Energy integrals, metric embeddings and absolutely summing operators

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Metric spaces arising from Euclidean spaces by a change of metric: some history

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\[(X, d) \xrightarrow{f} (X, f(d)), \text{ where } f(d)(x,y) := f(d(x,y)).\]
For the metric space $(\mathbb{R}, |·|)$, Wilson considered the function $f(t) = t^{1/2}$. Denote $d_{1/2} := f(|·|) \Rightarrow d_{1/2}(x, y) = |x - y|^{1/2}$. He showed that $(\mathbb{R}, d_{1/2})$ may be isometrically imbedded in a separable Hilbert space. In other words, he proved that there exist a distance preserving (isometry) mapping $j: (\mathbb{R}, d_{1/2}) \to (\ell_2, \|·\|_{\ell_2})$. That is, $\|j(x) - j(y)\|_{\ell_2} = d_{1/2}(x, y) = |x - y|^{1/2}, \forall x, y \in \mathbb{R}$. 
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They proved that, for $0 < \alpha < 1$, $f(t) = t^\alpha$ becomes a suitable metric transformation.
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What happens for the $n$-dimensional real space $\mathbb{R}^n$?

Is the metric space $\left(\mathbb{R}^n, d_\alpha\right)$ isometrically imbeddable in $\ell_2$, where $d_\alpha(x, y) = \|x - y\|^\alpha$?
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What happens for the $n$-dimensional real space $\mathbb{R}^n$? Is the metric space $(\mathbb{R}^n, d_\alpha)$ isometrically imbeddable in $\ell_2$, where $d_\alpha(x, y) = \|x - y\|^\alpha$?


**Theorem (Schoenberg)**

For $0 < \alpha < 1$, the metric space $(\mathbb{R}^n, d_\alpha)$ is imbeddable in $\ell_2$.
Moreover, by combining Schoenberg’s proof and a classic result of Menger, we have that for every compact set $K \subset \mathbb{R}^n$ the metric space $(K, d_\alpha)$ may be imbeddable in the surface of a Hilbert sphere.
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**Classic result**

For every compact set \( K \subset \mathbb{R}^n \), there exist a positive number \( r \) and a distance preserving mapping

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j : (K, d_\alpha) \rightarrow (rS_{\ell_2}, \| \cdot \|_{\ell_2})
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Classic results

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$$\rho_\alpha(K) := \inf r$$
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$$\rho_\alpha(K) := \inf r \sim \text{least possible radius (}\alpha\text{-Schoenberg’s radii of } K)$$
A connection with another area

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Link

Metric Geometry $\leftrightarrow$ Potential Theory
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For a real number \( p \) (for us, \( 0 < p < 2 \)), we define

\[
I_p(\mu; K) := \int_K \int_K \|x - y\|^p d\mu(x) d\mu(y)
\]

as the \( p \)-energy integral given by \( \mu \).

And define,

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M_p(K) := \sup \{ I_p(\mu; K) : \mu \text{ is a signed Borel measure on } K \text{ of total mass one} \}
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as the \( p \)-maximal energy of \( K \).
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Energy Integrals: some definitions

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**Theorem (Alexander-Stolarsky)**

Let $K \subset \mathbb{R}^n$ be a compact set. Then,

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\rho_\alpha(K) = \sqrt{\frac{M_{2\alpha}(K)}{2}}
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The connection!


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We will be focused on computing the value of $M_{2\alpha}(K)$. 
Denote by $B_n$ the unit ball in $\mathbb{R}^n$.

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- $M_1(B_n) =$??? $\Rightarrow$ remained unknown for a very long time.

**Theorem (Hinrichs, Nickolas and Wolf)**

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M_1(B_n) = \frac{\pi^{1/2} \Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}.
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**Question**

- What is the value of \( M_p(B_n) \), for \( 0 < p < 2 \)?

The number \( \frac{\pi^{1/2} \Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \) is exactly \( \pi^{1/2} \) (id: \( \ell^2 \to \ell^2 \)).

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Theorem (Carando, G., Pinasco: Int Math Res Notices)

\[ M_p(B_n) = \frac{\pi^{1/2} \Gamma\left(\frac{n+p}{2}\right)}{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{n}{2}\right)} M_p([-1, 1]) \]
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Using \( \lim_{m \to \infty} \frac{\Gamma(m+c)}{\Gamma(m)m^c} = 1 \), and the previous result we get:

Corollary

\[ \rho_\alpha(B_n) \asymp n^{\frac{\alpha}{2}}. \]
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In the case where the convex set is an ellipsoid \( \mathcal{E} \)?

\[ M_p(\mathcal{E}) = M_p(T(B_n)) = \pi_p(T: \ell^n_2 \rightarrow \ell^n_2)^p M_p([-1, 1]) \]
How is the $p$-summing norm related with this problem?
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**Lemma**

*For every* $x \in \mathbb{R}^n$, *we have*

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\|Tx\|^p = \pi_p(T : \ell_2^n \to \ell_2^n)^p \int_{S^{n-1}} |\langle x, t \rangle|^p d\nu(t),
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*where $\nu$ is a probability measure on the unit sphere $S^{n-1}$.*
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**Lemma**

*For every $x \in \mathbb{R}^n$, we have*

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*If $T$ is the identity...*

$$\|x\|_p^p = \pi_p(id : \ell_2^n \to \ell_2^n)^p \int_{S^{n-1}} |\langle x, t \rangle|^p d\lambda(t),$$

*where $\lambda$ is just the normalized Lebesgue surface measure on the sphere $S^{n-1}$.***
$$M_p(B_n) = \pi_p(id_{\ell_2^n})^p M_p([-1, 1])$$

**Upper bound: sketch**

Let $\mu$ be a signed borel measure on $B_n$ of total mass one.

$$I_p(\mu; B_n) := \int_{B_n} \int_{B_n} \|x - y\|^p d\mu(x) d\mu(y)$$
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\[
= \pi_p(id_{\ell_2^n})^p \int_{S^{n-1}} \left[ \int_{-1}^{1} \int_{-1}^{1} |u - v|^p d\mu_t(u) d\mu_t(v) \right] d\lambda(t)
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Classic results

$M_p(B_n) = \pi_p(id_{\ell^2_n})^p M_p([-1, 1])$

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\[ \leq \pi_p(id_{\ell^2_n})^p \int_{S^{n-1}} M_p([-1, 1]) d\lambda(t) \]
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$$= \pi_p(id_{\ell^2_n})^p \int_{S^{n-1}} \left[ \int_{-1}^1 \int_{-1}^1 |u - v|^p d\mu_t(u) d\mu_t(v) \right] d\lambda(t)$$

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\[ M_p(B_n) = \pi_p=id_{\ell_2}^p M_p([-1, 1]) \]

How to get equality?
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How to get equality? Recall that for any \( \mu \),

\[ I_p(\mu; B_n) = \pi_p(id_{\ell_2^n})^p \int_{S^{n-1}} \left[ \int_{-1}^{1} \int_{-1}^{1} |u - v|^p d\mu_t(u)d\mu_t(v) \right] d\lambda(t) \]
How to get equality? Recall that for any $\mu$,

$$I_p(\mu; B_n) = \pi_p(id_{\ell_2^n})^p \int_{S^{n-1}} \left[ \int_{-1}^{1} \int_{-1}^{1} |u - v|^p d\mu_t(u) d\mu_t(v) \right] d\lambda(t)$$

We found a sequence $(\mu^k)_k \in \mathbb{N}$ of signed measures of total mass one $B_n$ such that

$$\int_{-1}^{1} \int_{-1}^{1} |u - v|^p d\mu^k_t(u) d\mu^k_t(v) \Rightarrow M_p([-1, 1]).$$
$M_p(B_n) = \pi_p(id_{\ell_2^n})^p M_p([-1, 1])$

How to get equality? Recall that for any $\mu$,

$$I_p(\mu; B_n) = \pi_p(id_{\ell_2^n})^p \int_{S^{n-1}} \left[ \int_{-1}^{1} \int_{-1}^{1} |u - v|^p d\mu_t(u) d\mu_t(v) \right] d\lambda(t)$$

We found a sequence $(\mu^k)_{k \in \mathbb{N}}$ of signed measures of total mass one $B_n$ such that

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Therefore, $M_p(B_n) \geq I_p(\mu^k; B_n) \to \pi_p(id_{\ell_2^n})^p M_p([-1, 1])$. 

Let $K \subset \mathbb{R}^n$ be a centrally symmetric convex body, then $K$ is just the unit ball of an $n$-dimensional Banach space $(E, \| \cdot \|_E)$. 

Question: How can we estimate the value of $\rho_\alpha(B_E)$, $0 < \alpha < 1$? Or, equivalently, how can we compute $M_p(B_E)$, $0 < p < 2$?
Let $K \subset \mathbb{R}^n$ be a centrally symmetric convex body, then $K$ is just the unit ball of an $n$-dimensional Banach space $(E, \| \cdot \|_E) \sim$ i.e., $K = B_E$. 

Bounds for other convex bodies

Let $K \subset \mathbb{R}^n$ be a centrally symmetric convex body, then $K$ is just the unit ball of an $n$-dimensional Banach space $(E, \| \cdot \|_E) \sim \text{i.e., } K = B_E$.

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**Question**

*How can we estimate the value of $\rho_\alpha(B_E)$, $0 < \alpha < 1$? Or, equivalently, how can we compute $M_p(B_E)$, $0 < p < 2$?*
Theorem

(General upper bound)

\[ M_p(B_E) \leq M_p([-1, 1]) \frac{\pi^{1/2}}{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{n}{2}\right)} \int_{S^{n-1}} \|t\|_{E'}^p d\lambda(t). \]
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This bound is expressed in terms of the mean width of \( B_E \), and is good enough in many cases!
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**Theorem (Carando, G., Pinasco)**

Let \( 1 < q \leq 2 \) then

\[ M_p(B_{\ell_q^n}) \asymp n^{\frac{p}{q'}} . \]

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**Theorem (Carando, G., Pinasco)**

Let \( 1 < q \leq 2 \) then

\[ M_p(B_{\ell^n_q}) \asymp n^{\frac{p}{q'}}. \]

In particular, \( \rho_\alpha(B_{\ell^n_q}) \asymp n^{\frac{\alpha}{q'}}. \)

**Remark:**

- Upper bounds were given using the previous result.
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Several open questions

- What is the asymptotic behavior of $\rho_\alpha(B_{\ell_q^n})$, for $2 \leq q \leq \infty$ or $q = 1$?
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- What is the asymptotic behavior of $\rho_\alpha(\mathcal{B}_{\ell_q^n})$, for $2 \leq q \leq \infty$ or $q = 1$?
- Is there a closed formula for $M_p([-1, 1])$?


