Sobolev and BV classes on infinite-dimensional domains

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SOBOLEV SPACES ON $\mathbb{R}^d$

1. Via Sobolev derivatives:

$$\int \varphi \partial_{x_i} f \, dx = - \int f \partial_{x_i} \varphi \, dx$$

$$W^{p,1}(\mathbb{R}^d) = \{ f \in L^p(\mathbb{R}^d) \colon \ \partial_{x_i} f \in L^p(\mathbb{R}^d) \}$$

with norm

$$\|f\|_{p,1} = \|f\|_{L^p} + \|\nabla f\|_{L^p}, \quad \nabla f = (\partial_{x_i} f).$$

2. Via completions:
$=$completion of $C_0^\infty(\mathbb{R}^d)$ or with respect to $\|f\|_{p,1}$

3. Via directional derivatives:
$W^{p,1}(\mathbb{R}^d)$ consists of functions $f \in L^p(\mathbb{R}^d)$ such that for every i there is a version of f such that the functions

$$t \mapsto f(x + te_i)$$

are absolutely continuous on compact intervals and

$$\|f\|_{p,1} < \infty,$$

where ∇f is formed by $\partial_{x_i} f$ defined via these versions.

INFINITE DIMENSIONS:
$X = \mathbb{R}^\infty$ or $X = \ell^2$
(for simplicity)
Examples: product-measures, Gaussian standard product-measure, Gibbs measures
Difficulty:
no canonical measure a là Lebesgue

Integration by parts:

$$\int \partial_{x_i}\varphi(x_1,\dots,x_n)\,\mu(dx) =$$

$$= -\int \varphi(x_1,\dots,x_n)\beta_i(x)\,\mu(dx)$$

for all $\varphi \in C_b^\infty(\mathbb{R}^n)$.
β_i is called the Fomin derivative of μ along $e_i = (0,\dots,0,1,0,\dots)$.

Example: $\mu = \prod_{i=1}^{\infty} \varrho_i(x_i)\, dx_i$, $\varrho_i \in W^{1,1}$, then

$$\beta_i = \partial_{x_i} \varrho_i / \varrho_i .$$

Let μ be standard Gaussian on $\mathbb{R}^\infty$. For $h = (h_n) \in H = \ell^2$ let

$$\widehat{h}(x) = \sum_{n=1}^{\infty} h_n x_n .$$

Then

$$\int \partial_h \varphi \, \mu = \int \widehat{h} \varphi \, \mu .$$

Let μ be a Borel probability measure on $\mathbb{R}^\infty$ having logarithmic derivatives β_i along the vectors e_i. Let $f \in L^p(\mu)$.
Let $\beta_i \in L^{p'}(\mu)$, $p' = p/(p-1)$. We say that f has a Sobolev derivative

$$\partial_{x_i} f = \partial_{e_i} f \in L^p(\mu)$$

if for all φ of class C_b^∞ in $x_1, \dots, x_n$ we have

$$\int \varphi \partial_{x_i} f \, \mu =$$

$$= -\int f \partial_{x_i} \varphi \, \mu - \int \varphi f \beta_i \, \mu.$$

In $\mathbb{R}^d$ for $\mu = \varrho\, dx$ with nice ϱ the second integral on the right is

$$\int \varphi f \frac{\partial_{x_i} \varrho}{\varrho}\, \varrho\, dx =$$

$$= \int \varphi f \partial_{x_i} \varrho\, dx.$$

On $\mathbb{R}^\infty$ only the "ratio" $\beta_i = \partial_{x_i}\varrho / \varrho$ makes sense.

Finally, the directional version has a natural analog.

Let $W^{p,1}(\mu)$ be the class of all f with finite norm

$$\|f\|_p + \|\nabla f\|_p, \quad \nabla f = (\partial_{x_i} f).$$

But what is the norm of ∇f?
For the standard Gaussian measure a natural (not the only possible) choice is the ℓ^2-norm.

IN GENERAL:
suppose $H \subset \mathbb{R}^\infty$ is a continuously embedded Hilbert space with norm $|h|_H$ in which the linear span of e_i is dense. Let

$$|\nabla f|_H := \sup\Big\{|\partial_h f|,\ |h|_H \le 1\Big\},$$

$$h = \sum_{i=1}^{n} h_i e_i.$$

$$\partial_h f = h_1 \partial_{e_1} f + \cdots + h_n \partial_{e_n} f.$$

Sobolev classes on infinite-dimensional domains:

What is a domain?

H-open set Ω: $(\Omega - x) \cap H$ is open in H for all x.

$\Omega = \left\{ x : \ \sum_{n=1}^{\infty} n^{-2} x_n^2 < \infty \right\}$ is not open, but H-open.

Sobolev classes on H-open convex domains:

Definition N3 (directional property) applies

Two other definitions: modifications needed
Difficulty: what is the replacement for the class of smooth finitely based functions? (concerns both integration by parts and completion)
ONE POSSIBILITY: take the class of functions φ with the property: for any straight line L intersecting Ω, $\varphi|_L$ has compact support in $L \cap \Omega$ and is smooth.

BV spaces

On $\mathbb{R}^d$: $f \in L^1(\mathbb{R}^d)$ is in BV if the derivatives $\partial_{x_i} f$ in the sense of distributions are bounded (signed) measures ν_i, i.e.

$$\int \partial_{x_i}\varphi \; f \, dx = - \int \varphi \, \nu_i$$

for all $\varphi \in C_0^\infty$.

The Skorohod derivative of a measure ν along e_i is a bounded measure $d_{e_i}\nu$ such that

$$\int \partial_{x_i}\varphi \, \nu = -\int \varphi \, d_{e_i}\nu$$

for all smooth φ in finitely many variables.

When f was Sobolev, $\nu_i = d_{e_i}\nu$ was

$$\nu_i = \partial_{x_i} f \cdot \mu + f\beta_i \cdot \mu.$$

NOW take

$$D_i f := \nu_i - f\beta_i \cdot \mu.$$

This is a finite measure if $f\beta_i \in L^1(\mu)$; in the case of the standard Gaussian measure $\beta_i(x) = -x_i$, so we need $x_i f \in L^1(\mu)$.

Next step: take vector measure with components $D_i f$.
With values in ℓ^2 ?
ℓ^2-valued measures: bounded variation and bounded semivariation.

$\eta\colon\ \mathcal{B} \to \ell^2$ vector measure
Variation:

$$Var(\eta) = \sup \|\eta(B_1)\| + \cdots + \|\eta(B_n)\|$$

over finite partitions of the space in $B_1, \ldots, B_n \in \mathcal{B}$.
Semivariation:

$$\|\eta\| = \sup \|\langle \ell, \eta \rangle\|$$

over functionals ℓ with unit norm

AGAIN μ standard Gaussian on $\mathbb{R}^\infty$.
Let SBV = all $f \in L^1(\mu)$ such that

$$f\widehat{h} \in L^1(\mu)\ \forall\, h \in H$$

and there is an H-valued measure Λf
of bounded SEMIVARIATION
such that the measure $f \cdot \mu$ has Skorohod
derivatives $d_{e_i}(f \cdot \mu)$ and

$$d_{e_i}(f \cdot \mu) = (\Lambda f, e_i) - x_i f \cdot \mu.$$

The class BV = those $f \in SBV$ for which Λf has bounded variation.

$BV \neq SBV$

both are Banach w.r.t. natural norms: for SBV

$$\|f\|_{L^1} + \sup_{|h|\leq 1} \|\widehat{h}f\|_{L^1} + \|\Lambda f\|,$$

similarly for BV

Indicator functions I_V: not in $W^{p,1}$ (but may belong to $H^{p,r}$)
CONVEX Borel V:
I_V may not be in BV for bounded convex Borel, but is in BV for open convex
BETTER for SBV: $I_V \in SBV$.

EXTENSION on $\mathbb{R}^d$:
V a bounded convex domain:
$f \in W^{p,1}(V)$ extends to
$\widehat{f} \in W^{p,1}(\mathbb{R}^d)$
there is a bounded linear
extension operator $f \mapsto \widehat{f}$

NOW

μ is the countable power of the standard Gaussian measure

$H = \ell^2$ the Cameron–Martin space

V a convex Borel set, $\mu(V) > 0$

M. Hino, Dirichlet spaces on H-convex sets in Wiener space, Bull. Sci. Math. 135 (2011) 667–683.

Functions with extensions
are dense in the Sobolev norm

THEOREM. There is open convex V and $f \in W^{p,1}(\mu, V)$ with no extension $\widehat{f} \in W^{p,1}(\mu)$.

One can also find V convex H-open with compact closure in $\mathbb{R}^\infty$.

THE CASE OF UNIT BALL IN HILBERT SPACE ???

REMARK.
If each $f \in W^{p,1}(\mu, V)$ extends to some $\widehat{f} \in W^{p,1}(\mu)$, then $\widehat{f}$ can be found with

$$\|\widehat{f}\|_{p,1} \leq C\|f\|_{p,1,V}.$$

THEOREM.

Every $f \in SBV(\mu, V) \cap L^\infty$ extended by 0 outside V belongs to SBV.

If $I_V \in BV$, the same is true for BV.

POSITIVE APPROACH:
Define $\widehat{W}^{p,1}(V)$ = those that are restrictions

$$\|f\|_{p,1,*} = \inf\{\|g\|_{p,1}: \quad g|_V = f\}.$$

Then $\widehat{W}^{p,1}(V)$ is Banach and each f in $\widehat{W}^{p,1}(V)$ is extendible.