Extension operators on balls and on spaces of finite sets

Antonio Avilés, joint work with Witold Marciszewski

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An extension operator is an operator $E : C(K) \longrightarrow C(L)$ that sends every $f \in C(K)$ to an extension.

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$$E(f)(x) = \int f \; dE^*(x)$$

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In the non-metric case, we define

$$\eta(K, L) = \inf\{\|E\| : E : C(K) \longrightarrow C(L) \text{ is an extension operator}\}$$

which might be ∞ if there is no such E exists.

Our compact spaces

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$$\{1,2,3\},\{1,2,4\},\{1,2,5\},\ldots \longrightarrow \{1,2\}$$

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1. 1, if $|\Gamma| \leq \aleph_0$.
2. $2n - 2m + 1$, if $|\Gamma| = \aleph_1$.
3. $\sum_{k=0}^{m} \binom{n}{k} \binom{n-k-1}{m-k}$, if $|\Gamma| \geq \aleph_\omega$.

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- The function $\{p < q\} \mapsto \frac{q-1}{q}\delta_{\{p\}} + \frac{1}{q}\delta_{\{q\}}$ gives an extension operator of norm 1 when $\Gamma = \mathbb{N}$.
- The function $\{x, y\} \mapsto \delta_{\{x\}} + \delta_{\{y\}} - \delta_{\emptyset}$ gives an extension operator of norm 3. This is optimal for sizes $\geq \aleph_1$.

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Theorem (A., Marciszewski)

$\eta(\sigma_m(\aleph_\omega), \sigma_n(\aleph_\omega))$ equals the least norm of a natural extension operator from σ_m to σ_n

There is essentially a unique formula for a natural extension operator from σ_m to σ_n:

$$A \mapsto \sum_{B \in [A]^{\leq m}} (-1)^{m-|B|} \binom{|A| - |B| - 1}{m - |B|} \delta_B$$

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Suppose $|\Gamma| \geq \aleph_1$. Let F be a function that sends each finte subsets of Γ to a another disjoint finite subset of Γ. Then, there exists $Z \subset \Gamma$ with $|Z| = 2$ such that $F(A) \cap Z = \emptyset$ for all $A \subset Z$.

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Another related Banach space problem

Problem (Enflo, Rosenthal 73)

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Balls in the Hilbert space

For every $r < s$ we can produce produce diagrams:

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Open Problem: A non-separable Miljutin theorem?

Is $C(B(\Gamma))$ isomorphic to $C(\sigma_1(\Gamma)^{\mathbb{N}})$?