Energy integrals, metric embeddings and absolutely summing operators

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Joint work with Daniel Carando and Damián Pinasco

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Metric spaces arising from Euclidean spaces by a change of metric: some history

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$$(X,d) \overset{f}{\rightsquigarrow} (X,f(d)), \ \textit{where} \ f(d)(x,y) := f(d(x,y)).$$

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That is,

$$\|j(x) - j(y)\|_{\ell_2} = d_{1/2}(x,y) = |x-y|^{1/2}, \ \forall x, y \in \mathbb{R}.$$

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They proved that, for $0 < \alpha < 1, f(t) = t^\alpha$ becomes a suitable metric transformation.

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I. Schoenberg, *On certain metric spaces arising from Euclidean spaces by a change of metric and their imbedding in Hilbert space*, Ann. of Math. 38 (1937), pp. 787-793.

Theorem (Schoenberg)

For $0 < \alpha < 1$, the metric space $\left(\mathbb{R}^n, d_\alpha\right)$ is imbeddable in ℓ_2.

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Moreover, by combining Schoenberg's proof and a classic result of Menger, we have that for every compact set $K \subset \mathbb{R}^n$ the metric space (K, d_α) may be imbeddable in the surface of a Hilbert sphere.

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Classic result

For every compact set $K \subset \mathbb{R}^n$, there exist a positive number r and a distance preserving mapping

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Link

Metric Geometry $\leftrightarrow$ Potential Theory

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The connection!

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Theorem (Alexander-Stolarsky)

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We will be focused on computing the value of $M_{2\alpha}(K)$.

Some results...

Denote by B_n the unit ball in $\mathbb{R}^n$.

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- $M_1(B_3) = 2$ (Alexander, Proc. AMS. '77)
- $M_1(B_n) = ???$ $\rightsquigarrow$ remained unknown for a very long time.

A. Hinrichs, P. Nickolas, R. Wolf, *A note on the metric geometry of the unit ball*, Math. Z. 268 (2011), pp. 887-896.

Theorem (Hinrichs, Nickolas and Wolf)

$$M_1(B_n) = \frac{\pi^{1/2}\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}.$$

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The number $\frac{\pi^{1/2}\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})}$ is exactly $\pi_1(id : \ell_2^n \to \ell_2^n)$.

Theorem (Carando, G., Pinasco: Int Math Res Notices)

$$M_p(B_n) = \frac{\pi^{1/2}\,\Gamma\left(\frac{n+p}{2}\right)}{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{n}{2}\right)}\, M_p([-1,1])$$

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Using $\lim_{m\to\infty} \frac{\Gamma(m+c)}{\Gamma(m)m^c} = 1$, and the previous result we get:

Corollary

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In the case where the convex set is an ellipsoid $\mathcal{E}$?

$$M_p(\mathcal{E}) = M_p(T(B_n)) = \pi_p(T : \ell_2^n \to \ell_2^n)^p M_p([-1,1])$$

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Lemma

For every $x \in \mathbb{R}^n$, we have

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If T is the identity...

$$\|x\|^p = \pi_p(id : \ell_2^n \to \ell_2^n)^p \int_{S^{n-1}} |\langle x, t\rangle|^p d\lambda(t),$$

where λ is just the normalized Lebesgue surface measure on the sphere S^{n-1}.

$M_p(B_n) = \pi_p(id_{\ell_2^n})^p M_p([-1,1])$

Upper bound: sketch

Let μ be a signed borel measure on B_n of total mass one.

$$I_p(\mu; B_n) := \int_{B_n}\int_{B_n} \|x-y\|^p d\mu(x)d\mu(y)$$

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How to get equality?

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How to get equality? Recall that for any μ,

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Therefore, $M_p(B_n) \geq I_p(\mu_k; B_n) \to \pi_p(id_{\ell_2^n})^p M_p([-1,1])$.

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Question

How can we estimate the value of $\rho_\alpha(B_E)$, $0 < \alpha < 1$? Or, equivalently, how can we compute $M_p(B_E)$, $0 < p < 2$?

Theorem

(General upper bound)

$$M_p(B_E) \leq M_p([-1,1]) \frac{\pi^{1/2}\,\Gamma\left(\frac{n+p}{2}\right)}{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{n}{2}\right)} \int_{S^{n-1}} \|t\|_{E'}^p d\lambda(t).$$

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- Is there a closed formula for $M_p([-1, 1])$?

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