Hölder's inequality on mixed L_p spaces and summability of multilinear operators

Nacib Albuquerque

Federal Rural University of Pernambuco

Relations Between Banach Space Theory and Geometric Measure Theory

The University of Warwick – Coventry, UK

10th June 2015

Motivation: interpolative puzzels

Let us suppose that we have the following two inequalities at hand, for certain complex scalar matrix $(a_{ij})_{i,j=1}^{N}$:

$$\sum_{i=1}^{N}\left(\sum_{j=1}^{N}|a_{ij}|^2\right)^{\frac{1}{2}} \leq \mathrm{C}_1 \text{ and } \sum_{j=1}^{N}\left(\sum_{i=1}^{N}|a_{ij}|^2\right)^{\frac{1}{2}} \leq \mathrm{C}_2$$

for some constant $\mathrm{C} > 0$ and all positive integers N.

Motivation: interpolative puzzels

Let us suppose that we have the following two inequalities at hand, for certain complex scalar matrix $(a_{ij})_{i,j=1}^N$:

$$\sum_{i=1}^N \left(\sum_{j=1}^N |a_{ij}|^2 \right)^{\frac{1}{2}} \leq \mathrm{C}_1 \text{ and } \sum_{j=1}^N \left(\sum_{i=1}^N |a_{ij}|^2 \right)^{\frac{1}{2}} \leq \mathrm{C}_2$$

for some constant $\mathrm{C} > 0$ and all positive integers N.

How can one find an optimal exponent r and a constant $\mathrm{C}_1 > 0$ such that

$$\left(\sum_{i,j=1}^N |a_{ij}|^r \right)^{\frac{1}{r}} \leq \mathrm{C}_3, \quad \text{for all positive integers} \quad N \ ?$$

Moreover, how can one get a good (small) constant C_3?

Using a consequence of Minkowski's inequality and applying Hölder's inequality successively:

$$
\begin{aligned}
\sum_{i,j=1}^{N} |a_{ij}|^{\frac{4}{3}} &= \sum_{i=1}^{N} \left(\sum_{j=1}^{N} |a_{ij}|^{\frac{2}{3}} \, |a_{ij}|^{\frac{2}{3}} \right) \\
&\leq \sum_{i=1}^{N} \left(\left(\sum_{j=1}^{N} |a_{ij}|^2 \right)^{\frac{1}{3}} \left(\sum_{j=1}^{N} |a_{ij}| \right)^{\frac{2}{3}} \right) \\
&\leq \left(\sum_{i=1}^{N} \left(\sum_{j=1}^{N} |a_{ij}|^2 \right)^{\frac{1}{2}} \right)^{\frac{2}{3}} \left(\sum_{i=1}^{N} \left(\sum_{j=1}^{N} |a_{ij}| \right)^2 \right)^{\frac{1}{3}} \\
&= \left[\sum_{i=1}^{N} \left(\sum_{j=1}^{N} |a_{ij}|^2 \right)^{\frac{1}{2}} \right]^{\frac{2}{3}} \left[\left(\sum_{i=1}^{N} \left(\sum_{j=1}^{N} |a_{ij}| \right)^2 \right)^{\frac{1}{2}} \right]^{\frac{2}{3}} \leq \mathrm{C}_1^{\frac{2}{3}} \cdot \mathrm{C}_2^{\frac{2}{3}}
\end{aligned}
$$

Using a consequence of Minkowski's inequality and applying Hölder's inequality successively:

$$\begin{aligned}\sum_{i,j=1}^{N} |a_{ij}|^{\frac{4}{3}} &= \sum_{i=1}^{N} \left(\sum_{j=1}^{N} |a_{ij}|^{\frac{2}{3}} \, |a_{ij}|^{\frac{2}{3}} \right) \\ &\leq \sum_{i=1}^{N} \left(\left(\sum_{j=1}^{N} |a_{ij}|^{2} \right)^{\frac{1}{3}} \left(\sum_{j=1}^{N} |a_{ij}| \right)^{\frac{2}{3}} \right) \\ &\leq \left(\sum_{i=1}^{N} \left(\sum_{j=1}^{N} |a_{ij}|^{2} \right)^{\frac{1}{2}} \right)^{\frac{2}{3}} \left(\sum_{i=1}^{N} \left(\sum_{j=1}^{N} |a_{ij}| \right)^{2} \right)^{\frac{1}{3}} \\ &= \left[\sum_{i=1}^{N} \left(\sum_{j=1}^{N} |a_{ij}|^{2} \right)^{\frac{1}{2}} \right]^{\frac{2}{3}} \left[\left(\sum_{i=1}^{N} \left(\sum_{j=1}^{N} |a_{ij}| \right)^{2} \right)^{\frac{1}{2}} \right]^{\frac{2}{3}} \leq \mathrm{C}_1^{\frac{2}{3}} \cdot \mathrm{C}_2^{\frac{2}{3}}\end{aligned}$$

Solution: "interpolation via Hölder's inequality".

Mixed L_p spaces

A. Benedek and R. Panzone introduce the mixed L_p spaces notion on:

"The space $L_{\mathbf{p}}$*, with mixed norm", Duke Math. J. 28 (1961), 301-324.*

Mixed L_p spaces

A. Benedek and R. Panzone introduce the mixed L_p spaces notion on:

"The space $L_{\mathbf{p}}$, with mixed norm", Duke Math. J. 28 (1961), 301-324.

Let (X_i, Σ_i, μ_i) , $i = 1, \ldots, m$ be σ-finite measurable spaces, let

$$(\mathbf{X}, \Sigma, \mu) := \left(\prod_{i=1}^{m} X_i, \prod_{i=1}^{m} \Sigma_i, \prod_{i=1}^{m} \mu_i \right)$$

be the product space and $\mathbf{p} := (p_1, \ldots, p_m) \in [1, \infty]^m$.

Mixed L_p spaces

A. Benedek and R. Panzone introduce the mixed L_p spaces notion on:

"The space $L_{\mathbf{p}}$, with mixed norm", Duke Math. J. 28 (1961), 301-324.

Let (X_i, Σ_i, μ_i), $i = 1, \ldots, m$ be σ-finite measurable spaces, let

$$(\mathbf{X}, \Sigma, \mu) := \left(\prod_{i=1}^{m} X_i, \prod_{i=1}^{m} \Sigma_i, \prod_{i=1}^{m} \mu_i \right)$$

be the product space and $\mathbf{p} := (p_1, \ldots, p_m) \in [1, \infty]^m$.

The space $L_{\mathbf{p}}(\mathbf{X})$ consists in all measurable functions $f : \mathbf{X} \to \mathbb{K}$ with the following property:

$f(x_1, \ldots, x_{m-1}, \cdot) \in L_{p_m}(X_m)$, i.e., $\|f\|_{p_m} := \|f(x_1, \ldots, x_{m-1}, \cdot)\|_{p_m} < \infty$,

for any $(x_1, \ldots, x_{m-1}) \in \prod_{i=1}^{n-1} X_i$ and, also $\|f\|_{p_m}$, results in a measurable function;

Mixed L_p spaces

A. Benedek and R. Panzone introduce the mixed L_p spaces notion on:

"The space $L_{\mathbf{p}}$, with mixed norm", Duke Math. J. 28 (1961), 301-324.

Let (X_i, Σ_i, μ_i), $i = 1, \ldots, m$ be σ-finite measurable spaces, let

$$(\mathbf{X}, \Sigma, \mu) := \left(\prod_{i=1}^{m} X_i, \prod_{i=1}^{m} \Sigma_i, \prod_{i=1}^{m} \mu_i \right)$$

be the product space and $\mathbf{p} := (p_1, \ldots, p_m) \in [1, \infty]^m$.

The space $L_{\mathbf{p}}(\mathbf{X})$ consists in all measurable functions $f : \mathbf{X} \to \mathbb{K}$ with the following property:

$$f(x_1, \ldots, x_{m-1}, \cdot) \in L_{p_m}(X_m), \text{ i.e., } \|f\|_{p_m} := \|f(x_1, \ldots, x_{m-1}, \cdot)\|_{p_m} < \infty,$$

for any $(x_1, \ldots, x_{m-1}) \in \prod_{i=1}^{n-1} X_i$ and, also $\|f\|_{p_m}$, results in a measurable function; this process is repeated successively: the resulting p_{m-1}-norm, p_{m-2}-norm,..., p_1-norm (in this order) are finite.

For instance, when all $p_i < \infty$ a measurable function $f : \mathbf{X} \to \mathbb{K}$ it is an element of $L_{\mathbf{p}}(\mathbf{X})$ if, and only if,

$$\|f\|_{\mathbf{p}} := \left(\int_{X_1} \left(\dots \left(\int_{X_m} |f|^{p_m} d\mu_m \right)^{\frac{p_{m-1}}{p_m}} \dots \right)^{\frac{p_1}{p_2}} d\mu_1 \right)^{\frac{1}{p_1}} < \infty.$$

For instance, when all $p_i < \infty$ a measurable function $f : \mathbf{X} \to \mathbb{K}$ it is an element of $L_{\mathbf{p}}(\mathbf{X})$ if, and only if,

$$\|f\|_{\mathbf{p}} := \left(\int_{X_1} \left(\dots \left(\int_{X_m} |f|^{p_m} d\mu_m \right)^{\frac{p_{m-1}}{p_m}} \dots \right)^{\frac{p_1}{p_2}} d\mu_1 \right)^{\frac{1}{p_1}} < \infty.$$

Some classical properties and results concerning the $L_{\mathbf{p}}$ spaces:

- $L_{\mathbf{p}}(\mathbf{X})$ is a Banach space;
- Monotone's convergence classical theorems;
- Lebesgue's dominated convergence theorem.

Mixed Hölder's inequality

We are interested in a *"simple"* result:

Theorem (Mixed Hölder's inequality)

Let $\mathbf{r} \in [1, \infty)^m$ *and* $, \mathbf{p}(1), \ldots, \mathbf{p}(N) \in [1, \infty]^m$ *be such that*

$$\frac{1}{r_j} = \frac{1}{p_j(1)} + \cdots + \frac{1}{p_j(N)}, \quad \text{for } j = 1, \ldots, m.$$

If $f_k \in L_{\mathbf{p}(k)}(\mathbf{X})$ *for* $k = 1, \ldots, N$, *then*

$$f_1 f_2 \cdots f_N \in L_{\mathbf{r}}(\mathbf{X})$$

and, moreover,

$$\|f_1 \cdots f_N\|_{\mathbf{r}} \leq \|f_1\|_{\mathbf{p}(1)} \cdots \|f_N\|_{\mathbf{p}(N)}.$$

Interpolative approach

Corollary [Mixed interpolative Hölder's inequality]

Let $\mathbf{r}, \mathbf{p}(1), \ldots, \mathbf{p}(N) \in [1, \infty]^m$ and $\theta_1, \ldots, \theta_N \in [0,1]$ be such that

$$\theta_1 + \cdots + \theta_N = 1$$

and

$$\frac{1}{r_j} = \sum_{k=1}^{N} \frac{\theta_k}{p_j(k)} = \frac{\theta_1}{p_j(1)} + \cdots + \frac{\theta_N}{p_j(N)}, \qquad \text{for } j = 1, \ldots, m.$$

If $f \in L_{\mathbf{p}(k)}(X)$ for $k = 1, \ldots, N$, then $f \in L_{\mathbf{r}}(X)$ and, moreover,

$$\|f\|_{\mathbf{r}} \leq \|f\|_{\mathbf{p}(1)}^{\theta_1} \cdots \|f\|_{\mathbf{p}(N)}^{\theta_N}.$$

Mixed norm sequence spaces

Let X be a Banach space and $\mathbf{p} \in [1, \infty)^m$. The mixed norm sequence space

$$\ell_{\mathbf{p}}(X) := \ell_{p_1}\left(\ell_{p_2}\left(\ldots\left(\ell_{p_m}(X)\right)\ldots\right)\right)$$

is formed by all multi-index vector valued matrices $(x_{\mathbf{i}})_{\mathbf{i} \in \mathbb{N}^m}$ with finite $\mathbf{p}$-norm that is,

$$\left\|(x_{\mathbf{i}})_{\mathbf{i}}\right\|_{\mathbf{p}} := \left(\sum_{i_1=1}^{\infty}\left(\ldots\left(\sum_{i_m=1}^{\infty}\|x_{\mathbf{i}}\|_X^{p_m}\right)^{\frac{p_{m-1}}{p_m}}\ldots\right)^{\frac{p_1}{p_2}}\right)^{\frac{1}{p_1}} < \infty.$$

When $X = \mathbb{K}$, we just write $\ell_{\mathbf{p}}$ instead of $\ell_{\mathbf{p}}(\mathbb{K})$.

Hölder's interpolative inequality for sequences

The next interpolation result on these mixed norm sequences spaces has a central role on the results we will present.

Corollary [Hölder's interpolative inequality for mixed $\ell_{\mathbf{p}}$ spaces]

Let m, n, N be positive integers, $\mathbf{r}, \mathbf{p}(1), \ldots, \mathbf{p}(N) \in [1, \infty]^m$ and $\theta_1, \ldots, \theta_N \in [0, 1]$ be such that $\theta_1 + \cdots + \theta_N = 1$ and

$$\frac{1}{r_j} = \sum_{k=1}^{N} \frac{\theta_k}{p_j(k)} = \frac{\theta_1}{p_j(1)} + \cdots + \frac{\theta_N}{p_j(N)}, \qquad \text{for } j = 1, \ldots, m.$$

Then, for all scalar matrix $\mathbf{a} := (\mathrm{a}_{\mathbf{i}})_{\mathbf{i} \in \mathcal{M}(m,n)}$, we have

$$\|\mathbf{a}\|_{\mathbf{r}} \leq \|\mathbf{a}\|_{\mathbf{p}(1)}^{\theta_1} \cdots \|\mathbf{a}\|_{\mathbf{p}(N)}^{\theta_N}.$$

Hölder's interpolative inequality for sequences

In particular, if each $\mathbf{p}(k) \in [1, \infty)$, the previous inequality means that

$$\left(\sum_{i_1=1}^{n} \left(\cdots \left(\sum_{i_m=1}^{n} |\mathrm{a}_{\mathbf{i}}|^{r_m} \right)^{\frac{r_{m-1}}{r_m}} \cdots \right)^{\frac{r_1}{r_2}} \right)^{\frac{1}{r_1}}$$

$$\leq \prod_{k=1}^{N} \left[\left(\sum_{i_1=1}^{n} \left(\cdots \left(\sum_{i_m=1}^{n} |\mathrm{a}_{\mathbf{i}}|^{p_m(k)} \right)^{\frac{p_{m-1}(k)}{p_m(k)}} \cdots \right)^{\frac{p_1(k)}{p_2(k)}} \right)^{\frac{1}{p_1(k)}} \right]^{\theta_k}.$$

Hölder's interpolative inequality for sequences

In particular, if each $\mathbf{p}(k) \in [1, \infty)$, the previous inequality means that

$$\left(\sum_{i_1=1}^{n} \left(\cdots \left(\sum_{i_m=1}^{n} |a_{\mathbf{i}}|^{r_m} \right)^{\frac{r_{m-1}}{r_m}} \cdots \right)^{\frac{r_1}{r_2}} \right)^{\frac{1}{r_1}}$$

$$\leq \prod_{k=1}^{N} \left[\left(\sum_{i_1=1}^{n} \left(\cdots \left(\sum_{i_m=1}^{n} |a_{\mathbf{i}}|^{p_m(k)} \right)^{\frac{p_{m-1}(k)}{p_m(k)}} \cdots \right)^{\frac{p_1(k)}{p_2(k)}} \right)^{\frac{1}{p_1(k)}} \right]^{\theta_k}.$$

Thanks anonymous referee!

Multilinear Bohnenblust-Hille's inequality

- **[1931]** H. F. Bohnenblust and E. Hille generalized the Littlewood's 4/3-inequality and solved de Harald Bohr's radius strip problem.

Theorem (Multilinear Bohnenblust-Hille's inequality)

For each positive integer $m \geq 1$, there exists a constant $C_m \geq 1$ such that

$$\left(\sum_{i_1,\ldots,i_m=1}^{\infty} \|A(e_{i_1},\ldots,e_{i_m})\|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq C_m \|A\|,$$

for all continuous m-linear forms $A : c_0 \times \cdots \times c_0 \to \mathbb{K}$. Moreover, the exponent $\frac{2m}{m+1}$ is optimal.

Multilinear Hardy-Littlewood's inequality

- **[1934]** G. Hardy and J. P. Littlewood provided an ℓ_p-version for the bilinear case (Littlewood's 4/3 inequality).
- **[1981]** T. Praciano-Pereira obtained a general result for multilinear forms on ℓ_p spaces.

Multilinear Hardy-Littlewood's inequality

- **[1934]** G. Hardy and J. P. Littlewood provided an ℓ_p-version for the bilinear case (Littlewood's 4/3 inequality).
- **[1981]** T. Praciano-Pereira obtained a general result for multilinear forms on ℓ_p spaces.

Let us define $X_p := \ell_p$, $1 \leq p < +\infty$ and $X_\infty := c_0$.

Theorem (Multilinear Hardy-Littlewood's inequality)

Let $\mathbf{p} \in [1, +\infty]^m$ with $\left|\frac{1}{\mathbf{p}}\right| := \frac{1}{p_1} + \cdots + \frac{1}{p_m} \leq \frac{1}{2}$. Then there exists a constant $C_{m,\mathbf{p}} \geq 1$ such that, for every continuous m-linear form $A : X_{p_1} \times \cdots \times X_{p_m} \to \mathbb{C}$,

$$\left(\sum_{i_1,\ldots,i_m=1}^{\infty} |A(e_{i_1}, \ldots, e_{i_m})|^{\frac{2m}{m+1-2\left|\frac{1}{\mathbf{p}}\right|}} \right)^{\frac{m+1-2\left|\frac{1}{\mathbf{p}}\right|}{2m}} \leq C_{m,\mathbf{p}} \|A\|.$$

After results...

- **[2009]** Defant and Sevilla-Peris;
- **[2013]** A., Bayart, Pellegrino and Seoane;
- **[2013]** Dimant and Sevilla-Peris.

Unifying result

Theorem (A., Bayart, Pellegrino, Seoane (2014))

Let $\mathbf{p} \in [1, +\infty]^m$ *and* $1 \leq s \leq q \leq \infty$ *be such that*

$$\left| \frac{1}{\mathbf{p}} \right| < \frac{1}{2} + \frac{1}{s} - \frac{1}{\min\{q, 2\}}.$$

If $\lambda := \left[\frac{1}{2} + \frac{1}{s} - \frac{1}{\min\{q,2\}} - \left| \frac{1}{\mathbf{p}} \right| \right]^{-1} > 0$ *and* $t_1, \ldots, t_m \in [\lambda, \max\{\lambda, s, 2\}]$ *are such that*

$$\frac{1}{t_1} + \cdots + \frac{1}{t_m} \leq \frac{1}{\lambda} + \frac{m-1}{\max\{\lambda, s, 2\}},$$

then there exists $C > 0$ *satisfying, for every continuous* m*-linear map* $A : X_{p_1} \times \cdots \times X_{p_m} \to X_s$,

$$\left(\sum_{i_1=1}^{+\infty} \left(\cdots \left(\sum_{i_m=1}^{+\infty} \|A(e_{i_1}, \ldots, e_{i_m})\|_{\ell_q}^{t_m} \right)^{\frac{t_{m-1}}{t_m}} \cdots \right)^{\frac{t_1}{t_2}} \right)^{\frac{1}{t_1}} \leq C\|A\|.$$

Moreover, the exponents are optimal except eventually if $q \leq 2$ *and* $\left| \frac{1}{\mathbf{p}} \right| > \frac{1}{2}$.

Tools for the proof (sufficiency)

- norm-mixed estimate for (ℓ_λ, ℓ_q) or cotype version of Khinchinte's inequality [Dimant and Sevilla-Peris (2013)];

Tools for the proof (sufficiency)

- norm-mixed estimate for (ℓ_λ, ℓ_q) or cotype version of Khinchinte's inequality [Dimant and Sevilla-Peris (2013)];
- Bennet-Carl inequality;

Tools for the proof (sufficiency)

- norm-mixed estimate for (ℓ_λ, ℓ_q) or cotype version of Khinchinte's inequality [Dimant and Sevilla-Peris (2013)];
- Bennet-Carl inequality;
- Interpolative Hölder's inequality.

Multiple summing operators point of view

From now on, $E_1, E_2, \ldots, F$ shall denote Banach spaces.

Multiple summing operators point of view

From now on, $E_1, E_2, \ldots, F$ shall denote Banach spaces.

Proposition [Bohnenblust-Hille re-written]

If $\mathbf{q} \in [1,2]^m$ is such that $\left|\frac{1}{\mathbf{q}}\right| \leq \frac{1}{2}$, then

$$\left(\sum_{j_1=1}^{\infty} \left(\cdots \left(\sum_{j_m=1}^{\infty} \left| T\left(x_{j_1}^{(1)}, \ldots, x_{j_m}^{(m)} \right) \right|^{q_m} \right)^{\frac{q_{m-1}}{q_m}} \cdots \right)^{\frac{q_1}{q_2}} \right)^{\frac{1}{q_1}}$$

$$\leq B_{m,(q_1,\ldots,q_m)}^{\mathbb{K}} \|T\| \prod_{k=1}^{m} \left\| \left(x_{j_k}^{(k)} \right)_{j_k=1}^{\infty} \right\|_{w,1},$$

for all bounded m–linear forms $T : E_1 \times \cdots \times E_m \to \mathbb{K}$ and all sequences $\left(x_{j_k}^{(k)} \right)_{j_k=1}^{\infty} \in \ell_1^w \left(E_k \right), k = 1, \ldots, m.$

Proposition [Hardy-Littlewood re-written]

Let $m \geq 1$, $\mathbf{p} \in [1,\infty]^m$. If $0 \leq \left|\frac{1}{\mathbf{p}}\right| \leq \frac{1}{2}$ and $\mathbf{q} \in \left[\left(1-\left|\frac{1}{\mathbf{p}}\right|\right)^{-1}, 2\right]^m$ are such that

$$\left|\frac{1}{\mathbf{q}}\right| \leq \frac{m+1}{2} - \left|\frac{1}{\mathbf{p}}\right|.$$

Then, for all continuous m–linear forms $T : E_1 \times \cdots \times E_m \to \mathbb{K}$,

$$\left(\sum_{i_1=1}^{\infty}\left(\cdots\left(\sum_{i_m=1}^{\infty}\left|T\left(x_{i_1}^{(1)},\ldots,x_{i_m}^{(m)}\right)\right|^{q_m}\right)^{\frac{q_{m-1}}{q_m}}\cdots\right)^{\frac{q_1}{q_2}}\right)^{\frac{1}{q_1}} \leq C_{m,\mathbf{p},\mathbf{q}}^{\mathbb{K}} \|T\| \prod_{k=1}^{m}\left\|\left(x_i^{(k)}\right)_{i=1}^{\infty}\right\|_{w,p_k^*},$$

regardless of the sequences $\left(x_{j_k}^{(k)}\right)_{i=1}^{\infty} \in \ell_{p_k^*}^{w}(E_k)$, $k=1,\ldots,m$.

Parttially multiple summnig operators: the designs

Parttially multiple summnig operators: the designs

For Banach spaces $E_1, \ldots, E_m$ and an element $x \in E_j$, for some $j \in \{1, \ldots, m\}$, the symbol $x \cdot e_j$ represents the vector $x \cdot e_j \in E_1 \times \cdots \times E_m$ such that the j-th coordinate is $x \in E_j$, and 0 otherwise.

Definition

Let $E_1, \ldots, E_m, F$ be Banach spaces, m, k be positive integers with $1 \leq k \leq m$, and $(\mathbf{p}, \mathbf{q}) := (p_1, \ldots, p_m, q_1, \ldots, q_k) \in [1, \infty)^{m+k}$. Let also $\mathcal{I} = \{I_1, \ldots, I_k\}$ a family of non-void disjoints subsets of $\{1, \ldots, m\}$ such that $\cup_{i=1}^{k} I_i = \{1, \ldots, m\}$, that is, $\mathcal{I}$ is a partition of $\{1, \ldots, m\}$. A multilinear operator $T : E_1 \times \cdots \times E_m \to F$ is $\mathcal{I}$–partially multiple $(\mathbf{q}; \mathbf{p})$–summing if there exists a constant $C > 0$ such that

$$\left(\sum_{i_1=1}^{\infty} \left(\cdots \left(\sum_{i_k=1}^{\infty} \left\| T \left(\sum_{n=1}^{k} \sum_{j \in I_n} x_{i_n}^{(j)} \cdot e_j \right) \right\|_F^{q_k} \right)^{\frac{q_{k-1}}{q_k}} \cdots \right)^{\frac{q_1}{q_2}} \right)^{\frac{1}{q_1}}$$

$$\leq C \prod_{j=1}^{m} \left\| \left(x_i^{(j)} \right)_{i=1}^{\infty} \right\|_{w, p_j}$$

Definition

for all $\left(x_i^{(j)}\right)_{i=1}^{\infty} \in \ell_{p_j}^{w}\left(E_j\right), j = 1, \ldots, m$. We represent the class of all $\mathcal{I}$–partially multiple $(\mathbf{q};\mathbf{p})$–summing operators by $\Pi_{(\mathbf{q};\mathbf{p})}^{k,m,\mathcal{I}}(E_1, \ldots, E_m; F)$. The infimum taken over all possible constants $C > 0$ satisfying the previous inequality defines a norm in $\Pi_{(\mathbf{q};\mathbf{p})}^{k,m,\mathcal{I}}(E_1, \ldots, E_m; F)$, which is denoted by $\pi_{(\mathbf{q};\mathbf{p})}^{\mathcal{I}}$.

Definition

for all $\left(x_i^{(j)}\right)_{i=1}^{\infty} \in \ell_{p_j}^{w}(E_j)$, $j = 1, \ldots, m$. We represent the class of all $\mathcal{I}$–partially multiple $(\mathbf{q};\mathbf{p})$–summing operators by $\Pi_{(\mathbf{q};\mathbf{p})}^{k,m,\mathcal{I}}(E_1, \ldots, E_m; F)$. The infimum taken over all possible constants $C > 0$ satisfying the previous inequality defines a norm in $\Pi_{(\mathbf{q};\mathbf{p})}^{k,m,\mathcal{I}}(E_1, \ldots, E_m; F)$, which is denoted by $\pi_{(\mathbf{q};\mathbf{p})}^{\mathcal{I}}$.

Note that when

- $k = 1$, we recover the class of absolutely $(q; p_1, \ldots, p_m)$–summing operators, with $q := q_1$;
- $k = m$ and $q_1 = \cdots = q_m =: q$, we recover the class of multiple $(q; p_1, \ldots, p_m)$–summing operators.

From now on, m, k are positive integers with $1 \leq k \leq m$, $(\mathbf{p}, \mathbf{q}) := (p_1, \ldots, p_m, q_1, \ldots, q_k) \in [1, \infty)^{m+k}$ and $\mathcal{I} = \{I_1, \ldots, I_k\}$ is a partition of $\{1, \ldots, m\}$.

BH partially summ. version

Theorem [Bohnenblust-Hille's partially summ. version]

Let $\mathbf{q} \in [1,2]^k$ such that $\left|\frac{1}{\mathbf{q}}\right| \leq \frac{k+1}{2}$. Then

$$\left(\sum_{i_1=1}^{\infty} \left(\cdots \left(\sum_{i_k=1}^{\infty} \left| T \left(\sum_{n=1}^{k} \sum_{j \in I_n} x_{i_n}^{(j)} \cdot e_j \right) \right|^{q_k} \right)^{\frac{q_{k-1}}{q_k}} \cdots \right)^{\frac{q_1}{q_2}} \right)^{\frac{1}{q_1}}$$

$$\leq B_{k,\mathbf{q}}^{\mathbb{K}} \|T\| \prod_{j=1}^{m} \left\| \left(x_i^{(j)} \right)_{i=1}^{\infty} \right\|_{w,1},$$

for all m–linear forms $T : E_1 \times \cdots \times E_m \to \mathbb{K}$ and all sequences $\left(x_i^{(j)} \right)_{i=1}^{\infty} \in \ell_1^w (E_j), j = 1, \ldots, m.$

In other words, when $\mathbf{q} \in [1,2]^k$ such that $\left|\frac{1}{\mathbf{q}}\right| \leq \frac{k+1}{2}$ we have the following coincidence result:

$$\Pi_{(\mathbf{q};\mathbf{1})}^{k,m,\mathcal{I}}(E_1, \ldots, E_m; F) = \mathcal{L}(E_1, \ldots, E_m; \mathbb{K}),$$

with $\mathbf{1} := (1, \overset{m \text{ times}}{\ldots}, 1)$.

HL partially summ. version

Theorem [Hardy-Littlewood's partially summ. version]

Let $1 \leq k \leq m$, $\mathbf{p} \in [1,\infty]^m$. If $0 \leq \left|\frac{1}{\mathbf{p}}\right| \leq \frac{1}{2}$ and $\mathbf{q} \in \left[\left(1 - \left|\frac{1}{\mathbf{p}}\right|\right)^{-1}, 2\right]^k$ are such that $\left|\frac{1}{\mathbf{q}}\right| \leq \frac{k+1}{2} - \left|\frac{1}{\mathbf{p}}\right|$, then, for all continuous m–linear forms $T : E_1 \times \cdots \times E_m \to \mathbb{K}$,

$$\left(\sum_{i_1=1}^{\infty}\left(\cdots\left(\sum_{i_k=1}^{\infty}\left|T\left(\sum_{n=1}^{k}\sum_{j\in I_n} x_{i_n}^{(j)} \cdot e_j\right)\right|^{q_k}\right)^{\frac{q_{k-1}}{q_k}}\cdots\right)^{\frac{q_1}{q_2}}\right)^{\frac{1}{q_1}}$$

$$\leq C_{k,m,\mathbf{p},\mathbf{q}}^{\mathbb{K}} \|T\| \prod_{j=1}^{m} \left\|\left(x_i^{(j)}\right)_{i=1}^{\infty}\right\|_{w,p_j^*},$$

regardless of the sequences $\left(x_i^{(j)}\right)_{i=1}^{\infty} \in \ell_{p_j^*}^{w}\left(E_j\right)$, $j = 1, \ldots, m$.

HL partially summ. version

Theorem [Hardy-Littlewood's partially summ. version]

Let $1 \leq k \leq m$, $\mathbf{p} \in [1,\infty]^m$. If $0 \leq \left|\frac{1}{\mathbf{p}}\right| \leq \frac{1}{2}$ and $\mathbf{q} \in \left[\left(1-\left|\frac{1}{\mathbf{p}}\right|\right)^{-1}, 2\right]^k$ are such that $\left|\frac{1}{\mathbf{q}}\right| \leq \frac{k+1}{2} - \left|\frac{1}{\mathbf{p}}\right|$, then, for all continuous m–linear forms $T : E_1 \times \cdots \times E_m \to \mathbb{K}$,

$$\left(\sum_{i_1=1}^{\infty}\left(\cdots\left(\sum_{i_k=1}^{\infty}\left|T\left(\sum_{n=1}^{k}\sum_{j\in I_n} x_{i_n}^{(j)} \cdot e_j\right)\right|^{q_k}\right)^{\frac{q_{k-1}}{q_k}}\cdots\right)^{\frac{q_1}{q_2}}\right)^{\frac{1}{q_1}}$$

$$\leq C_{k,m,\mathbf{p},\mathbf{q}}^{\mathbb{K}} \|T\| \prod_{j=1}^{m} \left\|\left(x_i^{(j)}\right)_{i=1}^{\infty}\right\|_{w,p_j^*},$$

regardless of the sequences $\left(x_i^{(j)}\right)_{i=1}^{\infty} \in \ell_{p_j^*}^{w}(E_j)$, $j = 1, \ldots, m$.

In other words,we have the coincidence

$$\Pi_{(\mathbf{q};\mathbf{p}^*)}^{k,m,\mathcal{I}}(E_1, \ldots, E_m; F) = \mathcal{L}(E_1, \ldots, E_m; \mathbb{K}),$$

with $\mathbf{p}^* := (p_1^*, \ldots, p_m^*)$.

References

This lecture is related to papers from 2013-2015 in collaboration with

- G. Araújo (João Pessoa, Brazil);
- F. Bayart (Clermont-Ferrand, France);
- D. Pellegrino (João Pessoa, Brazil);
- D. Nuñez-Alarcón (Recife, Brazil);
- P. Rueda (Valence, Spain);
- J. Seoane (Madrid, Spain);

Thank you very much!