Approximate Ramsey properties for finte dimensional ℓ_p-spaces

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Notation

- X Banach space, F finite dimensional Banach space,
- $\mathrm{Emb}(F, X)$ denotes the class of linear **isometric** embeddings of F into X, equipped with the distance

$$d(T, U) = \|T - U\|$$

induced by the operator norm.
- $\mathrm{Isom}(X)$ is the group of linear (surjective) isometries on X, usually equipped with SOT.

The main result

Theorem (Ramsey theorem for embeddings between ℓ_p^n's)

Given $0 < p < \infty$, *integers* d, m, r, *and* $\epsilon > 0$ *there exists* n *such that whenever* $c : \mathrm{Emb}(\ell_p^d, \ell_p^n) \to r$ *is a coloring of the set of all isometric embeddings* $\mathrm{Emb}(\ell_p^d, \ell_p^n)$ *of* ℓ_p^d *into* ℓ_p^n *into* r*-many colors, there is some isometric embedding* $\gamma : \ell_p^m \to \ell_p^n$ *and some color* $i < r$ *such that*

$$\gamma \circ \mathrm{Emb}(\ell_p^d, \ell_p^m) \subset (c^{-1}\{i\})_\epsilon.$$

The main result

The main result may also be stated as a result of stabilization of Lipschitz maps:

Theorem

Given $0 < p < \infty$*, integers* d, m*, and* $K, \epsilon > 0$ *there exists* $n \in \mathbb{N}$ *such that for any* K*-Lipschitz function* $L : \mathrm{Emb}(\ell_p^d, \ell_p^n) \to \mathbb{R}$*, there exists* $\gamma \in \mathrm{Emb}(\ell_p^m, \ell_p^n)$ *such that*

$$\mathrm{Osc}(L_{|}\gamma \circ \mathrm{Emb}(\ell_p^d, \ell_p^m)) < \epsilon.$$

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4. Embeddings versus copies; the case $d = 1$.

Borsuk-Ulam antipodal theorem

We can relate our Ramsey result to an equivalent form of Borsuk-Ulam called Lyusternik-Schnirel'man theorem (1930):

Theorem (a form of Borsuk-Ulam)

If the unit sphere $\mathcal{S}^{n-1}$ of ℓ_2^n is covered by n open sets, then one of them contains a pair $\{-x, x\}$ of antipodal points.

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By the fact that every finite open cover of a finite dimensional sphere is the ϵ-fattening of some smaller open cover, for some $\epsilon > 0$, our result for $d = 1, m = 1$ may be seen as a version of Borsuk-Ulam theorem ($n = n_p(1, 1, r, \epsilon)$).

Consequence on L_p spaces

Using classical results of Rudin (1976) - see also Lusky (1978):

Proposition

Assume $0 < p < +\infty, p \neq 4, 6, 8, \ldots$. *Then* $L_p(0,1)$ *is "approximately ultrahomogeneous", meaning that for any finite-dimensional subspace* F *of* $L_p(0,1)$*, for any* $\epsilon > 0$*, for any* $t \in \mathrm{Emb}(F, L_p(0,1))$*, there exists a surjective isometry* $T \in \mathrm{Isom}(L_p(0,1))$ *such that*

$$\|T_F - t\| \leq \epsilon.$$

As as consequence of a result of B. Randrianantoanina (1998), we may observe that this proposition is false for $p = 4, 6, 8, ...$; actually:

Proposition

If $p = 4, 6, 8, ...$ then for every $M > 1$, there exists a finite dimensional subspace F of $L_p(0,1)$ and $t \in \mathrm{Emb}(F, L_p(0,1))$ such any extension T of t on $L_p(0,1)$ satisfies $\|T\| \geq M$.

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A positive result holds for all $0 < p < +\infty$:

Consequence on L_p spaces

Theorem

For any $0 < p < +\infty$, $L_p[0,1]$ is of almost ℓ_p^n-disposition: for any $\ell_p^m \subset L_p([0,1])$, any embedding $j \in \mathrm{Emb}(\ell_p^m, \ell_p^n)$, there exists an embedding $T \in \mathrm{Emb}(\ell_p^n, L_p(0,1))$ such that

$$\| Id_{|\ell_p^m} - T \circ j \| \leq \epsilon.$$

Observation

$L_p(0,1)$ is the unique separable $\mathcal{L}_p$-space of almost ℓ_p^n-disposition.

Extreme amenability

The almost ℓ_p^n-disposition of $L_p(0,1)$ plus the approximate Ramsey property of embeddings between ℓ_p^n's imply an alternative proof of the following results of Gromov-Milman (1983) and Giordano-Pestov (2003).

Theorem

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The point is that $L_p(0,1)$ looks like the Fraïssé limit of class of ℓ_p^n's together with isometric embeddings....

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There exists a renorming of $C(0,1)$ as an M-space with almost transitive norm.

Defining a "disjoint copy of ℓ_∞^n" in an M-space as an isometric copy of ℓ_∞^n generated by disjoint vectors, we may improve this to:

Theorem (the "lattice Gurarij")

There exists a renorming of $C(0,1)$ as an M-space $\mathbb{G}_\ell$ which is "approximately disjointly homogeneous": i.e. for any $\epsilon > 0$, any isometry t between two disjoint copies F and F' of ℓ_∞^n, there is a surjective, disjoint preserving, linear isometry T on $\mathbb{G}_\ell$ such that

$$\|T_{|F} - t\| \leq \epsilon.$$

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Observation

Therefore Gowers' Ramsey theorem about block subspaces of c_0 *has a finite dimensional version with subspaces of* ℓ_∞^n*'s generated by* **disjoint vectors***, rather than with subspaces generated by* **successive vectors***.*

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- we look at matrices (n, d) of isometric embeddings of ℓ_1^d into ℓ_1^n. Since $n >> d$, and up to a discretization, some lines of the matrix will repeat themselves. Therefore we may associate
  - elements of $\mathrm{Emb}(\ell_1^d, \ell_1^n)$ with
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  - partitions of n into $k = k(d, \epsilon)$ pieces indexed by possible values of the lines
- with more work, and using a result of Matoušek-Rödl, we shall do it in a coherent way, e.g. composing surjections essentially corresponds to composing partitions...

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- however in our context we only need a Ramsey Theorem for ϵ**-equipartitions.**
- this is proved by **concentration of measure** (one of the colours has measure $\geq 1/r$, so "almost all" elements have the same colour up to ϵ, ...)

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