Null sequences which are defined by ℓ_p spaces

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Relations Between Banach Space Theory and Geometric Measure Theory

We establish numerous equivalences for a sequence (x_n) in a Banach space X to be a (p, r)-null sequence.

The talk is based on the following paper:

K. Ain, E. Oja, On (p, r)-null sequences and their relatives, Math. Nachr. (2015) DOI: 10.1002/mana.201400300.

The (p, r)-compactness

Throughout:
X Banach space, B_X its closed unit ball.
$1 \le p < \infty$, $1 \le r \le p^*$ $(1/p + 1/p^* = 1)$.

Definition (A–Lillemets–Oja, 2012)

$K \subset X$ is relatively (p, r)-compact if $K \subset (p, r)\text{-conv}(x_n) := \{\sum_n a_n x_n : (a_n) \in B_{\ell_r}\}$ (where $(a_n) \in B_{c_0}$ if $r = \infty$) for some $(x_n) \in \ell_p(X)$.

Definition (A–Lillemets–Oja, 2012)

$T \in \mathcal{K}_{(p,r)}(X, Y)$, i.e., linear $T : X \to Y$ is (p, r)-compact if $T(B_X) \subset Y$ is relatively (p, r)-compact in Y.

- (p, p^*)-compactness $=$ p-compactness (Sinha–Karn, 2002)

p-null sequences

Definition (Delgado–Piñeiro, 2011)

$(x_n) \subset X$ is p-null if for every $\varepsilon > 0$ there exist $(z_k) \in \varepsilon B_{\ell_p(X)}$ and $N \in \mathbb{N}$ such that $x_n \in (p, p^*)$-conv(z_k), $n \geq N$.

Theorem (Delgado–Piñeiro–Oja)

$(x_n) \subset X$ is p-null $\Leftrightarrow$ (x_n) is null and relatively p-compact.

Proof of Theorem relies

- in [Delgado–Piñeiro, 2011] on a version of approximation property depending on p – special case of X,
- in [Oja, 2012] on the description of the space of p-null sequences as Chevet–Saphar tensor product,
- in [Lassalle–Turco, 2014] on the use of the Carl–Stephani theory.

Motivation

- Prove the Theorem in some general (p, r)-case.
- Use the Carl–Stephani theory in a more efficient way.
- Use the developed direct method for further extended cases.

Preliminaries

$\boldsymbol{b}$ - the class of all bounded subsets in all Banach spaces
$\boldsymbol{g}(X) :=$ subsets of X that are of type $\boldsymbol{g}$ where $\boldsymbol{g} \subset \boldsymbol{b}$

- $\boldsymbol{w} :=$ all relatively weakly compact subsets in all Banach spaces
- $\boldsymbol{k} :=$ all relatively compact subsets in all Banach spaces
- $\boldsymbol{k}_{(p,r)} :=$ all relatively (p,r)-compact subsets in all Banach spaces

For an operator ideal $\mathcal{A}$, $\mathcal{A}(\boldsymbol{g}) \subset \boldsymbol{b}$ is given by
$\mathcal{A}(\boldsymbol{g})(X) := \{E \subset X : E \subset T(F)$ for some $F \in \boldsymbol{g}(Y)$ and $T \in \mathcal{A}(Y,X)\}$

Proposition (Grothendieck)

$\boldsymbol{k} = \overline{\mathcal{F}}(\boldsymbol{b}) = \mathcal{K}(\boldsymbol{b})$.

Theorem

$\boldsymbol{k}_{(p,r)} = \mathcal{N}_{(p,1,r^*)}(\boldsymbol{k}) = \mathcal{K}_{(p,r)}(\boldsymbol{k})$.

Carl-Stephani theory

Let $\mathcal{A}$ be an operator ideal

Definition (Carl–Stephani, 1985)

$(x_n) \subset X$ is $\mathcal{A}$-null if there exist a Banach space Y, $(y_n) \in c_0(Y)$, and $T \in \mathcal{A}(Y, X)$ such that $x_n = Ty_n\ \forall n$.

Definition (Carl–Stephani, 1985)

$K \subset X$ is $\mathcal{A}$-compact if $K \in \mathcal{A}(\boldsymbol{k})(X)$.

Theorem (Lassalle–Turco, 2012)

$(x_n) \subset X$ is $\mathcal{A}$-null $\Leftrightarrow$ (x_n) is null and $\mathcal{A}$-compact.

(p, r)-null sequences

Definition (A–Oja, 2012)

$(x_n) \subset X$ is (p, r)-null if for every $\varepsilon > 0$ there exist $(z_k) \in \varepsilon B_{\ell_p(X)}$ and $N \in \mathbb{N}$ such that $x_n \in (p, r)\text{-conv}(z_k)$, $n \geq N$.

- (p, p^*)-null $=$ p-null

Definition

$(x_n) \subset X$ uniformly (p, r)-null if there exists $(z_k) \in B_{\ell_p(X)}$ with the following property: for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $x_n \in \varepsilon\,(p, r)\text{-conv}(z_k)$ for all $n \geq N$.

Main theorem

Theorem (Delgado–Piñeiro–Oja)

$(x_n) \subset X$ is p-null $\Leftrightarrow$ (x_n) is null and relatively p-compact.

Theorem

For $(x_n) \subset X$ TFAE:

1. (x_n) *is* (p, r)*-null,*
2. (x_n) *is null and relatively* (p, r)*-compact,*
3. (x_n) *is null and* $\mathcal{N}_{(p,1,r^*)}$*-compact,*
4. (x_n) *is null and* $\mathcal{K}_{(p,r)}$*-compact,*
5. (x_n) *is* $\mathcal{N}_{(p,1,r^*)}$*-null,*
6. (x_n) *is* $\mathcal{K}_{(p,r)}$*-null,*
7. (x_n) *is uniformly* (p, r)*-null.*

Definition

$K \subset X$ is relatively unconditionally (p, r)-compact (i.e., $K \in \boldsymbol{u}_{(p,r)}(X)$) if $K \subset (p, r)\text{-conv}(x_n) := \{\sum_{n=1}^{\infty} a_n x_n : (a_n) \in B_{\ell_r}\}$ for some $(x_n) \in \ell_p^u(X)$.

Definition

Linear $T : Y \to X$ is unconditionally (p, r)-compact (i.e., $T \in \mathcal{U}_{(p,r)}(Y, X)$) if $T(B_Y) \in \boldsymbol{u}_{(p,r)}(X)$.

Definition

$(x_n) \subset X$ unconditionally (p, r)-null if for every $\varepsilon > 0$ there exist $N \in \mathbb{N}$ and $(z_k) \in \ell_p^u(X)$ with $\|(z_k)\|_p^w \leq \varepsilon$ such that $x_n \in (p, r)\text{-conv}(z_k)$ for all $n \geq N$.

Definition

$(x_n) \subset X$ uniformly unconditionally (p, r)-null if there exists $(z_k) \in B_{\ell_p^u(X)}$ with the following property: for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $x_n \in \varepsilon\,(p, r)\text{-conv}(z_k)$ for all $n \geq N$.

Theorem

For $(x_n) \subset X$ TFAE:

1. (x_n) *is unconditionally* (p, r)*-null,*
2. (x_n) *is null and relatively unconditionally* (p, r)*-compact,*
3. (x_n) *is null and* $\mathcal{N}_{(\infty, p^*, r^*)}$*-compact,*
4. (x_n) *is null and* $\mathcal{U}_{(p,r)}$*-compact,*
5. (x_n) *is* $\mathcal{N}_{(\infty, p^*, r^*)}$*-null,*
6. (x_n) *is* $\mathcal{U}_{(p,r)}$*-null,*
7. (x_n) *is uniformly unconditionally* (p, r)*-null.*

Theorem (Kim 2014)

$(x_n) \subset X$ *is unconditionally p-null* $\Leftrightarrow$ (x_n) *is null and relatively unconditionally p-compact.*

Definition

$K \subset X$ is relatively weakly (p, r)-compact (i.e., $K \in \boldsymbol{w}_{(p,r)}(X)$) if $K \subset (p, r)\text{-conv}(x_n) := \{\sum_{n=1}^{\infty} a_n x_n : (a_n) \in B_{\ell_r}\}$ for some $(x_n) \in \ell_p^w(X)$

Definition

Linear $T : Y \to X$ is weakly (p, r)-compact (i.e., $T \in \mathcal{W}_{(p,r)}(Y, X)$) if $T(B_Y) \in \boldsymbol{w}_{(p,r)}(X)$.

Definition

$(x_n) \subset X$ is weakly (p, r)-null if for every $x^* \in X^*$ and every $\varepsilon > 0$ there exist $(z_k) \in \ell_p^w(X)$ and $N \in \mathbb{N}$ such that $|x^*(x_n)| \le \varepsilon$ and $x_n \in (p, r)\text{-conv}(z_k)$ for all $n \ge N$.

Definition

$(x_n) \subset X$ is uniformly weakly (p, r)-null if there exists $(z_k) \in \ell_p^w(X)$ with the following property: for every $x^* \in X^*$ and every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x^*(x_n)| \le \varepsilon$ and $x_n \in (p, r)\text{-conv}(z_k)$ for all $n \ge N$.

Definition

$(x_n) \subset X$ is weakly $\mathcal{A}$-null if there exist a Banach space Y, a weakly null sequence (y_n) in Y, and $T \in \mathcal{A}(Y, X)$ such that $x_n = Ty_n$ for all $n \in \mathbb{N}$.

Definition

$K \subset X$ is weakly $\mathcal{A}$-compact if $K \in \mathcal{A}(\boldsymbol{w})(X)$ where $\boldsymbol{w}$ denotes the class of all relatively weakly compact sets.

Theorem

Let $1 \leq p < \infty$ and $1 < r \leq p^*$ with $r < \infty$ if $p = 1$. For $(x_n) \subset X$ TFAE:

1. *(x_n) is weakly (p, r)-null,*
2. *(x_n) is weakly null and relatively weakly (p, r)-compact,*
3. *(x_n) is weakly null and weakly $\mathcal{W}_{(p,r)}$-compact,*
4. *(x_n) is weakly $\mathcal{W}_{(p,r)}$-null,*
5. *(x_n) is uniformly weakly (p, r)-null.*

Thank you!