Hölder’s inequality on mixed $L_p$ spaces and summability of multilinear operators

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**Relations Between Banach Space Theory and Geometric Measure Theory**
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Motivation: interpolative puzzles

Let us suppose that we have the following two inequalities at hand, for certain complex scalar matrix \((a_{ij})_{i,j=1}^N:\)

\[
\sum_{i=1}^N \left( \sum_{j=1}^N |a_{ij}|^2 \right)^{\frac{1}{2}} \leq C_1 \quad \text{and} \quad \sum_{j=1}^N \left( \sum_{i=1}^N |a_{ij}|^2 \right)^{\frac{1}{2}} \leq C_2
\]

for some constant \(C > 0\) and all positive integers \(N\).
Let us suppose that we have the following two inequalities at hand, for certain complex scalar matrix \((a_{ij})_{i,j=1}^N\):

\[
\frac{1}{2} N \left( \sum_{j=1}^N |a_{ij}|^2 \right) \leq C_1 \quad \text{and} \quad \frac{1}{2} \left( \sum_{i=1}^N \sum_{j=1}^N |a_{ij}|^2 \right) \leq C_2
\]

for some constant \(C > 0\) and all positive integers \(N\).

How can one find an optimal exponent \(r\) and a constant \(C_1 > 0\) such that

\[
\left( \sum_{i,j=1}^N |a_{ij}|^r \right)^{\frac{1}{r}} \leq C_3, \quad \text{for all positive integers } N?
\]

Moreover, how can one get a good (small) constant \(C_3\)?
Using a consequence of Minkowski’s inequality and applying Hölder’s inequality successively:

\[
\sum_{i,j=1}^{N} |a_{ij}|^{\frac{4}{3}} = \sum_{i=1}^{N} \left( \sum_{j=1}^{N} |a_{ij}|^{\frac{2}{3}} |a_{ij}|^{\frac{2}{3}} \right)
\]

\[
\leq \sum_{i=1}^{N} \left( \left( \sum_{j=1}^{N} |a_{ij}|^{2} \right)^{\frac{1}{3}} \left( \sum_{j=1}^{N} |a_{ij}|^{\frac{2}{3}} \right) \right)
\]

\[
\leq \left( \sum_{i=1}^{N} \left( \sum_{j=1}^{N} |a_{ij}|^{2} \right)^{\frac{1}{2}} \right)^{\frac{2}{3}} \left( \sum_{i=1}^{N} \left( \sum_{j=1}^{N} |a_{ij}|^{\frac{2}{3}} \right) \right)^{\frac{1}{3}}
\]

\[
= \left[ \sum_{i=1}^{N} \left( \sum_{j=1}^{N} |a_{ij}|^{2} \right)^{\frac{1}{2}} \right]^{\frac{2}{3}} \left[ \left( \sum_{i=1}^{N} \left( \sum_{j=1}^{N} |a_{ij}|^{\frac{2}{3}} \right) \right)^{\frac{1}{2}} \right]^{\frac{2}{3}} \leq C_1^{\frac{2}{3}} \cdot C_2^{\frac{2}{3}}
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Using a consequence of Minkowski’s inequality and applying Hölder’s inequality successively:

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\]

\[
\leq \sum_{i=1}^{N} \left( \left( \sum_{j=1}^{N} |a_{ij}|^{2} \right)^{\frac{1}{3}} \left( \sum_{j=1}^{N} |a_{ij}| \right)^{\frac{2}{3}} \right)
\]

\[
\leq \left( \sum_{i=1}^{N} \left( \sum_{j=1}^{N} |a_{ij}|^{2} \right)^{\frac{1}{2}} \right)^{\frac{2}{3}} \left( \sum_{i=1}^{N} \left( \sum_{j=1}^{N} |a_{ij}| \right)^{2} \right)^{\frac{1}{3}}
\]

\[
= \left[ \sum_{i=1}^{N} \left( \sum_{j=1}^{N} |a_{ij}|^{2} \right)^{\frac{1}{2}} \right]^{\frac{2}{3}} \left[ \left( \sum_{i=1}^{N} \left( \sum_{j=1}^{N} |a_{ij}| \right)^{2} \right)^{\frac{1}{2}} \right]^{\frac{2}{3}} \leq C_1^{\frac{2}{3}} \cdot C_2^{\frac{2}{3}}
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Solution: “interpolation via Hölder’s inequality”.

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Hölder’s inequality and operators summability
A. Benedek and R. Panzone introduce the mixed $L_p$ spaces notion on:

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Let $(X_i, \Sigma_i, \mu_i)$, $i = 1, \ldots, m$ be $\sigma$-finite measurable spaces, let

$$(X, \Sigma, \mu) := \left( \prod_{i=1}^{m} X_i, \prod_{i=1}^{m} \Sigma_i, \prod_{i=1}^{m} \mu_i \right)$$

be the product space and $p := (p_1, \ldots, p_m) \in [1, \infty]^m$. 

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be the product space and $\mathbf{p} := (p_1, \ldots, p_m) \in [1, \infty]^m$.

The space $L_\mathbf{p}(X)$ consists in all measurable functions $f : X \to \mathbb{K}$ with the following property:

$f(x_1, \ldots, x_{m-1}, \cdot) \in L_{p_m}(X_m)$, i.e., $\|f\|_{p_m} := \| f(x_1, \ldots, x_{m-1}, \cdot) \|_{p_m} < \infty$,

for any $(x_1, \ldots, x_{m-1}) \in \prod_{i=1}^{n-1} X_i$ and, also $\|f\|_{p_m}$, results in a measurable function;
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The space $L_p(X)$ consists in all measurable functions $f : X \to \mathbb{K}$ with the following property:

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for any $(x_1, \ldots, x_{m-1}) \in \prod_{i=1}^{m-1} X_i$ and, also $\|f\|_{p_m}$, results in a measurable function; this process is repeated successively: the resulting $p_{m-1}$-norm, $p_{m-2}$-norm, $\ldots$, $p_1$-norm (in this order) are finite.
For instance, when all \( p_i < \infty \) a measurable function \( f : X \to \mathbb{K} \) it is an element of \( L_p(X) \) if, and only if,

\[
\|f\|_p := \left( \int_{X_1} \left( \cdots \left( \int_{X_m} |f|^{p_m} \, d\mu_m \right)^{\frac{p_m-1}{p_m}} \cdots \right)^{\frac{p_1}{p_2}} \, d\mu_1 \right)^{\frac{1}{p_1}} < \infty.
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$$

Some classical properties and results concerning the $L_p$ spaces:

- $L_p(X)$ is a Banach space;
- Monotone’s convergence classical theorems;
- Lebesgue’s dominated convergence theorem.
We are interested in a “simple” result:

**Theorem (Mixed Hölder’s inequality)**

Let $\mathbf{r} \in [1, \infty)^m$ and $\mathbf{p}(1), \ldots, \mathbf{p}(N) \in [1, \infty]^m$ be such that

$$\frac{1}{r_j} = \frac{1}{p_j(1)} + \cdots + \frac{1}{p_j(N)}, \quad \text{for } j = 1, \ldots, m.$$  

If $f_k \in L_{\mathbf{p}(k)}(X)$ for $k = 1, \ldots, N$, then

$$f_1 f_2 \cdots f_N \in L_{\mathbf{r}}(X)$$

and, moreover,

$$\|f_1 \cdots f_N\|_{\mathbf{r}} \leq \|f_1\|_{\mathbf{p}(1)} \cdots \|f_N\|_{\mathbf{p}(N)}.$$
Corollary [Mixed interpolative Hölder’s inequality]

Let \( r, p(1), \ldots, p(N) \in [1, \infty]^m \) and \( \theta_1, \ldots, \theta_N \in [0, 1] \) be such that

\[
\theta_1 + \cdots + \theta_N = 1
\]

and

\[
\frac{1}{r_j} = \sum_{k=1}^{N} \frac{\theta_k}{p_j(k)} = \frac{\theta_1}{p_j(1)} + \cdots + \frac{\theta_N}{p_j(N)}, \quad \text{for } j = 1, \ldots, m.
\]

If \( f \in L_{p(k)}(X) \) for \( k = 1, \ldots, N \), then \( f \in L_r(X) \) and, moreover,

\[
\|f\|_r \leq \|f\|_{p(1)}^{\theta_1} \cdots \|f\|_{p(N)}^{\theta_N}.
\]
Let $X$ be a Banach space and $p \in [1, \infty)^m$. The mixed norm sequence space

$$
\ell_p(X) := \ell_{p_1}(\ell_{p_2}(\ldots(\ell_{p_m}(X))\ldots))
$$

is formed by all multi-index vector valued matrices $(x_i)_{i \in \mathbb{N}^m}$ with finite $p$-norm that is,

$$
\| (x_i)_{i} \|_p := \left( \sum_{i_1=1}^{\infty} \left( \ldots \left( \sum_{i_m=1}^{\infty} \| x_i \|_{X}^{p_m} \right)^{\frac{p_m-1}{p_m}} \ldots \right)^{\frac{p_1}{p_2}} \right)^{\frac{1}{p_1}} < \infty.
$$

When $X = \mathbb{K}$, we just write $\ell_p$ instead of $\ell_p(\mathbb{K})$. 
Hölder’s interpolative inequality for sequences

The next interpolation result on these mixed norm sequences spaces has a central role on the results we will present.

**Corollary [Hölder’s interpolative inequality for mixed \( \ell_p \) spaces]**

Let \( m, n, N \) be positive integers, \( r, p(1), \ldots, p(N) \in [1, \infty]^m \) and \( \theta_1, \ldots, \theta_N \in [0, 1] \) be such that \( \theta_1 + \cdots + \theta_N = 1 \) and

\[
\frac{1}{r_j} = \sum_{k=1}^{N} \frac{\theta_k}{p_j(k)} = \frac{\theta_1}{p_j(1)} + \cdots + \frac{\theta_N}{p_j(N)}, \quad \text{for } j = 1, \ldots, m.
\]

Then, for all scalar matrix \( a := (a_i)_{i \in \mathcal{M}(m,n)} \), we have

\[
\|a\|_r \leq \|a\|_{p(1)}^{\theta_1} \cdots \|a\|_{p(N)}^{\theta_N}.
\]
Hölder’s interpolative inequality for sequences

In particular, if each $p(k) \in [1, \infty)$, the previous inequality means that

$$
\left( \sum_{i_1=1}^{n} \left( \cdots \left( \sum_{i_m=1}^{n} |a_{i_1}|^{r_m} \right)^{\frac{r_m-1}{r_m}} \cdots \right)^{\frac{1}{r_1}} \right) \left( \sum_{i_1=1}^{N} \left( \cdots \left( \sum_{i_m=1}^{n} |a_{i_1}|^{p_{m}(k)} \right)^{\frac{p_{m-1}(k)}{p_{m}(k)}} \cdots \right)^{\frac{1}{p_{1}(k)}} \right)^{\theta_k} \leq \prod_{k=1}^{N} \left( \sum_{i_1=1}^{n} \left( \cdots \left( \sum_{i_m=1}^{n} |a_{i_1}|^{p_{m}(k)} \right)^{\frac{p_{m-1}(k)}{p_{m}(k)}} \cdots \right)^{\frac{1}{p_{1}(k)}} \right)^{\theta_k}.
$$

Thanks anonymous referee!
Hölder’s interpolative inequality for sequences

In particular, if each \( p(k) \in [1, \infty) \), the previous inequality means that

\[
\left( \sum_{i_1=1}^{n} \left( \ldots \left( \sum_{i_m=1}^{n} |a_i| r_m \right)^{r_m - 1 \over r_m} \ldots \right)^{r_1 \over r_2} \right)^{1 \over r_1} \leq \prod_{k=1}^{N} \left[ \left( \sum_{i_1=1}^{n} \left( \ldots \left( \sum_{i_m=1}^{n} |a_i| p_m(k) \right)^{p_m - 1(k) \over p_m(k)} \ldots \right)^{p_1(k) \over p_2(k)} \right)^{1 \over p_1(k)} \right]^{\theta_k}.
\]

Thanks anonymous referee!

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Hölder’s inequality and operators summability

Theorem (Multilinear Bohnenblust-Hille’s inequality)

For each positive integer \( m \geq 1 \), there exists a constant \( C_m \geq 1 \) such that

\[
\left( \sum_{i_1, \ldots, i_m=1}^\infty \|A(e_{i_1}, \ldots, e_{i_m})\|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq C_m \|A\|,
\]

for all continuous \( m \)-linear forms \( A : c_0 \times \cdots \times c_0 \to \mathbb{K} \). Moreover, the exponent \( \frac{2m}{m+1} \) is optimal.
Multilinear Hardy-Littlewood’s inequality

- [1934] G. Hardy and J. P. Littlewood provided an $\ell_p$-version for the bilinear case (Littlewood’s 4/3 inequality).
- [1981] T. Praciano-Pereira obtained a general result for multilinear forms on $\ell_p$ spaces.
G. Hardy and J. P. Littlewood provided an $\ell_p$-version for the bilinear case (Littlewood’s 4/3 inequality).

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Let us define $X_p := \ell_p, 1 \leq p < +\infty$ and $X_\infty := c_0$.

**Theorem (Multilinear Hardy-Littlewood’s inequality)**

Let $\mathbf{p} \in [1, +\infty]^m$ with $\left|\frac{1}{\mathbf{p}}\right| := \frac{1}{p_1} + \cdots + \frac{1}{p_m} \leq \frac{1}{2}$. Then there exists a constant $C_{m, \mathbf{p}} \geq 1$ such that, for every continuous $m$-linear form $A : X_{p_1} \times \cdots \times X_{p_m} \to \mathbb{C}$,

$$
\left( \sum_{i_1, \ldots, i_m=1}^{\infty} \left| A(e_{i_1}, \ldots, e_{i_m}) \right|^2 \right)^{\frac{2m}{m+1-2\left|\frac{1}{\mathbf{p}}\right|}} \leq C_{m, \mathbf{p}} \|A\|.
$$
After results...

- [2009] Defant and Sevilla-Peris;
- [2013] A., Bayart, Pellegrino and Seoane;
- [2013] Dimant and Sevilla-Peris.
Theorem (A., Bayart, Pellegrino, Seoane (2014))

Let \( p \in [1, +\infty]^m \) and \( 1 \leq s \leq q \leq \infty \) be such that
\[
\left| \frac{1}{p} \right| < \frac{1}{2} + \frac{1}{s} - \frac{1}{\min\{q, 2\}}.
\]

If \( \lambda := \left[ \frac{1}{2} + \frac{1}{s} - \frac{1}{\min\{q, 2\}} - \left| \frac{1}{p} \right| \right]^{-1} > 0 \) and \( t_1, \ldots, t_m \in [\lambda, \max\{\lambda, s, 2\}] \) are such that
\[
\frac{1}{t_1} + \cdots + \frac{1}{t_m} \leq \frac{1}{\lambda} + \frac{m - 1}{\max\{\lambda, s, 2\}},
\]
then there exists \( C > 0 \) satisfying, for every continuous \( m \)-linear map \( A : X_{p_1} \times \cdots \times X_{p_m} \to X_s \),
\[
\left( \sum_{i_1=1}^{+\infty} \left( \sum_{i_m=1}^{+\infty} \left\| A(e_{i_1}, \ldots, e_{i_m}) \right\|_{\ell_q}^{t_m} \right)_{t_m}^{\frac{t_m-1}{t_m}} \right)_{t_2}^{\frac{t_1}{t_2}} \leq C \|A\|.
\]

Moreover, the exponents are optimal except eventually if \( q \leq 2 \) and \( \left| \frac{1}{p} \right| > \frac{1}{2} \).
Tools for the proof (sufficiency)

- norm-mixed estimate for \((\ell_\lambda, \ell_q)\) or cotype version of Khinchinte’s inequality [Dimant and Sevilla-Peris (2013)];
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- Bennet-Carl inequality;
- Interpolative Hölder’s inequality.
From now on, $E_1, E_2, \ldots, F$ shall denote Banach spaces.
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**Proposition [Bohnenblust-Hille re-written]**

If $q \in [1, 2]^m$ is such that $\frac{1}{q} \leq \frac{1}{2}$, then

$$
\left( \sum_{j_1=1}^{\infty} \left( \ldots \left( \sum_{j_m=1}^{\infty} \left| T \left( x_{j_1}^{(1)}, \ldots, x_{j_m}^{(m)} \right) \right|^{q_m} \right)^{\frac{q_m-1}{q_m}} \right)^{\frac{q_1}{q_2}} \right)^{\frac{1}{q_1}}
$$

is less than or equal to $B_{m,(q_1,\ldots,q_m)}^K \|T\| \prod_{k=1}^{m} \left\| \left( x_{j_k}^{(k)} \right)_{j_k=1}^{\infty} \right\|_{w,1}$,

for all bounded $m$–linear forms $T : E_1 \times \cdots \times E_m \to K$ and all sequences

$$
\left( x_{j_k}^{(k)} \right)_{j_k=1}^{\infty} \in \ell_1^w (E_k), \ k = 1, \ldots, m.
$$
Proposition [Hardy-Littlewood re-written]

Let $m \geq 1$, $p \in [1, \infty]^m$. If $0 \leq \left| \frac{1}{p} \right| \leq \frac{1}{2}$ and $q \in \left[ \left( 1 - \left| \frac{1}{p} \right| \right)^{-1}, 2 \right]^m$ are such that

$$\left| \frac{1}{q} \right| \leq \frac{m + 1}{2} - \left| \frac{1}{p} \right|.$$ 

Then, for all continuous $m$–linear forms $T : E_1 \times \cdots \times E_m \rightarrow \mathbb{K}$,

$$\left( \sum_{i_1=1}^{\infty} \left( \cdots \left( \sum_{i_m=1}^{\infty} \left| T \left( x_{i_1}^{(1)}, \ldots, x_{i_m}^{(m)} \right) \right| q_m \right)^{\frac{q_m-1}{q_m}} \cdots \right) \right)^{\frac{1}{\frac{q_1}{q_1}}} \leq C_{m, p, q}^{\mathbb{K}} \| T \| \prod_{k=1}^{m} \left\| \left( x_i^{(k)} \right)_{i=1}^{\infty} \right\|_{w, p_k^*},$$

regardless of the sequences $\left( x_{j_k}^{(k)} \right)_{i=1}^{\infty} \in \ell^{w}_{p_k^*}(E_k)$, $k = 1, \ldots, m$. 
Partially multiple summing operators: the designs

For Banach spaces $E_1, \ldots, E_m$ and an element $x \in E_j$, for some $j \in \{1, \ldots, m\}$, the symbol $x \cdot e_j$ represents the vector $x \cdot e_j \in E_1 \times \cdots \times E_m$ such that the $j$-th coordinate is $x \in E_j$, and 0 otherwise.

Definition

Let $E_1, \ldots, E_m, F$ be Banach spaces, $m, k$ be positive integers with $1 \leq k \leq m$, and $(p, q) := (p_1, \ldots, p_m, q_1, \ldots, q_k) \in [1, \infty)^{m+k}$. Let also $I = \{I_1, \ldots, I_k\}$ a family of non-void disjoints subsets of $\{1, \ldots, m\}$ such that $\bigcup_{i=1}^k I_i = \{1, \ldots, m\}$, that is, $I$ is a partition of $\{1, \ldots, m\}$. A multilinear operator $T: E_1 \times \cdots \times E_m \to F$ is $I$-partially multiple $(q; p)$-summing if there exists a constant $C > 0$ such that

$$
\left\| \sum_{i_1=1}^{\infty} \cdots \sum_{i_k=1}^{\infty} \left\| \sum_{j \in I_{i_1}} \sum_{j \in I_{i_2}} \cdots \sum_{j \in I_{i_k}} x(j) i_{i_1} \cdot e_{j} \right\|_{F}^{q_k} \right\|_{q_{k-1}} \cdots \left\| \sum_{j \in I_{i_1}} x(j) i_{i_1} \cdot e_{j} \right\|_{q_1} \leq C m \prod_{j=1}^{\infty} \left\| x(j) i_{i_1} \right\|_{w,p_{i_1}}
$$
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**Definition**

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$$\left( \sum_{i_1=1}^{\infty} \cdots \left( \sum_{i_k=1}^{\infty} \left\| T \left( \sum_{n=1}^{k} \sum_{j \in I_n} x_{i_n}^{(j)} \cdot e_j \right) \right\|_{F}^{q_k} \right)^{\frac{q_k - 1}{q_k}} \cdots \right)^{\frac{q_1}{q_2}} \left( \prod_{j=1}^{m} \left\| (x_{i_j}^{(j)})_{i=1}^{\infty} \right\|_{w,p_j}^{q_1} \right) \leq C \prod_{j=1}^{m} \left\| (x_{i_j}^{(j)})_{i=1}^{\infty} \right\|_{w,p_j}^{q_1}$$

Hölder’s inequality and operators summability
Definition

for all \( (x_i^{(j)})_{i=1}^\infty \in \ell_{p_j}^w (E_j) \), \( j = 1, \ldots, m \). We represent the class of all \( \mathcal{I} \)-partially multiple \((q; p)\)-summing operators by \( \Pi_{(q; p)}^{k,m} (E_1, \ldots, E_m; F) \). The infimum taken over all possible constants \( C > 0 \) satisfying the previous inequality defines a norm in \( \Pi_{(q; p)}^{k,m} (E_1, \ldots, E_m; F) \), which is denoted by \( \pi_{(q; p)}^{\mathcal{I}} \).
Definition

for all \( \left( x_i^{(j)} \right)_{i=1}^{\infty} \in \ell_{p_j}^w (E_j) \), \( j = 1, \ldots, m \). We represent the class of all \( \mathcal{I} \)–partially multiple \((q; p)\)–summing operators by \( \Pi_{(q; p)}^{k, m, \mathcal{I}} (E_1, \ldots, E_m; F) \). The infimum taken over all possible constants \( C > 0 \) satisfying the previous inequality defines a norm in \( \Pi_{(q; p)}^{k, m, \mathcal{I}} (E_1, \ldots, E_m; F) \), which is denoted by \( \pi_{(q; p)}^{\mathcal{I}} \).

Note that when

- \( k = 1 \), we recover the class of absolutely \((q; p_1, \ldots, p_m)\)–summing operators, with \( q := q_1 \);
- \( k = m \) and \( q_1 = \cdots = q_m =: q \), we recover the class of multiple \((q; p_1, \ldots, p_m)\)–summing operators.

From now on, \( m, k \) are positive integers with \( 1 \leq k \leq m \), \((p, q) := (p_1, \ldots, p_m, q_1, \ldots, q_k) \in [1, \infty)^{m+k} \) and \( \mathcal{I} = \{I_1, \ldots, I_k\} \) is a partition of \( \{1, \ldots, m\} \).
Theorem [Bohnenblust-Hille’s partially summ. version]

Let \( q \in [1, 2]^k \) such that \( \frac{1}{q} \leq \frac{k+1}{2} \). Then

\[
\left( \sum_{i_1=1}^{\infty} \left( \cdots \left( \sum_{i_k=1}^{\infty} \left| T \left( \sum_{n=1}^{k} \sum_{j \in I_n} x_{i_n}^{(j)} \cdot e_j \right) \right| \right)^q \frac{q_k-1}{q_k} \right) \cdots \right) \left( \frac{q_1}{q_2} \right) \frac{1}{q_1} \leq B_{k, q}^{\mathbb{K}} \| T \| \prod_{j=1}^{m} \left\| \left( x_{i_1}^{(j)} \right)^{\infty} \right\|_{w, 1},
\]

for all \( m \)-linear forms \( T : E_1 \times \cdots \times E_m \to \mathbb{K} \) and all sequences

\[
\left( x_{i_1}^{(j)} \right)^{\infty} \in \ell^w_1 (E_j), \; j = 1, \ldots, m.
\]

In other words, when \( q \in [1, 2]^k \) such that \( \frac{1}{q} \leq \frac{k+1}{2} \) we have the following coincidence result:

\[
\Pi_{(q; 1)}^{k,m,F} (E_1, \ldots, E_m; F) = \mathcal{L} (E_1, \ldots, E_m; \mathbb{K}),
\]

with \( 1 := (1, m \text{ times}, 1) \).
Theorem [Hardy-Littlewood’s partially summ. version]

Let $1 \leq k \leq m$, $p \in [1, \infty]^m$. If $0 \leq \left| \frac{1}{p} \right| \leq \frac{1}{2}$ and $q \in \left[ \left( 1 - \left| \frac{1}{p} \right| \right)^{-1}, 2 \right]^k$ are such that $\left| \frac{1}{p} \right| \leq \frac{k+1}{2} - \left| \frac{1}{p} \right|$, then, for all continuous $m$–linear forms $T : E_1 \times \cdots \times E_m \to \mathbb{K}$,

$$\left( \sum_{i_1=1}^{\infty} \cdots \left( \sum_{i_k=1}^{\infty} \right) \left| T \left( \sum_{n=1}^{k} \sum_{j \in I_n} x_{i_n}^{(j)} \cdot e_j \right) \right| q_k \right) \left( \left( \frac{q_k}{q_1} \right)^{q_k} \left( \frac{q_1}{q_2} \right)^{q_1} \right) \leq C_{k,m,p,q}^{\|T\| \prod_{j=1}^{m} \left\| \left( x_{i}^{(j)} \right)_{i=1}^{\infty} \right\|_{w, p_j^*}^{ \prod_{j=1}^{m} \left\| x_{i}^{(j)} \right\|_{w, p_j^*}}$$

regardless of the sequences $(x_{i}^{(j)})_{i=1}^{\infty} \in \ell_{w, p_j}^{p_j^*} (E_j)$, $j = 1, \ldots, m$. 

In other words, we have the coincidence $\prod_{k,m}(q;p^*)_{E_1, \ldots, E_m} = L_{E_1, \ldots, E_m}$, with $p^* : = (p_1^*, \ldots, p_m^*)$. 

Nacib Albuquerque

Hölder’s inequality and operators summability
Theorem [Hardy-Littlewood’s partially summ. version]

Let \(1 \leq k \leq m\), \(p \in [1, \infty]^m\). If \(0 \leq \left| \frac{1}{p} \right| \leq \frac{1}{2}\) and \(q \in \left[ \left(1 - \left| \frac{1}{p} \right| \right)^{-1}, 2 \right]\) are such that \(\left| \frac{1}{q} \right| \leq \frac{k+1}{2} - \left| \frac{1}{p} \right|\), then, for all continuous \(m\)-linear forms \(T : E_1 \times \cdots \times E_m \to \mathbb{K}\),

\[
\left( \sum_{i_1=1}^{\infty} \cdots \left( \sum_{i_k=1}^{\infty} \left| T \left( \sum_{n=1}^{k} \sum_{j \in I_n} x_{i_n}^{(j)} \cdot e_j \right) \right|^{q_k} \right) \right)^{\frac{q_k-1}{q_k}} \cdot \left( \frac{q_1}{q_2} \right)^{\frac{1}{q_1}} \leq C_{k,m,p,q}^{\mathbb{K}} \left\| T \right\| \prod_{j=1}^{m} \left\| \left( x_{i_j}^{(j)} \right)^{\infty}_{i_1=1} \right\|_{w,p_j^*},
\]

regardless of the sequences \(\left( x_{i_j}^{(j)} \right)^{\infty}_{i_1=1} \in \ell^w_{p_j^*} (E_j), j = 1, \ldots, m\).

In other words, we have the coincidence

\[
\Pi_{(q;p^*)}^{k,m}(E_1, \ldots, E_m; F) = \mathcal{L} (E_1, \ldots, E_m; \mathbb{K}),
\]

with \(p^* := (p_1^*, \ldots, p_m^*)\).
This lecture is related to papers from 2013-2015 in collaboration with

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Thank you very much!