σ-Porosity of the set of strict contractions in a space of non-expansive mappings

Christian Bargetz

joint work with Michael Dymond

Relations Between Banach Space Theory
and Geometric Measure Theory

8–12 June 2015

The setting

Let X be a Banach space and $C \subset X$ a closed, convex and bounded set.
We consider the space

$$\mathcal{M} = \{f \colon C \to C \colon \forall x, y \in C \colon \|f(x) - f(y)\| \leq \|x - y\|\}$$

equipped with the metric

$$d(f, g) = \sup_{x \in C} \|f(x) - g(x)\|$$

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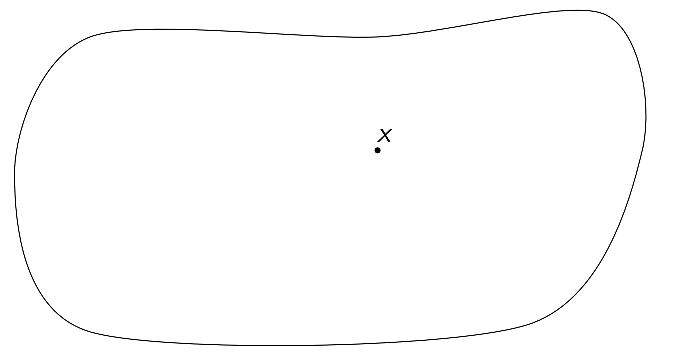

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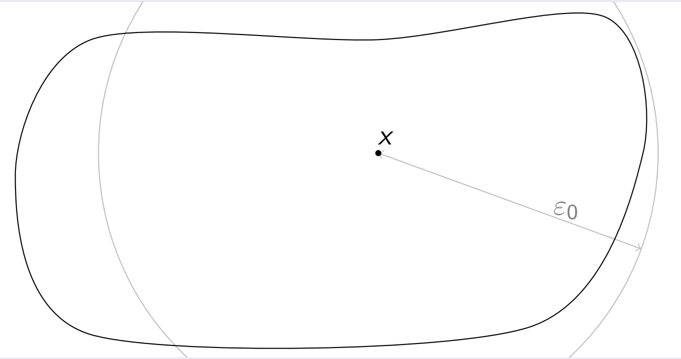

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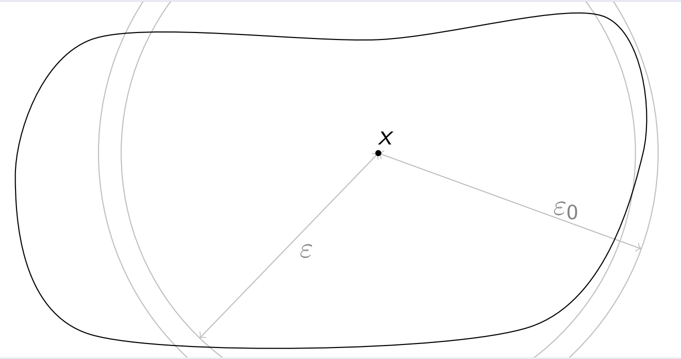

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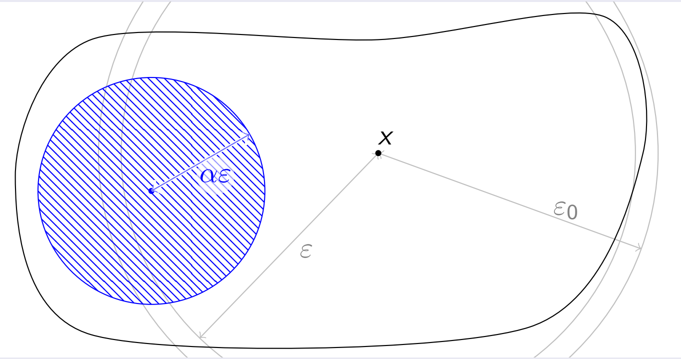

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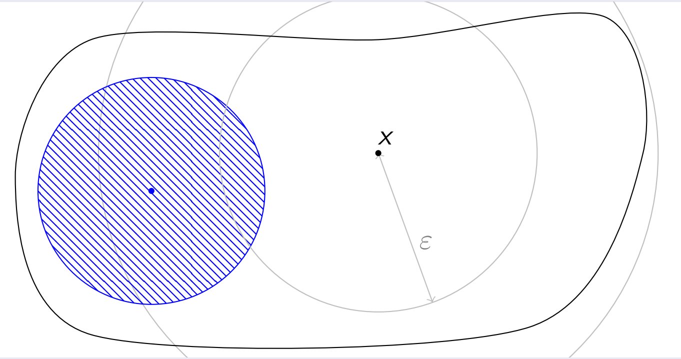

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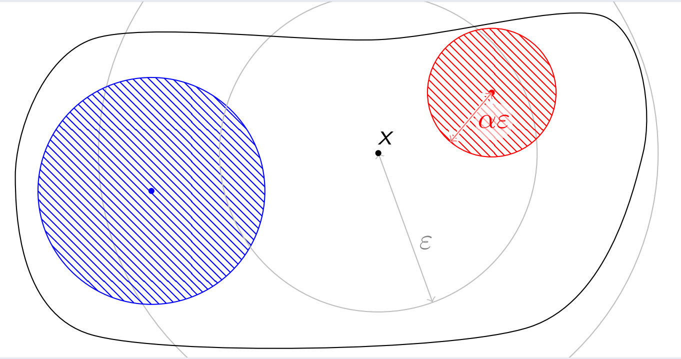

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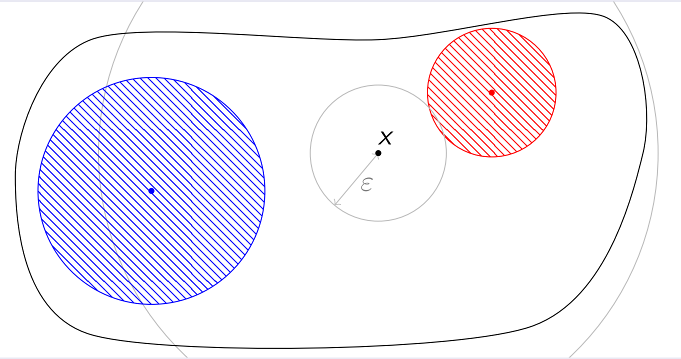

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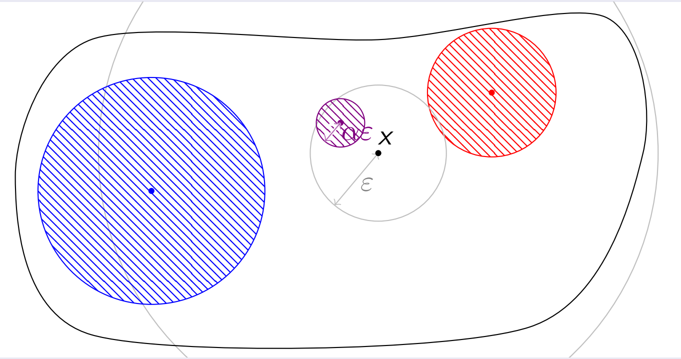

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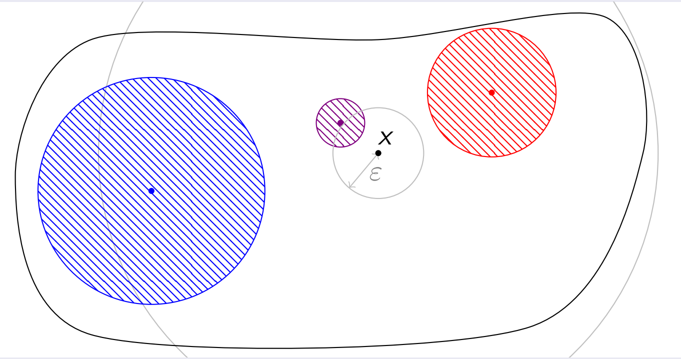

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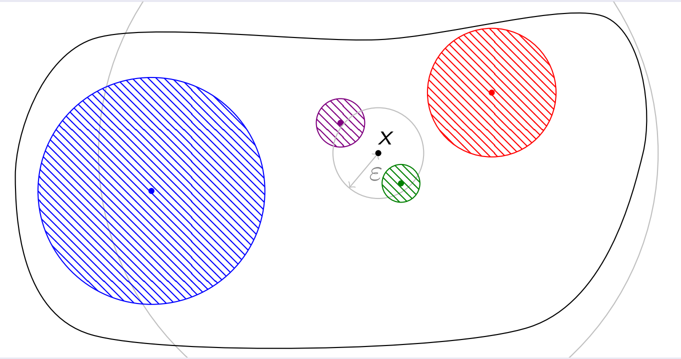

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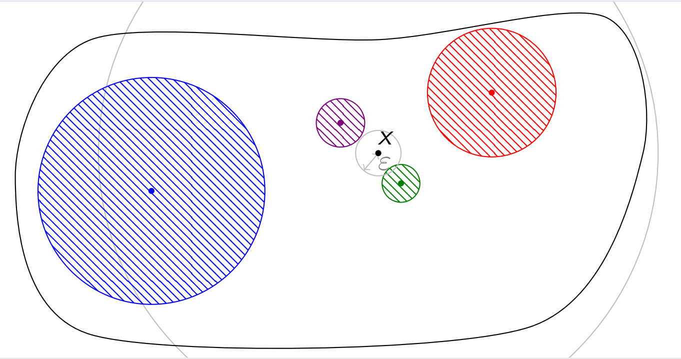

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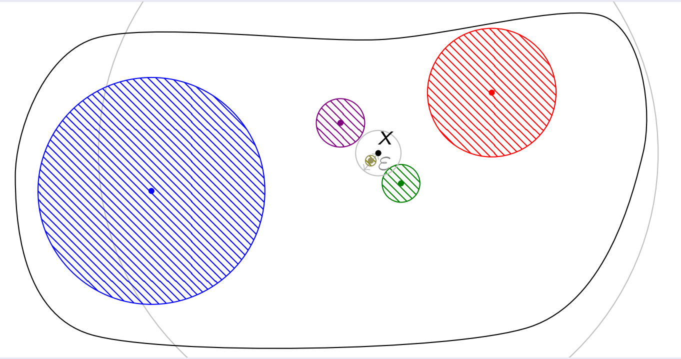

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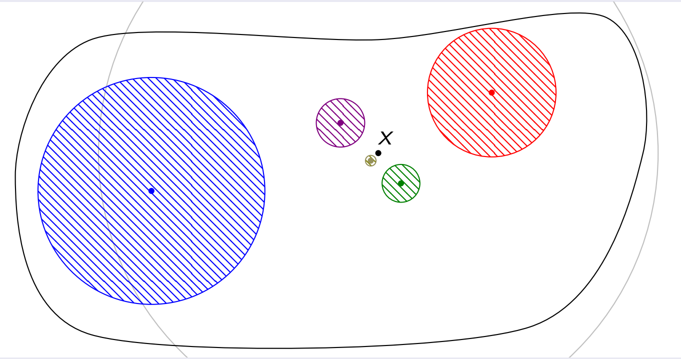

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Note that σ-porous sets are of first category in the sense of the Baire category theorem.

The Hilbert space case

Theorem (De Blasi and Myjak, 1989)

If X is a Hilbert space the set $\mathcal{N}$ of strict contractions on C is a σ-porous subset of $\mathcal{M}$.

Proof sketch.

Take a sequence $(L_k)_{k\in\mathbb{N}}$ with $L_n \nearrow 1$ and set

$$\mathcal{N}_k = \{f \in \mathcal{M} \colon \operatorname{Lip}(f) \leq L_k\}.$$

Given $f \in \mathcal{N}_k$ and $\varepsilon > 0$ set g to the identity on a small ball around the fixed point x^* of f and to f outside a bigger ball around x^* then use Kirszbraun's extension theorem to get a close enough midpoint of a ball outside $\mathcal{N}_k$. □

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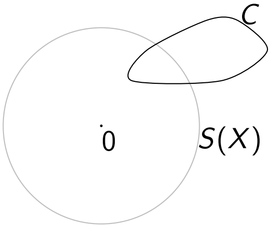
The general case

Theorem (B. and Dymond, 2015)

Let X be a Banach space and $C \subset X$ a closed, convex and bounded set. Then the set $\mathcal{N}$ of all strict contractions is a σ-porous subset of $\mathcal{M}$.

Sketch of the proof



We define

$$\mathcal{N}^p_{a,b} = \left\{ f \in \mathcal{N} : a < \operatorname{Lip}(f, \Gamma) \leq b, \ \operatorname{Lip}(f) \leq 1 - \frac{1}{p} \right\}.$$

Fix $f \in \mathcal{N}^p_{a,b}$. Choose $x_0 \in \Gamma$ such that

$$\liminf_{t \to 0+} \frac{\|f(x_0 + te) - f(x_0)\|}{t} \geq a.$$

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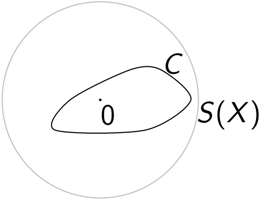

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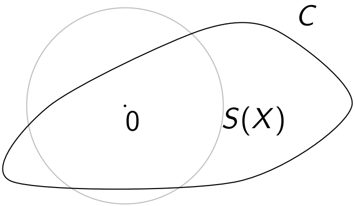

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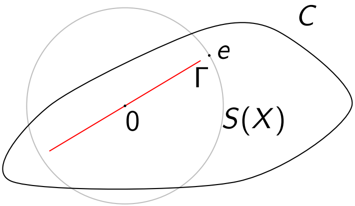

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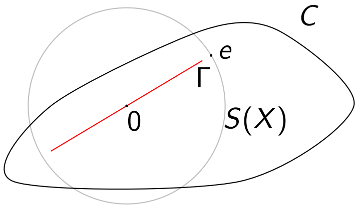

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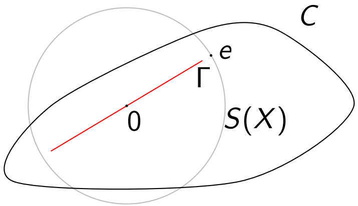

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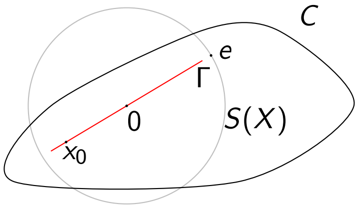

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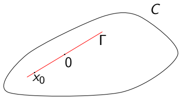

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$$g(x) = f(x + \sigma\phi_\varepsilon(e^*(x - x_0))(e - (x - x_0))).$$

where $\sigma\phi_\varepsilon(e^*(x - x_0))(e - (x - x_0))$ stretches along Γ to increase the Lipschitz constant.

Setting $R = \mathrm{diam}(C)$, if $b - a$ is small enough, we obtain

- $g \in \mathcal{M}$
- $d(f, g) \leq R\varepsilon$
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Hence $\mathcal{N}^p_{a,b}$ is porous.

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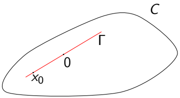

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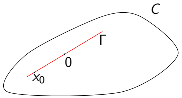

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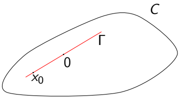

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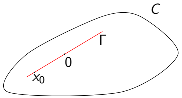

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Sketch of the proof, part III

Additionally to the condition that $b-a$ has to be small enough, it has to be big enough so that we can cover the whole interval $(0,1)$. Writing

$$\mathcal{N}=\left(\bigcup_{k,p}\mathcal{N}^p_{a_{k,p},b_{k,p}}\right)\cup\left(\bigcup_{k,p}\mathcal{N}^p_{a'_{k,p},b'_{k,p}}\right)\cup\mathcal{N}_0.$$

for suitable sequences $(a_{k,p})_{k,p}$, $(b_{k,p})_{k,p}$, $(a'_{k,p})_{k,p}$ and $(b'_{k,p})_{k,p}$ and

$$\mathcal{N}_0=\{f\in\mathcal{M}\colon f|_\Gamma=\text{const.}\}$$

finishes the proof. □

Sketch of the proof, part III

Additionally to the condition that $b-a$ has to be small enough, it has to be big enough so that we can cover the whole interval $(0,1)$. Writing

$$\mathcal{N} = \left(\bigcup_{k,p} \mathcal{N}^{p}_{a_{k,p},b_{k,p}}\right) \cup \left(\bigcup_{k,p} \mathcal{N}^{p}_{a'_{k,p},b'_{k,p}}\right) \cup \mathcal{N}_0.$$

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The case of separable Banach spaces

If X is a separable Banach space we get the following stronger result:

Theorem (B. and Dymond, 2015)

Let X be a separable Banach space. Then there exists a σ-porous set $\widetilde{\mathcal{N}} \subset \mathcal{M}$ such that for every $f \in \mathcal{M} \setminus \widetilde{\mathcal{N}}$, the set

$$R(f) = \{x \in C : \quad \mathrm{Lip}(f, x) = 1\}$$

is a residual subset of C.

Put differently, this Theorem says that outside of a negligible set, all mappings in the space $\mathcal{M}$ have the maximal possible Lipschitz constant 1 at typical points of their domain C.

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Outlook

Denote by g_ε the function

$$g_\varepsilon \colon C \to C, x \mapsto f(x + \sigma\phi_\varepsilon(e^*(x - x_0))(e - (x - x_0))).$$

The curve

$$[0, \varepsilon_0) \to \mathcal{C}(X; X), \varepsilon \mapsto g_\varepsilon$$

is Lipschitz.

Question

Can such a curve be chosen differentiable, to get information on the directions from which the midpoints g_ε are approaching f?

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