Quasi-isometries of graphs and groups, random walks, and harmonic functions

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Question by W. at Ljublana-Leoben graph theory seminar on Mt. Vogel (198?):

Is there a locally finite, vertex-transitive graph that does not look vaguely like a Cayley graph of a group?


Is there a locally finite, vertex-transitive graph that is not quasi-isometric a Cayley graph of some finitely generated group?


Outline

- Review involved concepts.
- Describe construction of counterexample by Diestel and Leader.
- Review results by Eskin, Fisher and Whyte on DL-graphs and related structures.
- Outline results on random walks and harmonic functions.
- Describe extended constructions and related results.
- Mention issues for future work.
Introduction

- All graphs in this talk are connected, locally finite and infinite, carry integer-valued graph metric.

- If $G$ is a finitely generated group and $S$ a finite, symmetric set of generators, then the Cayley graph $X(G, S)$ has vertex set $G$, and

  $$x \sim y \iff y = xs, \ s \in S.$$  

- A quasi-isometry (rough isometry) between metric spaces $(X_1, d_1)$ and $(X_2, d_2)$ is $\varphi : X_1 \rightarrow X_2$ with

  $$A^{-1}d_1(x_1, y_1) - B \leq d_2(\varphi x_1, \varphi y_1) \leq A d_1(x_1, y_1) + B \ \forall \ x_1, y_1 \in X_1$$

  $$d(y_2, \varphi X_1) \leq B \ \forall \ y_2 \in X_2.$$  

- Bi-Lipschitz, if $B = 0$.
Examples

- Any two Cayley graphs of the same f.g. group are bi-Lipschitz.
- If $G_1$ and $G_2$ have a common subgroup with finite index in each of the two then they are quasi-isometric.
- The integer lattices $\mathbb{Z}^{d_1}$ and $\mathbb{Z}^{d_2}$ are not quasi-isometric when $d_1 \neq d_2$.
- Let $T$ be a tree with $2 \leq \text{deg}(\cdot) \leq M$ and a finite upper bound on the lengths of all unbranched paths, then $T$ is quasi-isometric with the regular tree with degree 3.
- A group is quasi-isometric with a tree if and only if it is virtually free Gromov (1988), Woess (1986/89).
Transitive graphs

- The **Automorphism group** of a graph is the group of neighbourhood perserving bijections of the vertex set.
- A graph is called (**vertex**) **transitive** if its automorphism group acts transitively.
- **Cayley graphs** of groups are transitive.
- Example of an (intrinsically infinite) **transitive non-Cayley graph**: grandmother graph:

  Start with **upper half plane drawing** of homogeneous tree $\mathbb{T}_p$ with degree $p + 1$. 
Grandmother graph

\[ h(x) = k :\iff x \in H_k \]
level function

\[ \text{Aff}\left(\Gamma\right) = \left\{ g \in \text{Aut}\left(\Gamma\right) : g(x^-) = (gx)^- \text{ for all } x \right\} \]
affine group of \(\Gamma\).

\[ = \text{Aut(}\text{grandma graph}) \implies \text{grandma graph is not a Cayley graph.} \]

But it is quasi-isometric to a Cayley graph!
Diestel-Leader graphs

- In the mid-early 1990ies, Diestel and Leader proposed a construction of transitive graphs which they conjectured to be non-q.i. to any Cayley graph. Conjecture published in 2001.

\[ DL(p, q) = \{ x_1 x_2 \in \mathbb{T}_p \times \mathbb{T}_q : h(x_1) + h(x_2) = 0 \} \]

Neighbourhood: \( x_1 x_2 \sim y_1 y_2 \iff x_i \sim y_i \quad (i = 1, 2) \)
More on DL graphs

- **Diestel and Leader** considered $\text{DL}(2, 3)$

- **Möller and P. Neumann** (2001, private communication by M.) observed (for $q = 2$) that $\text{DL}(q, q)$ is a Cayley graph of the lamplighter group $\mathbb{Z}(q) \wr \mathbb{Z}$.

- Solution of quasi-isometry question:

  **Theorem.** [Eskin, Fisher and Whyte, 2012]

  If $q \neq p$ then $\text{DL}(p, q)$ is not quasi-isometric with any finitely generated group.
Key to the proof

- A quasi-isometry $\text{DL} \rightarrow \text{DL}$ is called height respecting if it permutes the “horizontal” level sets $H_k \times H_{-k}$ up to bounded distance.

Theorem. [Eskin, Fisher and Whyte, 2012]

If $q \neq p$ then every $(A, B)$-quasi isometry is at $C(A, B)$-bounded distance from a height respecting one.

Implies that horizontal levels (and their distances) are distorted only up to uniform bounds.
Another q.i.-result:

**Theorem.** [ESKIN, FISHER AND WHYTE, 2012 + 2013]

\( DL(p, q) \) is quasi-isometric with \( DL(p', q') \) if and only if \( p \) and \( p' \) are powers of a common integer, \( p \) and \( p' \) are powers of a common integer, and \( \log p' / \log p = \log q' / \log q \).
Two sister structures / horocyclic products

Treebolic space

► $T_p$ metric graph, edges $\equiv$ intervals of length 1
► Extend level function $h$ linearly to interior points of edges
  (now real-valued)
► $\mathbb{H}_q$ ($q > 1$ real) sliced hyperbolic plane:

\[ h(z) = \log_q y \]
Treebolic space

$$HT(p, q) = \left\{ \beta = (w, z) \in T_p \times H_q : h(w) = h(z) \right\}$$

"Horocyclic product"

If $q = p$, the Baumslag-Solitar group

$$BS(p) = \left\{ \begin{pmatrix} p^m & k/p^l \\ 0 & 1 \end{pmatrix} : k, l, m \in \mathbb{Z} \right\} = \langle a, b \mid a b = b^p a \rangle$$

acts on $HT(p, p)$ by isometries & with compact quotient.
Treebolic space

- Quasi-isometry classification of $\text{BS}(p)$ ($2 \leq p \in \mathbb{Z}$) by [Farb and Mosher, 1998 + 1999] uses action on $\text{HT}(p, p)$.

Theorem.
$\text{HT}(p, p)$ is quasi-isometric with $\text{HT}(p', p')$ if and only if $p$ and $p'$ are powers of a common integer.

- If $p \neq q$ then there is no finitely generated group of isometries that acts with compact quotient on $\text{HT}(p, q)$.

Conjecture.
In that case, $\text{HT}(p, q)$ is not quasi-isometric to any finitely generated group. [Almost sure.]
Sol-manifold / -group

- Hyperbolic upper half plane \( \mathbb{H} = \{ x + i w : x, w \in \mathbb{R}, w > 0 \} \)

  \( \rightarrow \) logarithmic model \( z = \log w \), coordinates \( (x, z) \in \mathbb{R}^2 \).

- Change curvature to \( -p^2 \) \( \rightarrow \) \( \mathbb{H}(p) \) is \( \mathbb{R}^2 \),

  length element \( ds^2 = d_p s^2 = e^{-2pz} \, dx^2 + dz^2 \).

- Height function of \( x = (x, z) \) is \( h(x) = z \)

\[
\text{Sol}(p, q) = \left\{ x_1 x_2 \in \mathbb{H}(p) \times \mathbb{H}(q) : h(x_1) + h(x_2) = 0 \right\}
\]

\[
= \left\{ (x, y, z) \in \mathbb{R}^3 : (x, z) \in \mathbb{H}(p), (y, -z) \in \mathbb{H}(q) \right\}
\]

with length element

\[
ds^2 = d_{p,q} s^2 = e^{-2pz} \, dx^2 + e^{2qz} \, dy^2 + dz^2.
\]
Sol-manifold / -group

\[ S = S(p, q) = \left\{ g = \begin{pmatrix} e^{pc} & a & 0 \\ 0 & 1 & 0 \\ 0 & b & e^{-qc} \end{pmatrix}, \quad a, b, c \in \mathbb{R} \right\} \]

- Lie group \( \equiv \text{Sol}(p, q) \), \( g \leftrightarrow (a, b, c) \).
- Isometric, fixed-point-free action on \( \text{Sol}(p, q) \) (\( \equiv \) group product):
  \[ (a, b, c) \cdot (x, y, z) = \left( e^{pc}x + a, e^{-qc}y + b, c + z \right). \]
- The group \( S(p, p) = S(1, 1) \) contains many co-compact lattices (discrete subgroups acting with compact quotient).
Quasi-isometry questions: solution by analogous methods as for DL\((p, q)\).

**Theorem.** [Eskin, Fisher and Whyte, 2012]

If \( q \neq p \) then \( \text{Sol}(p, q) \) is not quasi-isometric with any (Cayley graph of a) finitely generated group.

**Theorem.** [Eskin, Fisher and Whyte, 2012 + 2013]

\( \text{Sol}(p, q) \) is quasi-isometric with \( \text{Sol}(p', q') \) if and only if \( p'/p = q'/q \).
Random processes, harmonic functions

Class of random processes adapted to the geometry, with vertical drift parameters.

- On $\text{DL}(p, q)$: random walk with transition matrix $P_\alpha$ ($0 < \alpha < 1$)

- On $\text{Sol}(p, q)$: Brownian motion induced by Laplacian

$$\mathcal{L}_a = \frac{1}{2} \left( e^{2pz} \frac{\partial^2}{\partial x^2} + e^{-2qz} \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) + a \frac{\partial}{\partial z}.$$
Random processes, harmonic functions

On $\mathbf{HT}(p, q)$: Brownian motion $\equiv$ process induced by Laplacian $\mathcal{L}_{\alpha, \beta}$ ($\alpha \in \mathbb{R}, \beta > 0$) that takes care of singularities at bifurcation lines.

For $\mathfrak{z} = (x + iy, w) \in \mathbf{HT}^o$

$$\mathcal{L}_{\alpha, \beta} f(\mathfrak{z}) = y^2(\partial_x^2 + \partial_y^2) f(\mathfrak{z}) + \alpha y \partial_y f(\mathfrak{z})$$

acting on suitable function space.

“Nice” functions in its domain must be

- continuous on $\mathbf{HT}$
- twice continuously differentiable on each $S_v$
- satisfy on each $L_v$ the Kirchhoff condition

$$\partial_y f_v(\mathfrak{z}^-) = \beta \cdot \sum_{u:u^- = v} \partial_y f_u(\mathfrak{z}^+) \quad \text{for} \quad f_v = f\big|_{S_v}$$
Harmonic functions

- In all three cases, the respective operator has natural projections onto the first as well as on the second of the two spaces that make up the horocyclic product.
- Also, there is the “vertical” projection onto the line.


Every positive harmonic function for the respective operator has the form

$$h(x_1 x_2) = h_1(x_1) + h_2(x_2),$$

where for $j = 1, 2$, $h_j$ is a non-negative harmonic function for the projected operator on the 1st, resp. 2nd one of the two spaces that make up the horocyclic product.
Methods

- Green kernel \( G(x, z) = \sum_{n=0}^{\infty} p^{(n)}(x, z) \), resp. = \( \int_{n=0}^{\infty} p_t(x, z) dt \)
  - is invariant under (transitive) group of “vertical” isometries,
  - satisfies uniform local Harnack inequality

\[
G(x, z)m(z) \leq C_d G(x, z') m(z')
\]
whenever \( d(z, z') \leq d \) and \( d(z, x), d(z', x) \geq 10(d + 1) \).

- \( h \) positive harmonic function is called minimal, if
  1. \( h(o) = 1 \) (\( o \) reference point on level 0)
  2. Whenever \( h \geq \bar{h} \geq 0 \) with \( \bar{h} \) harmonic, then \( \bar{h}/h = \text{const} \).

- Every minimal harmonic function is a limit of Martin kernels:

\[
h = \lim_{n \to \infty} K(\cdot, z_n), \quad \text{where} \quad d(o, z_n) \to \infty \quad \text{and} \quad K(x, z) = \frac{G(x, z)}{G(o, z)}
\]
Methods

- If \( z_n = z_{n,1} z_{n,2} \), first suppose \( \inf_n h(z_{n,1}) = c > -\infty \).

  Let \( \tau_1 \) be a level-isometry of the first factor (\( T \) or \( H \)). Then \( \tau(z_1 z_2) = \tau(z_1)z_2 \) is an isometry of the horocyclic product.

- Geometry \( \Rightarrow d(\tau z_n, z_n) = d_1(\tau_1 z_{n,1}, z_{n,1}) \leq d = d_c \), whence

  \[
  K(\tau x, z_n) = \frac{G(\tau x, z_n)}{G(\tau x, \tau z_n)} \frac{G(\tau x, \tau z_n)}{G(0, z_n)} \leq C_d K(x, z_n)
  \]

- \( \Rightarrow h(\tau x) \leq C_d h(x) \Rightarrow h(\tau x)/h(x) = \text{const.} \)

- Additional use of Harnack inequality

  \( \Rightarrow h(\tau x) = h(x), \quad h(x_1 x_2) \) depends only on \( x_2 \).
Methods

- Second, if $\sup_n h(z_{n,1}) < \infty$, then analogously $h(x_1 x_2)$ depends only on $x_1$.

- Every harmonic function $h \geq 0$ is an integral over minimal ones:
  \[
  h = \int_{M_{\text{min}}} K(\cdot, \xi) \, d\nu^h(\xi)
  \]

- Martin boundary $M$: boundary in compactification where
  - each $K(x, \cdot)$ extends continuously;
  - extended functions separate boundary points.

  $M_{\text{min}} = \{ \xi \in M : K(\cdot, \xi) \text{ minimal} \}$.

- $M_{\text{min}} \setminus M_{\text{min},1} \subset M_{\text{min},2}$  \Rightarrow

  \[
  h = \int_{M_{\text{min},1}} K(\cdot, \xi) \, d\nu^h(\xi) + \int_{M_{\text{min}} \setminus M_{\text{min},1}} K(\cdot, \xi) \, d\nu^h(\xi)
  \]
More on minimal harmonic funcs & Martin bdry

- For $P_\alpha$ on $DL(p,q)$ ($0 < \alpha < 1$):

  - All minimal harmonic functions (on the two trees) are known explicitly \cite{Woess, 2005}.
  - The Martin compactification is fully described \cite{Brofferio and Woess, 2005}.

  It depends on the vertical drift $a = 2\alpha - 1$. 

probability $\frac{\alpha}{p}$ on each edge

probability $\frac{1-\alpha}{q}$ on each edge
More on minimal harmonic fctns & Martin bdry

For $\mathcal{L}_a = \frac{1}{2} \left( e^{2pz} \partial_x^2 + e^{-2qz} \partial_y^2 + \partial_z^2 \right) + a \partial_z$ on $\text{Sol}(p, q)$ with vertical drift parameter $a$:

- All minimal harmonic functions (on the two hyperbolic planes) are known explicitly [Brofferio, Salvatori and Woess, 2011]. They are modified Poisson kernels.
- Compare also with [Raugi, 1996] (for random walks).
- We do not (yet) have the full Martin compactification $\equiv$ directions of convergence of Martin kernels.
More on minimal harmonic functions & Martin bdry

- For $\mathcal{L}_{\alpha,\beta}$ on $HT(p, q)$ ($\alpha \in \mathbb{R}, \beta > 0$):

  For $z = (x + iy, w) \in HT^0$

  $$\mathcal{L}_{\alpha,\beta} f(z) = y^2 (\partial_x^2 + \partial_y^2) f(z) + \alpha y \partial_y f(z)$$

  $$\partial_y f_V(z-) = \beta \cdot \sum_{u: u^- = v} \partial_y f_u(z+), \quad f_v = f|_{S_v}$$

- All minimal harmonic functions coming from $\mathbb{T}$ are known explicitly, but those coming from “sliced” $\mathbb{H}$ are known explicitly (modified Poisson kernels) only when $\beta p = 1$. [Bendikov, Saloff-Coste, Salvatori and Woess, 2014/15].

- We do not (yet) have the full Martin compactification.

In all three cases, the weak Liouville property holds (all bounded harmonic functions are constant; the Poisson boundary is trivial) if and only if $a = 0$ (vertical drift).
Martin compactification

- **Geometric compactification**
  - $\hat{T}$ end compactification
  - $\hat{H}$ hyperbolic compactification ($\equiv$ closed unit disk)

- **Horocyclic compactification**
  - $T$ has special end $\rho$. $H$ has special bdry point $\infty =: \rho$.

Both cases: replace $\rho$ by

- $\rho_k$, $k \in \mathbb{Z} \cup \{\pm \infty\}$ (tree, discrete case), resp.
- $\rho_t$, $t \in [-\infty, \infty]$ (metric tree, resp. hyp. plane)

- topology refines geometric compactification; $z_n \to \rho_t$ if $z_n \to \rho$ in geometric compactification, and $h(z_n) \to t$.
Martin compactification

Geometric / horocyclic compactification of horocyclic product: closure in the direct product of the geometric / horocyclic compactifications of the two factor spaces.
Conjecture. In all three cases: dependence on vertical drift

If \( a = 0 \) then Martin compactification = geometric compactification.

If \( a \neq 0 \) then Martin compactification = horocyclic compactification.

For \( P_\alpha \) on \( DL(p, q) \), conjecture is a
Theorem of [Brofferio and Woess, 2005]
More work & future work

- Horocyclic product of more than 2 trees:

\[ DL(p_1, \ldots, p_r) = \left\{ x_1 \cdots x_r \in \prod_{p_1} \times \cdots \times \prod_{p_r} : h(x_1) + \cdots + h(x_r) = 0 \right\} \]

with suitable neighbourhood relation.

- Comprehensive study by [Bartholdi, Neuhauser and Woess, 2008].

Comprises prominent group whose Cayley graph is \( DL(p, p, p) \), see recent work of [Amchislavska and Riley, 2014/15].

- Open question: is \( DL(2, 2, 2, 2) \) a Cayley graph?
More work & future work

- **Levelled trees**: $\mathbb{T}_{p,r}$ – each vertex has $p$ incoming and $r$ outgoing edges.

- **Levelled product with “sliced” $\mathbb{H}_q$** – if $p < r$ then **non-amenable Baumslag-Solitar group** $\text{BS}(p, r) = \langle a, b \mid a b^p = b^r a \rangle$ acts on resulting treebolic space with $q = r/p$.

- [Cuno and Sava, 2015]: **Poisson boundary** of random walks on $\text{BS}(p, r)$.