On smooth approximation and smooth convex extensions of convex functions

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Relations between Banach Space Theory and Geometric Measure Theory

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Whitney’s Theorems of 1934

Theorem (Whitney's approximation theorem)

Let $U \subseteq \mathbb{R}^n$ be open, $f \in C^k(U)$, $k = 0, 1, 2, ...,$ and $\varepsilon : U \to (0, \infty)$ continuous. Then there exists $g : U \to \mathbb{R}$ real analytic such that

$$|D^j f - D^j g| \leq \varepsilon \text{ on } U \text{ for every } j = 0, ..., k.$$

The proof combines integral convolutions with the heat kernel, and what we could call real-analytic approximations to partitions of unity.

Theorem (Whitney's extension theorem, for m finite)

Let $C \subset \mathbb{R}^n$ be closed. A necessary and sufficient condition, for a function $f : C \to \mathbb{R}$ and a family of functions $\{f_\alpha\}_{|\alpha| \le m}$ defined on C satisfying $f = f_0$ and

$$f_\alpha(x) = \sum_{|\beta| \le m - |\alpha|} \frac{f_{\alpha+\beta}(y)}{\beta!}(x-y)^\beta + R_\alpha(x,y)$$

for all $x, y \in C$ and all multi-indices α with $|\alpha| \le m$, to admit a C^m extension F to all of $\mathbb{R}^n$ such that $D^\alpha F = f_\alpha$ on C for all $|\alpha| \le m$, is that

$$\lim_{|x-y| \to 0} \frac{R_\alpha(x,y)}{|x-y|^{m-|\alpha|}} = 0 \qquad (W^m)$$

uniformly on compact subsets of C, for every $|\alpha| \le m$.

If, instead of the functions f_α, for every $y \in C$ we are given a polynomial $P_y : \mathbb{R}^n \to \mathbb{R}$ with degree$(P_y) \leq m$ and $P_y(y) = f(y)$ (P_y=candidate for Taylor polynomial of f of order m at y), then Whitney's condition (W^m) can be reformulated by saying that

$$\lim_{\delta \to 0^+} \rho_m(K, \delta) = 0 \text{ for each compact subset } K \text{ of } C,$$

where we denote

$$\rho_m(K, \delta) = \sup\{\frac{\|D^j P_y(z) - D^j P_z(z)\|}{|y - z|^{m-j}} \ : j = 0, ..., m, \ y, z \in K, 0 < |y-z| \leq \delta\}$$

If this condition is met, then Whitney's theorem provides us with a function $F \in C^m(\mathbb{R}^n)$ such that $D^j F(y) = D^j P_y(y)$ for every $j = 0, ..., m$ and $y \in C$. In other words, each P_y is the Taylor polynomial of order m of F at y.

Sketch of the proof of Whitney's extension theorem.

Step 1. Construct a family of *Whitney cubes*: a covering $\mathcal{F} = \{Q_j\}_{j\in\mathbb{N}}$ of $U = \mathbb{R}^n \setminus C$ by dyadic cubes with pairwise disjoint interiors and such that

1. $\mathrm{diam}(Q_j) \leq \mathrm{dist}(Q_j, C) \leq 4\mathrm{diam}(Q_j)$ for every j;
2. for every $Q \in \mathcal{F}$ there are at most $N := 12^n$ cubes in $\mathcal{F}$ which touch Q;
3. for every $x \in U$ there are at most N cubes Q_j^* which contain x,

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Step 2. Construct a *Whitney partition of unity*: a family of C^∞ functions $\varphi_j : \mathbb{R}^n \to [0,1]$ such that

1. $\varphi_j = 1$ on Q_j, and $\varphi_j = 0$ on Q_j^*;
2. $\sum_j \varphi_j = 1$ on U;
3. for each multi-index α there exists a constant $A_\alpha > 0$ such that

$$\left|\frac{\partial^{|\alpha|}\varphi_k}{\partial x^\alpha}(x)\right| \leq A_\alpha \mathrm{diam}(Q_k)^{-|\alpha|},$$

for every $x \in U$ and $k \in \mathbb{N}$.

Step 3. Define the extension $F : \mathbb{R}^n \to \mathbb{R}$ by

$$F(x) = \begin{cases} f(x) & \text{if } x \in C; \\ \sum_{j=1}^{\infty} P_{y_j}(x)\varphi_j(x) & \text{if } x \notin C, \end{cases}$$

where the $y_j \in C$ are such that $\mathrm{dist}(C, Q_j) = \mathrm{dist}(y_j, Q_j)$. Check that F does the job.

Let us call $\{P_y^m\}_{y\in C, m\in\mathbb{N}\cup\{0\}}$ a *compatible family of polynomials for C^∞ extension* of a function f defined on C, where P_y^m is a polynomial of degree up to m such that $P_y^m(y) = f(y)$, if for every $k > j$ the polynomial P_y^j is the Taylor polynomial of order j at y of the polynomial P_y^k. In other words, $\{P_y^m\}_{y\in C, m\in\mathbb{N}\cup\{0\}}$ is compatible if for every $k > j$ the polynomial P_y^j is obtained from P_y^k by discarding all of its homogeneous terms of order greater than j.

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Theorem (Whitney's extension theorem, for $m = \infty$)

If $\{P^m_y\}_{y\in C, m\in\mathbb{N}\cup\{0\}}$ is a compatible family of polynomials for C^∞ extension of a function f such that for each $m \in \mathbb{N}$ the subfamily $\{P^m_y\}_{y\in C}$ satisfies Whitney's condition (W^m), then there is a function $F \in C^\infty(\mathbb{R}^n)$ such that P^m_y is the Taylor polynomial of order m of F at y (denoted by $J^m_y F$), for every $y \in C$ and $m \in \mathbb{N}$. (The converse is trivially true.)

Sketch of the proof of Whitney's extension theorem in the case $m = \infty$.
Steps 1 and 2 are the same as in the case $m \in \mathbb{N}$.
Step 3. Define F by

$$F(x) = \begin{cases} f(x) & \text{if } x \in C; \\ \sum_{j=1}^{\infty} P_{y_j}^{m_j}(x)\varphi_j(x) & \text{if } x \notin C, \end{cases}$$

where the $y_j \in C$ are such that $\operatorname{dist}(C, Q_j) = \operatorname{dist}(y_j, Q_j)$, and the $m_j \nearrow \infty$ are carefully chosen.

The Whitney Extension Problem

Problem (Whitney's extension problem)

Let n, m positive integers. If C is an arbitrary subset of $\mathbb{R}^n$ and we are given a function $f : C \to \mathbb{R}$, how can we decide whether there exists $F \in C^m(\mathbb{R}^n)$ such that $F = f$ on C?

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- C. Fefferman (2006) solved it in general. He had previously solved (2005) the analogous problem for the class $C^{m,\omega}(\mathbb{R}^n)$.

Theorem (C. Fefferman, 2006)

Let n, m be positive integers. If C is an arbitrary subset of $\mathbb{R}^n$ and $f : C \to \mathbb{R}$ is an arbitrary function, then f has an extension $F \in C^m(\mathbb{R}^n)$ if and only if for every $x \in C$ the stable Glaeser refinement *of f at x is nonempty.*

Smooth approximations of Lipschitz and C^1 functions on Banach spaces

Smooth approximations and partitions of unity in Banach spaces have been studied, among others, by Kurzweil, Eells, Bonic, Frampton, Toruńczyk... For instance, every separable Banach space with a C^k smooth bump function has C^k smooth partitions of unity. Therefore it is possible to uniformly approximate continuous functions by C^k functions on such spaces.

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Unfortunately, the method of approximation by partitions of unity tends to obliterate good properties of functions (for instance: Lipschitzness in infinite-dimensional spaces, and convexity even in $\mathbb{R}^n$).

It follows from the work of Moulis (1971) that if one is able to approximate a Lipschitz functions defined on a ball by C^k smooth Lipschitz functions, with a control of the Lipschitz constants of the approximating functions, then one is also able to approximate C^1 functions by C^k smooth functions, in the C^1 fine topology.

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Moulis was able to do this for the classical spaces ℓ_2, c_0, ℓ_p, $p \geq 2$, with $k \geq 2$ appropriately defined in each case (so that the space has a C^k smooth norm). In fact her proof works also for all separable spaces with unconditional bases.

For instance:

Theorem (Moulis, 1971)

Let U be an open subset of a separable Banach space with an unconditional basis and a C^k smooth Lipschitz bump function, let Y be a normed space, and let $f \in C^1(U,Y)$, $\varepsilon \in C(U)$ with $\varepsilon > 0$. Then there exists $g \in C^k(U,Y)$ such that $|f-g| \leq \varepsilon$ and $\|Df - Dg\| \leq \varepsilon$.

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For instance, if $X = \ell_2$ then $g \in C^\infty(U, Y)$.
Roughly, the main idea of the proof is to approximate the identity on X by a C^k smooth Lipschitz mapping whose image is locally contained in finite-dimensional subspaces, on which one can use integral convolution tenhniques, and then to glue all local approximations together by means of a partition of unity.

Thus, apart from being a natural question, it is useful to know when Lipschitz functions can be approximated by smooth Lipschitz functions, with a control of the Lipschitz constants of the approximating functions.

Thus, apart from being a natural question, it is useful to know when Lipschitz functions can be approximated by smooth Lipschitz functions, with a control of the Lipschitz constants of the approximating functions. R. Fry was the first one to get a result in this direction beyond the spaces with unconditional bases.

Theorem (R. Fry, 2004)

Let X be a separable Banach space with a C^k smooth Lipschitz bump function, and let $f : X \to \mathbb{R}$ be uniformly continuous. Then for every ε there exists a Lipschitz function $g \in C^k(X)$ such that $|f - g| \leq \varepsilon$.

The essential point in Fry's approach was to construct what, later on, P. Hajek and M. Johanis very adequately called a C^k *sup-partition of unity* $\{\psi_j(x)\}$, and to replace the usual quotient of sums

$$\frac{\sum_j a_j\psi_j(x)}{\sum_j \psi_j(x)}$$

with the expression

$$\frac{\|\{a_j\psi_j(x)\}_{j=1}^{\infty}\|_{c_0}}{\|\{\psi_j(x)\}_{j=1}^{\infty}\|_{c_0}},$$

where $\|\cdot\|_{c_0}$ is a C^∞ equivalent norm in c_0 with the property that $\|\{\lambda_j\}_{j=1}^{\infty}\|_{c_0} \leq \|\{\mu_j\}_{j=1}^{\infty}\}\|_{c_0}$ whenever $0 \leq \lambda_j \leq \mu_j$ for all j.

Theorem (Existence of smooth Lipschitz sup-partitions of unity. Fry, 2004)

Let X be a separable Banach space with a Lipschitz C^k smooth bump function ($k \in \mathbb{N} \cup \{\infty\}$). Then there exists $M > 0$ such that for every $\delta > 0$ there exists a family of C^k functions $\{\psi_j\}_{j\in\mathbb{N}}$ such that:

- $diam(supp(\psi_j)) \leq \delta$;
- $Lip(\psi_j) \leq M/\delta$ *for all* j;
- $0 \leq \psi_j \leq 1$ *for every* j;
- *for every* x *there exist* $n_x \in \mathbb{N}$ *and a neighborhood* U_x *of* x *so that* $\psi_j = 0$ *on* U_x *for all* $j \geq n_x$;
- *for every* $x \in X$ *there exists* $j_x \in \mathbb{N}$ *such that* $\psi_{j_x}(x) = 1$.

However, Fry's method of approximation

$$\frac{\|\{a_j\psi_j(x)\}_{j=1}^{\infty}\|_{c_0}}{\|\{\psi_j(x)\}_{j=1}^{\infty}\|_{c_0}},$$

had one important fault concerning the control of Lipschitz constants: it gives

$$\mathrm{Lip}(g) \leq \frac{C}{\varepsilon}\|f\|_{\infty}\mathrm{Lip}(f),$$

while we should like to have

$$\mathrm{Lip}(g) \leq C\mathrm{Lip}(f),$$

where C is a constant independent of f and ε.

Hájek and Johanis were able to surmount this difficulty, and they proved very general results on smooth approximation of Lipschitz functions and of C^1 functions (even functions taking values in some Banach spaces):

Theorem (Hájek-Johanis, 2010)

Let X be a separable Banach space that admits a C^k smooth Lipschitz bump function. Let Y be a Banach space. If at least one of the spaces X, Y is a separable $C(K)$, then there is a constant $M > 0$ such that for every Lipschitz function $f : X \to Y$ and every continuous function $\varepsilon : X \to (0, \infty)$ there exists a C^k smooth Lipschitz function $g : X \to Y$ such that

- $\|f - g\|_Y \leq \varepsilon$, *and*
- $Lip(g) \leq M Lip(f)$.

Hájek-Johanis's proof combines, among other things, bi-Lipschitz embeddings into $c_0(\Gamma)$, some results of Lindenstrauss's on absolute Lipschitz retracts, sup-partitions of unity, and their following previous result (which relies on integral convolutions defined locally with respect finitely many coordinates in $c_0(\Gamma)$):

Theorem (Hájek-Johanis)

Let Γ be an arbitrary set, Y be a Banach space, $M \subset c_0(\Gamma)$, $U \subset c_0(\Gamma)$ be a uniform neighbourhood of M, $f : U \to Y$ be an L-Lipschitz mapping, and let $\varepsilon > 0$. Then there is a mapping $g \in C^\infty(c_0(\Gamma), Y)$ which locally depends on finitely many coordinates, such that $\|f - g\| \le \varepsilon$ on M, and g is L-Lipschitz on M.

Hájek and Johanis also applied these results to obtain approximation of C^1 functions by C^k functions in the C^1-fine topology.

Theorem (Hájek-Johanis, 2010)

Let X be a separable Banach space that admits a C^k smooth Lipschitz bump. Let Y be a Banach space. If at least one of the spaces X, Y is a separable $C(K)$, or if at least one of X, Y has an unconditional basis and a separable dual, then, for every $U \subset X$ open, for every $f \in C^1(U,Y)$ and every continuous function $\varepsilon : U \to (0,\infty)$ there exists $g \in C^k(U,Y)$ such that

- $\|f - g\|_Y \leq \varepsilon$, *and*
- $\|Df - Dg\| \leq \varepsilon$.

As for the approximation of Lipschitz functions by real analytic-functions, we proved:

Theorem (A-Fry-Keener, 2012)

Let X be a separable Banach space with a separating polynomial. Then there exists $M > 0$ such that for every Lipschitz function $f : X \to \mathbb{R}$, and every $\varepsilon > 0$, there exists a Lipschitz, real analytic function $g : X \to \mathbb{R}$ such that

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The proof was later simplified by M. Johanis.

Another application of the smooth approximation of Lipschitz functions: C^1 extensions of functions defined on closed subsets of Banach spaces

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Let X be a Banach space with a separable dual. Let $Y \subset X$ be a closed subspace, and $f \in C^1(Y)$ smooth function. Then there exists $F \in C^1(X)$ such that $F = f$ on Y.

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The proof combines:

- the Bartle-Graves selector theorem (to construct a continuous $G : Y \to X^*$ which is a candidate for DF restricted to Y).
- ideas from the proof of the classical Tietze extension theorem. Namely, an inductive construction of a series defining the extension function, making use of
- the mentioned results on smooth approximation of Lipschitz functions, so that the series of derivatives converges too.

By substituting, in this scheme, the use of the Bartle-Graves theorem with a suitable infinite-dimensional analogue $(\mathcal{W}^1)$ of Whitney's extension condition (W^1), M. Jiménez-Sevilla and Luis Sánchez-González proved:

Theorem (Jiménez-Sevilla and Sánchez-González, 2011)

Let C be a closed subset of a Banach space X with a separable dual. Let $f : C \to \mathbb{R}$ and $G : C \to X^*$ be continuous mappings so that for every $x \in C$ and every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$y, z \in C \cap B(x, \delta),\ 0 < |y - z| \leq \delta \implies \frac{|f(z) - f(y) - G(y)(x - y)|}{|x - y|} \leq \varepsilon. \qquad (\mathcal{W}^1)$$

Then there exists $F \in C^1(X)$ such that $F = f$ on C.

The converse is also true (a simple exercise).

Smooth convex approximations of convex functions on $\mathbb{R}^n$

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- Partitions of unity can be used to pass from approximation on compact sets to global approximation only if the function f is strongly convex.

A function $f : U \to \mathbb{R}$ is said to be *strongly convex* if for every $x \in U$ there exist $r > 0$ and $\varphi \in C^2(B(x, r))$ with $D^2\varphi > 0$ such that $f - \varphi$ is still convex on $B(x, r)$.

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For instance, if f and g are C^2 with $D^2 f > 0$, $D^2 g > 0$ then $x \mapsto \max\{f(x), g(x)\}$ is strongly convex. On the other hand, $x \mapsto x^4$ is convex, but not strongly convex.

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For instance, if f and g are C^2 with $D^2 f > 0$, $D^2 g > 0$ then $x \mapsto \max\{f(x), g(x)\}$ is strongly convex. On the other hand, $x \mapsto x^4$ is convex, but not strongly convex.
Therefore, the global smooth approximation problem for convex functions $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ cannot be solved only by these classical methods if f is not Lipschitz or strongly convex.

Theorem (A, 2013)

Let $U \subseteq \mathbb{R}^d$ be open and convex. For every convex function $f : U \to \mathbb{R}$ and every $\varepsilon > 0$ there exists a real-analytic convex function $g : U \to \mathbb{R}$ such that $f - \varepsilon \leq g \leq f$.

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The proof combines:

- A new method of gluing semilocal smooth convex approximations into global smooth convex approximations;
- Whitney's approximation theorem, and
- some insight into the global structure of convex functions $f : \mathbb{R}^n \to \mathbb{R}$.

Meaning that the global geometrical behaviour of a convex function $f : \mathbb{R}^d \to \mathbb{R}$ is rather rigid:

Theorem (A, 2013)

Let $f : \mathbb{R}^d \to \mathbb{R}$ be a convex function. The following conditions are equivalent:

1. *f cannot be uniformly approximated by strictly convex functions.*
2. *f cannot be uniformly approximated by strongly convex functions.*
3. *There exist $k < d$, a linear projection $P : \mathbb{R}^d \to \mathbb{R}^k$, a convex function $c : \mathbb{R}^k \to \mathbb{R}$ and a linear function $\ell : \mathbb{R}^d \to \mathbb{R}$ such that $f = c \circ P + \ell$.*
4. *f cannot be written in the form $f = \ell + c$, where ℓ is linear and $\lim_{|x| \to \infty} c(x) = \infty$.*

Smooth convex approximations of convex functions on Banach spaces

Classical methods: the inf-convolutions with squares of norms

$$f_\lambda(x) = \inf_{y \in X}\{f(y) + \frac{1}{2\lambda}|x-y|^2\}$$

work fine on Banach spaces X with dual LUR norms, providing C^1 convex approximations of functions on bounded sets whenever f is bounded on bounded sets.

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work fine on Banach spaces X with dual LUR norms, providing C^1 convex approximations of functions on bounded sets whenever f is bounded on bounded sets. By combining this method with a refinement of the new gluing procedure for convex approximations mentioned before, one can show:

Theorem (A-Mudarra, 2014)

Let U be an open convex subset of a Banach space X with a separable dual, and let $f : U \to \mathbb{R}$ be convex and continuous (not necessarily bounded on bounded sets). Then, for every $\varepsilon > 0$ there exists a convex function $g \in C^1(X)$ such that $|f - g| \leq \varepsilon$ on U.

What about higher order smoothness?

Open Problem

Can every continuous convex function $f : \ell_2 \to \mathbb{R}$ be approximated by C^∞ smooth convex functions, uniformly on ℓ_2?

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It would suffice to solve this problem in the special case when f is convex and Lipschitz.

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Theorem (Hajek-Talponen 2014, resp. Deville-Fonf-Hajek 1998)

Let X be a separable Banach space with a C^k smooth equivalent norm, $k \in \mathbb{N} \cup \{\infty\}$ (resp. let X be $L^{2n}[0,1]$, or ℓ_{2n}, or c_0). For every bounded convex body C in X and every $\varepsilon > 0$ there exists a C^k smooth (resp. real analytic) convex body D such that $C \subset D \subset C + \varepsilon B(0,1)$.

C^1 convex extensions of functions

Problem

Let $C \subset \mathbb{R}^n$ be closed, and let $f : C \to \mathbb{R}$, $G : C \to \mathbb{R}^n$ be continuous mappings satisfying Whitney's extension condition (W^1), that is,

$$\lim_{x,y\in C,\, |x-y|\to 0^+} \frac{f(x)-f(y)-\langle G(y), x-y\rangle}{|x-y|} = 0,$$

uniformly on compact subsets of C. What additional conditions on f, G, if any, will guarantee that there exists a convex *function $F \in C^1(\mathbb{R}^n)$ such that $F = f$ on C and $\nabla F = G$ on C?*

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Previous results of M. Ghomi (2002) and, independently, M. Yan (2014), imply that if C is compact and convex, and if $f \in C^2$ satisfies $D^2 f > 0$ on a neighbourhood of C then this is always possible, with no further assumptions on f, G.

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However, positive definiteness of $D^2 f$ is a very strong condition, far from necessary.

Obstructions: if C is not compact, even in a most nice situation (for instance, when we assume C to be convex with a smooth boundary, and f is assumed to have C^∞ smooth extensions with strictly positive Hessian on a neighborhood of C) there may not be any such F.

These are the instructions to find a counterexample:

These are the instructions to find a counterexample:

1. Avoid using the shower.

2. Get in a bathtub...

... preferrably without a girl.

The following example is due to Schulz and Schwartz (1979).

Example

Let $C = \{(x,y) \in \mathbb{R}^2 \, : \, x > 0, xy \geq 1\}$, *and define*

$$f(x,y) = -2\sqrt{xy}$$

for every $(x,y) \in C$. *The set* C *is convex and closed, with a nonempty interior, and* f *is convex on a neighborhood of* C. *However,* f *does not have any convex extension to all of* $\mathbb{R}^2$.

A variation of this bathtub-like example shows that the obstruction persists if we require that $D^2 f > 0$ on a neighborhood of C.

Example

Let $C = \{(x, y) \in \mathbb{R}^2 : x > 0, xy \geq 1\}$*, and define*

$$f(x, y) = -2\sqrt{xy} + \frac{1}{x+1} + \frac{1}{y+1}$$

for every $(x, y) \in C$*. The set* C *is convex and closed, with a nonempty interior, and* f *has a strictly positive Hessian on a neighborhood of* C*. However,* f *does not have any convex extension to all of* $\mathbb{R}^2$*.*

However, if we require that C be compact, there are geometrical conditions which, together with (W^1), are necessary and sufficient for f to have a convex extension $F \in C^1(\mathbb{R}^n)$ such that $F = f$ on C and $\nabla F = G$ on C.

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$$(C) \qquad f(x) - f(y) \geq \langle G(y), x - y \rangle \text{ for all } x, y \in C;$$

$$(CW^1) \quad f(x) - f(y) = \langle G(y), x - y \rangle \implies G(x) = G(y) \text{ for all } x, y \in C.$$

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Theorem (A-Mudarra, 2015)

Let C be a compact (not necessarily convex) subset of $\mathbb{R}^n$. Let $f : C \to \mathbb{R}$ be an arbitrary function, and $G : C \to \mathbb{R}^n$ be a continuous mapping. Then there exists a convex function $F \in C^1(\mathbb{R}^n)$ with $F = f$ and $\nabla F = G$ on C if and only if f and G satisfy the conditions (C), (W^1), and (CW^1) on C.

Note that (W^1) is a local condition, while (C) and (CW^1) are not.

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Corollary (A-Mudarra, 2015)

Let C be a compact convex subset of $\mathbb{R}^n$ with non-empty interior. Let $f : C \to \mathbb{R}$ be a convex function, and $G : C \to \mathbb{R}^n$ be a continuous mapping satisfying Whitney's extension condition (W^1) on C. Then there exists a convex function $F \in C^1(\mathbb{R}^n)$ such that $F(y) = f(y)$ and $\nabla F(y) = G(y)$ for every $y \in C$.

Idea of the proof of the Theorem:
Using Whitney's extension theorem, we may assume that $f \in C^1(\mathbb{R}^n)$, with $\nabla f = G$ on C, and that f satisfies conditions (C) and (CW^1) on C.

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Consider $m(f) : \mathbb{R}^n \to \mathbb{R}$ defined by

$$m(f)(x) = \sup_{y \in C}\{f(y) + \langle \nabla f(y), x - y\rangle\}.$$

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(In the case when C is convex and has nonempty interior, it is easy to see that if $h : \mathbb{R}^n \to \mathbb{R}$ is convex and $h = f$ on C, then $m(f) \leq h$. Thus, in this case, $m(f)$ is the minimal convex extension of f to all of $\mathbb{R}^n$, which explains the notation.

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(In the case when C is convex and has nonempty interior, it is easy to see that if $h : \mathbb{R}^n \to \mathbb{R}$ is convex and $h = f$ on C, then $m(f) \leq h$. Thus, in this case, $m(f)$ is the minimal convex extension of f to all of $\mathbb{R}^n$, which explains the notation. However, if C is convex but has empty interior then there is no minimal convex extension operator.)

If the function $m(f)$ were differentiable on $\mathbb{R}^n$, there would be nothing else to say. Unfortunately, there are examples showing that $m(f)$ need not be differentiable outside C, even when C is convex and f satisfies (CW^1).

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Lemma

Let $f \in C^1(\mathbb{R}^n)$, let C be a compact subset of $\mathbb{R}^n$ (not necessarily convex), and assume that f satisfies (C) and (CW^1) on C. Then, for each $x_0 \in C$, the function $m(f)$ is differentiable at x_0, with $\nabla m(f)(x_0) = \nabla f(x_0)$.

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Lemma

Let $f \in C^1(\mathbb{R}^n)$, let C be a compact subset of $\mathbb{R}^n$ (not necessarily convex), and assume that f satisfies (C) and (CW^1) on C. Then, for each $x_0 \in C$, the function $m(f)$ is differentiable at x_0, with $\nabla m(f)(x_0) = \nabla f(x_0)$.

Now, the differentiability of $m(f)$ on ∂C can be used, in combination with Whitney's approximation theorem, to construct a (not necessarily convex) differentiable function g such that $g = f$ on C, $g \geq m(f)$ on $\mathbb{R}^n$, and $\lim_{|x|\to\infty} g(x) = \infty$.

Indeed, define

$$H(x) = |f(x) - m(f)(x)| + 2d(x, C)^2,$$

where $d(x, C)$ stands for the distance from x to C.

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It's easy to see that H is differentiable on C, with $\nabla H(x_0) = 0$ for every $x_0 \in C$.
Then, according to Whitney's approximation theorem, we can find $\varphi \in C^\infty(\mathbb{R}^n \setminus C)$ such that

$$|\varphi(x) - H(x)| \leq d(x, C)^2 \quad \text{for every } x \in \mathbb{R}^n \setminus C.$$

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Define $\widetilde{\varphi} : \mathbb{R}^n \to \mathbb{R}$ by $\widetilde{\varphi} = \varphi$ on $\mathbb{R}^n \setminus C$ and $\widetilde{\varphi} = 0$ on C.
Again, it's easy to see that $\widetilde{\varphi}$ is differentiable on all of $\mathbb{R}^n$, and has a null gradient on C.

Put $g := f + \widetilde{\varphi}$. The function g is differentiable on $\mathbb{R}^n$, and coincides with f on C.

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$$g(x) \geq f(x) + H(x) - d(x, C)^2 = f(x) + |f(x) - m(f)(x)| + d(x, C)^2 \geq m(f)(x) + d(x, C)^2.$$

In particular $g \geq m(f)$.

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In particular $g \geq m(f)$. On the other hand, we know that $m(f)$ is Lipschitz on $\mathbb{R}^n$, and therefore $m(f)$ may decay only linearly at infinity, while $d(\cdot, C)^2$ grows quadratically at infinity. Hence the above inequality also implies that $\lim_{|x|\to\infty} g(x) = \infty$.

Now we use a differentiability property of the convex envelope of a function $\psi : \mathbb{R}^n \to \mathbb{R}$, defined by

$$\mathrm{conv}(\psi)(x) = \sup\{h(x) \ : \ h \text{ is convex }, h \leq \psi\}.$$

Namely,

Theorem (Kirchheim-Kristensen, 2001)

If $\psi : \mathbb{R}^n \to \mathbb{R}$ *is differentiable and* $\lim_{|x|\to\infty} \psi(x) = \infty$*, then* $conv(\psi) \in C^1(\mathbb{R}^n)$.

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If we define $F = \mathrm{conv}(g)$ we thus get that F is convex on $\mathbb{R}^n$ and $F \in C^1(\mathbb{R}^n)$. And it's easy to see that $F = f$ and $\nabla F = G$ on C. $\square$

C^∞ convex extensions of functions

Problem

Let $C \subset \mathbb{R}^n$ be compact and convex, and let $f : C \to \mathbb{R}$, $G : C \to \mathbb{R}^n$ be continuous mappings satisfying Whitney's extension condition (W^m). What additional conditions on f and $\{P_y^m\}_{y \in C}$, if any, will guarantee that there exists a convex *function $F \in C^m(\mathbb{R}^n)$ such that $F = f$ on C and P_y^m is the Taylor polynomial of order m of f at y, for each $y \in C$?*

Definition

Let $m \in \mathbb{N}$, $m \geq 2$. We will say that f, together with a family of polynomials $\{P_y^m\}_{y\in C}$ of degree up to m such that $P_y^m(y) = f(y)$, satisfy the condition (CW^m) provided that

$$\liminf_{t\to 0^+} \frac{1}{t^{m-2}} \left(D^2 P_y^m(y)(v)^2 + \cdots + \frac{t^{m-2}}{(m-2)!} D^m P_y^m(y)(w^{m-2}, v^2) \right) \geq 0$$

uniformly on $y \in C, w, v \in \mathbb{S}^{n-1}$.

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This condition is necessary for f to have a C^m convex extension F with Taylor polynomials P_y^m at each $y \in C$, as is easily seen by writing the Taylor expansion of D^2F at points $y \in C$ and using that $D^2F \geq 0$ on $\mathbb{R}^n$.

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This condition is necessary for f to have a C^m convex extension F with Taylor polynomials P_y^m at each $y \in C$, as is easily seen by writing the Taylor expansion of $D^2 F$ at points $y \in C$ and using that $D^2 F \geq 0$ on $\mathbb{R}^n$. We do not know whether this condition is sufficient as well if C has nonempty interior.

We do know that the condition is insufficient if $\mathrm{int}(C) = \emptyset$.

However, in the case $m = \infty$ we have a complete answer to our extension problem for C convex and compact (no matter whether or not C has empty interior).

Theorem (A-Mudarra, 2015)

Let C be a compact convex subset of $\mathbb{R}^n$. Let $f : C \to \mathbb{R}$ be a function, and let $\{P_y^m\}_{y \in C, m \in \mathbb{N}}$ be a compatible family of polynomials for C^∞ extension of f. Then f has a convex, C^∞ extension F to all of $\mathbb{R}^n$, with $J_y^m F = P_y^m$ for every $y \in C$ and $m \in \mathbb{N}$, if and only if $\{P_y^m\}_{y \in C}$ satisfies (W^m) and (CW^m) on C, for every $m \in \mathbb{N}$, $m \geq 2$.

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First step. We estimate the possible lack of convexity of f outside C: by using the conditions (CW^m), a Whitney partition of unity, and some ideas from the proof of the Whitney extension theorem in the C^∞ case, we construct a function $\eta \in C^\infty(\mathbb{R})$ such that $\eta \geq 0$, $\eta^{-1}(0) = (-\infty, 0]$, and $\min_{|v|=1} D^2 f(x)(v)^2 \geq -\eta\left(d(x, C)\right)$ for every $x \in \mathbb{R}^n$.

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First step. We estimate the possible lack of convexity of f outside C: by using the conditions (CW^m), a Whitney partition of unity, and some ideas from the proof of the Whitney extension theorem in the C^∞ case, we construct a function $\eta \in C^\infty(\mathbb{R})$ such that $\eta \geq 0$, $\eta^{-1}(0) = (-\infty, 0]$, and $\min_{|v|=1} D^2 f(x)(v)^2 \geq -\eta\left(d(x, C)\right)$ for every $x \in \mathbb{R}^n$.

Second step. Then we compensate the lack of convexity of f outside C with the construction of a convex function $\psi \in C^\infty(\mathbb{R}^n)$ such that $\psi \geq 0$, $\psi^{-1}(0) = C$, and $\min_{|v|=1} D^2\psi(x)(v)^2 \geq 2\eta\left(d(x, C)\right)$. Then, by setting $F := f + \psi$ we conclude the proof.

In order to construct this function ψ, we consider a function $g \in C^\infty(\mathbb{R})$ with $g^{-1}(0) = (-\infty, 0]$ and $g''(t) > 0$ for all $t > 0$, and with some other essential properties coming from the estimations done in the first step that we ignore here.

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We may assume that $0 \in C$. Now, for every $w \in \mathbb{S}^n$, define $h(w) = \max_{z \in C} \langle z, w \rangle$, the support function of C. Then define the function $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}$ by

$$\varphi(x) = \int_{\mathbb{S}^n} g\left(\langle x, w \rangle - h(w)\right)\, dw \quad \text{for every} \quad x \in \mathbb{R}^n.$$

We have $\varphi^{-1}(0) = C$ and φ is convex, with

$$D^2\varphi(x)(v)^2 = \int_{\mathbb{S}^n} g''(\langle x, w \rangle - h(w))\langle w, v \rangle^2\, dw.$$

For given $x \in \mathbb{R}^n \setminus C$ and $v \in \mathbb{S}^{n-1}$ we find a region $W = W(x, v)$ of $\mathbb{S}^{n-1}$ of sufficient volume (depending only, and conveniently, on $d(x, C)$) on which we have good lower estimates for $g''(\langle x, w\rangle - h(w))\langle w, v\rangle^2$. This only involves a careful selection of angles and directions, and provides the lower estimates on $D^2\varphi(x)(v^2)$ that we need.

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Unfortunately, this region is a hyperspherical cap whose volume is of the order of $d(x, C)^{n-1}$, and this is essentially the reason why the proof only works for $m = \infty$.
For m finite, the proof can be adapted to get an extension result with loss of differentiability.

Namely,

Theorem

Let C be a compact convex subset of $\mathbb{R}^n$. Let $f : C \to \mathbb{R}$ be a function, $m \in \mathbb{N}$ with $m \geq n+3$, and let $\{P_y^m\}_{y \in C}$ be a family of polynomials of degree less than or equal to m and $P_y^m(y) = f(y)$ for every $y \in C$. Assume that $\{P_y^m\}_{y \in C}$ satisfies (W^m) and (CW^m). Then there exists a convex function $F \in C^{m-n-1}(\mathbb{R}^n)$ such that $J_y^{m-n-1} F = P_y^{m-n-1}$ for every $y \in C$.

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This result is probably not optimal, at least in the case when C has nonempty interior. However, examples show that if C has empty interior then one cannot expect to find smooth convex extensions (of functions satisfying (W^m) and (CW^m) on C) without experiencing a certain loss of differentiability.

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The examples also show that in $\mathbb{R}^2$ this loss amounts to at least two orders of smoothness, and that the situation does not improve as m grows large (unless $m = \infty$).

What about similar results in Banach spaces?

There are important obstructions even when C is a subspace and one only looks for continuous convex extensions of $f : C \to \mathbb{R}$.

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Borwein, Montesinos and Vanderwerff showed that : there are infinite-dimensional Banach spaces X, closed *subspaces* $E \subset X$ and continuous convex functions $f : E \to \mathbb{R}$ which have no continuous convex extensions to X.

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However, they showed that if X is a Banach space and X/E is separable, then each continuous convex function on E admits a continuous convex extension to X.

Vesely and Zajicek showed that this is still true even if X is a noncomplete normed space.

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Theorem (V. Zizler, 1989)

A separable $C(K)$ space is isomorphic to c_0 if and only if for every Banach space X containing it, every equivalent Gâteaux differentiable norm on $C(K)$ extends to an equivalent Gâteaux differentiable norm on X.

Open Problems

1. *Let C be a compact convex body of $\mathbb{R}^n$, and let $f : C \to \mathbb{R}$ be convex and satisfy (W^2). Is it true that there always exists a convex function $F \in C^2(\mathbb{R}^n)$ such that $F = f$ on C?*

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   (This is the most general form of the Whitney extension problem for convex functions.)

Thank you for your attention!