x)}{\sigma \sqrt{n}}, \quad U_d(x) := \mathbb{E}_x|x - S_{T_1}|,$$

where $T_1$ is the first moment of jump over either $-d$ or $d$. 
3. Results for the basic case

Let \( T_k \) be the moment of the \( k \)th jump over \( \{-d, d\} \) \textit{from the outside}; let \( H_k := S_{T_k}, k \geq 0 \) be the overshoots; denote the \# of jumps over \((-d, d)\) before it is hit as \( \kappa := \min(k \geq 1 : |H_k| < d) \).

**Theorem 1**

Let \( S_n \) be a random walk with \( \mathbb{E}X_1 = 0, \mathbb{E}X_1^2 := \sigma^2 \in (0, \infty) \).

Then for any \( d > 0 \) and any \( x \) from the state space \( M \),

\[
p_n(x) \sim \sqrt{\frac{2}{\pi}} \frac{V_d(x)}{\sigma \sqrt{n}}, \quad V_d(x) := \mathbb{E}_x \left[ \sum_{i=1}^{\kappa} |H_i - H_{i-1}| \right].
\]

Moreover, this holds uniformly for \( x = o(\sqrt{n}) \). Further,

- \( V_d(x) \) is harmonic for the walk killed as it enters \((-d, d)\);
- \( 0 < U_d(x) \leq V_d(x) < \infty \) for \( |x| \geq d \);
- \( V_d(x) \sim |x| \) as \( x \to \infty \).
4. Ideas of the proof

1. *It costs to jump over:*

There exists a $\gamma \in (0, 1)$ such that

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\mathbb{P}_x(|H_1| \geq d) \leq \gamma.
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This follows since $H_1$ converge weakly as $x \to \pm \infty$ to the overshoots over “infinitely remote” levels.
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2. Regularity of $p_n(x)$ in both $x$ and $n$ is needed.

**Lemma:** For any $x \in \mathbb{R}$ and $n \geq 1$, $p_n(x) \leq C|x|n^{-1/2}$.

Roughly, $\mathbb{E}_x p_{n-T_1}(H_1) 1\{|H_1| \geq d, T_1 \leq n\}$ is controlled by $\mathbb{E}_x |H_1|$. 
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3. The mechanism of stabilisation:
For any $\alpha \in (0, 1)$ it holds that

$$\mathbb{E}_x |H_1| \leq \alpha|x| + K(\alpha), \quad |x| \geq d.$$ 

This follows from the known $\mathbb{E}_x |H_1| = o(|x|)$ as $|x| \to \infty$. 

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5. General sets

Denote $T'_k$ the moments of jumps over $\{\inf B, \sup B\}$; $H'_k := S'_{T'_k}$ the overshoots; and put $\kappa' := \min\{k \geq 1 : T'_k \geq \tau_B\}$.

Theorem 2

Assume that $E X_1 = 0$, $E X_2^1 = \sigma^2 \in (0, \infty)$, and $B$ is a bounded Borel set with the non-empty $\text{Int} M(B)$. Then for any $x \in M$, $p'_n(x) \sim \sqrt{2 V_B(x)} \sigma \sqrt{\pi} n$,

$$V_B(x) := E_x \left[ \kappa' \sum_{i=1}^{\kappa'} |H'_i - H'_{i-1}| \right] \right]$$.

Moreover, this holds uniformly for $x = o(\sqrt{n})$. It is true that $0 < V_B(x) < \infty$ for $x \not\in \text{Conv}(B)$ and clearly, $V(-d, d)(x) = V_d(x)$.
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Heuristics
1. It costs exponentially in time to stay within $\text{Conv}(B) \setminus B$.
2. Each return from $B^c$ to $\text{Conv}(B) \setminus B$ costs multiplicatively.
3. The rest is as in the basic case.
6. Conditional functional limit theorem
Define $\hat{S}_n(t)$: for $t = k/n$ with a $k \in \mathbb{N}$ put $\hat{S}_n(k/n) := S_k/(\sigma \sqrt{n})$, and define the other values by linear interpolation.

**Theorem 3**
Under assumptions of Thm 2, for any $x \in M$ such that $V_B(x) > 0$,

$$\text{Law}_x(\hat{S}_n(\cdot)|\tau_B > n) \xrightarrow{D} \text{Law}(\rho W_+) \quad \text{in } C[0,1],$$

where $W_+$ is a Brownian meander, $\rho$ is a r.v. independent of $W_+$ with the distribution given by $\mathbb{P}(\rho = \pm 1) = \frac{1}{2} \pm \frac{x - \mathbb{E}_x S_{\tau_B}}{2V_B(x)}$.

For integer-valued asymptotically $\alpha$-stable walks ($1 \leq \alpha \leq 2$) the weak convergence was proved by Belkin '72.
7. Applications to the largest problem
Define the largest gap (maximal spacing) within the range of $S_n$:

$$\text{Gap}(\{S_k\}_{k \geq 1}) := G_n := \max_{1 \leq k \leq n-1} (S_{(k+1,n)} - S_{(k,n)})$$

where $m_n := S_{(1,n)} \leq S_{(2,n)} \leq \cdots \leq S_{(n,n)} =: M_n$ denote the elements of $S_1, \ldots, S_n$ arranged in the weakly ascending order.

**Theorem 4**
If $\mathbb{E}X_1 = 0$, $\text{Var}(X_1) < \infty$, then

$$G_n \xrightarrow{D} G,$$

where $G$ is a non-degenerate proper random variable.