On the structure of universal differentiability sets.

Michael Dymond

Universität Innsbruck

Relations between Banach Space Theory and
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Sets containing a point of differentiability of every Lipschitz function.

A set $S \subseteq \mathbb{R}^d$ is called a *universal differentiability set* if S contains a point of differentiability of every Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}$.

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A set $E \subseteq \mathbb{R}^d$ is called a *non-universal differentiability set* if there exists a Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}$ such that f is nowhere differentiable in E.

Examples.

- By Rademacher's theorem, any subset of $\mathbb{R}^d$ of positive Lebesgue measure is a universal differentiability set.
- Universal differentiabilty sets in $\mathbb{R}$ are precisely the sets of positive Lebesgue measure.

Existence of exceptional universal differentiability sets.

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- (D., Maleva 2014) For $d \geq 1$, $\mathbb{R}^d$ contains compact UDSs with Minkowski dimension one.
- (Preiss, Speight 2014) (Alberti, Csörnyei, Preiss 2010) (Csörnyei, Jones) $\mathbb{R}^d$ contains Lebesgue null UDSs for Lipschitz mappings $f : \mathbb{R}^d \to \mathbb{R}^l$ if and only if $l < d$.

Connection to porosity.

Definition

A subset P of $\mathbb{R}^d$ is called porous if there exists $c \in (0,1)$ such that for every $x \in P$ and every $\epsilon > 0$ there exists $h \in \mathbb{R}^d$ with $\|h - x\| \leq \epsilon$ and $B(h, c\|h - x\|) \cap P = \emptyset$. P is called σ-porous if P can be expressed as a countable union of porous sets.

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- Every porous subset of $\mathbb{R}^d$ is a *non-universal differentiability set.*

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- Every porous subset of $\mathbb{R}^d$ is a *non-universal differentiability set*.
- (Kirchheim, Preiss, Zajíček, 2001) Every σ-porous subset of $\mathbb{R}^d$ is a *non-universal differentiability set*.

Can a universal differentiability set be decomposed?

Given a universal differentiability set $S \subseteq \mathbb{R}^d$, is it possible to write $S = A \cup B$ where A and B are non-universal differentiability sets?

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Equivalently: Given a universal differentiabilty set $S \subseteq \mathbb{R}^d$ is it possible to find a pair of Lipschitz functions $f, g : \mathbb{R}^d \to \mathbb{R}$ such that f and g have no common points of differentiability in S.

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Simultaneous differentiability of Lipschitz functions.

1. (Lindenstrauss, Tišer, Preiss, 2012) Every pair (f, g) of Lipschitz functions on a Hilbert space have a common point of differentiability.

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Simultaneous differentiability of Lipschitz functions.

1. (Lindenstrauss, Tišer, Preiss, 2012) Every pair (f, g) of Lipschitz functions on a Hilbert space have a common point of differentiability.
2. **Open Question**: Does every triple (f, g, h) of Lipschitz functions on a Hilbert space have a common point of differentiability?

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The answer is yes.

1. (Csörnyei, Preiss, Tišer, 2004), (Alberti, Csörnyei, Preiss, 2010) Case $d = 2$.

Structural results for universal differentiability sets.

Theorem (D., 2014)

1. *Let* $S = A \cup B \subseteq \mathbb{R}^d$ *be a universal differentiability set where* A *is a closed subset of* S*. Then either* A *or* B *is a universal differentiability set.*

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- Recall: (Csörnyei, Preiss, Tišer, 2004) There exists a universal differentiability set $S = A \cup B \subseteq \mathbb{R}^d$ such that both A and B are non-universal differentiability sets.

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- (Csörnyei, Preiss, Tišer, 2004) There exists a universal differentiability set $S = A \cup B \subseteq \mathbb{R}^d$ such that A is a G_δ set and both A and B are non-universal differentiability sets.

The kernel of a universal differentiability set.

Theorem (D., 2014)

Let S be a universal differentiability set and define

$$\ker(S) = S \setminus \{x \in S : \exists r > 0 \text{ s.t. } B(x,r) \cap S \text{ is a non-UDS}\}.$$

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1. $\ker(S)$ *is a universal differentiability set.*
2. $\ker(\ker(S)) = \ker(S)$.

Open questions.

1. Does every universal differentiability set contain a closed universal differentiability set?

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1. Does every universal differentiability set contain a closed universal differentiability set?
2. Does every subset of $\mathbb{R}^d$ with positive Lebesgue measure contain a universal differentiability set of Lebesgue measure zero?

Thank you for listening.