Isometrically universal structures in Banach spaces

Joanna Garbulińska-Węgrzyn

Institute of Mathematics
Jan Kochanowski University

8-12 June 2015

Results

(1) It is well-known that the (separable) Banach space $C[0;1]$ is isometrically universal for the class of all separable Banach space, there is no available proof of the corresponding fact for p-Banach spaces with $p < 1$. We presented the construction, uniqueness, and universality of the p-Gurariĭ space, an analogue of the Gurariĭ space for the class of p-Banach spaces, where $0 < p \leq 1$ is fixed.

Results

(2) The problem of existence of a universal Banach space with a finite-dimensional decomposition was investigated by Pełczyński and Kadec. We presented an isometric version of the known result on the existence of a complementably universal Banach space with a FDD.

(3) We showed that the Gurariĭ space has a norm one linear operator which is isometrically universal for the class of all nonexpansive operators between separable Banach spaces, in the sense that its restrictions to closed subspaces cover, up to isometry, the whole class of all nonexpansive operators between separable Banach spaces.

Results

(2) The problem of existence of a universal Banach space with a finite-dimensional decomposition was investigated by Pełczyński and Kadec. We presented an isometric version of the known result on the existence of a complementably universal Banach space with a FDD.

(3) We showed that the Gurariĭ space has a norm one linear operator which is isometrically universal for the class of all nonexpansive operators between separable Banach spaces, in the sense that its restrictions to closed subspaces cover, up to isometry, the whole class of all nonexpansive operators between separable Banach spaces.

Definitions

(1) Let X, Y be Banach spaces, $\varepsilon \in (0,1)$. A linear operator $f : X \to Y$ is an *ε-isometry* if

$$(1-\varepsilon)\cdot\|x\| \leq \|f(x)\| \leq (1+\varepsilon)\cdot\|x\|$$

holds for every $x \in X$. Analogously we can define ε-isometry for nonexpansive operators.

(2) Let $p \in (0,1]$. A *p-normed* (respectively, *p-Banach*) *space* is a quasinormed (respectively, quasi-Banach) space whose quasinorm is a p-norm, that is, it satisfies the inequality $\|x+y\|^p \leq \|x\|^p + \|y\|^p$.

(3) Aoki and Rolewicz showed that every quasinorm is equivalent to a p-norm for some $p \in (0,1]$ in the sense that they induce the same topology.

Definitions

(1) Let X, Y be Banach spaces, $\varepsilon \in (0,1)$. A linear operator $f : X \to Y$ is an *ε-isometry* if

$$(1-\varepsilon)\cdot\|x\| \leq \|f(x)\| \leq (1+\varepsilon)\cdot\|x\|$$

holds for every $x \in X$. Analogously we can define ε-isometry for nonexpansive operators.

(2) Let $p \in (0,1]$. A *p-normed* (respectively, *p-Banach*) *space* is a quasinormed (respectively, quasi-Banach) space whose quasinorm is a p-norm, that is, it satisfies the inequality $\|x+y\|^p \leq \|x\|^p + \|y\|^p$.

(3) Aoki and Rolewicz showed that every quasinorm is equivalent to a p-norm for some $p \in (0,1]$ in the sense that they induce the same topology.

Definitions

(1) Let X, Y be Banach spaces, $\varepsilon \in (0,1)$. A linear operator $f : X \to Y$ is an *ε-isometry* if

$$(1-\varepsilon)\cdot\|x\| \leq \|f(x)\| \leq (1+\varepsilon)\cdot\|x\|$$

holds for every $x \in X$. Analogously we can define ε-isometry for nonexpansive operators.

(2) Let $p \in (0,1]$. A *p-normed* (respectively, *p-Banach*) *space* is a quasinormed (respectively, quasi-Banach) space whose quasinorm is a p-norm, that is, it satisfies the inequality $\|x+y\|^p \leq \|x\|^p + \|y\|^p$.

(3) Aoki and Rolewicz showed that every quasinorm is equivalent to a p-norm for some $p \in (0,1]$ in the sense that they induce the same topology.

Definitions

(4) A separable Banach space $\mathbb{G}$ is called a *Gurariĭ space* if for an arbitrary $\varepsilon \in (0, 1)$, arbitrary finite-dimensional Banach spaces $X \subset Y$ and an arbitrary isometric embedding $f : X \to \mathbb{G}$, there is a linear extension $\tilde{f} : Y \to \mathbb{G}$ of f satisfying

$$(1 - \varepsilon) \cdot \|x\| \leq \|\tilde{f}(x)\| \leq (1 + \varepsilon) \cdot \|x\|$$

for all $x \in Y$.

(5) A monotone FDD in a Banach space E is determined by an increasing chain $\{E_n\}_{n \in \omega}$ of finite-dimensional subspaces of E such that each E_n is 1-complemented in E_{n+1} and $\bigcup_{n \in \omega} E_n$ is dense in E.

Definitions

(4) A separable Banach space $\mathbb{G}$ is called a *Gurariĭ space* if for an arbitrary $\varepsilon \in (0,1)$, arbitrary finite-dimensional Banach spaces $X \subset Y$ and an arbitrary isometric embedding $f : X \to \mathbb{G}$, there is a linear extension $\tilde{f} : Y \to \mathbb{G}$ of f satisfying

$$(1-\varepsilon)\cdot\|x\| \leq \|\tilde{f}(x)\| \leq (1+\varepsilon)\cdot\|x\|$$

for all $x \in Y$.

(5) A monotone FDD in a Banach space E is determined by an increasing chain $\{E_n\}_{n\in\omega}$ of finite-dimensional subspaces of E such that each E_n is 1-complemented in E_{n+1} and $\bigcup_{n\in\omega} E_n$ is dense in E.

Amalgamation

Amalgamation Lemma

Let $f : Z \to X$, $g : Z \to Y$ be nonexpansive linear operators between Banach spaces and let $W(f,g) = (X \oplus_1 Y)/\Delta(f,g)$, $\Delta(f,g)$ is the closure of the linear space $\{(f(z), -g(z)) \in X \oplus_1 Y : z \in Z\}$. Then the diagram

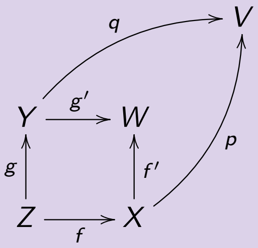
$$
\begin{array}{ccc}
Y & \xrightarrow{g'} & W \\
{\scriptstyle g}\uparrow & & \uparrow{\scriptstyle f'} \\
Z & \xrightarrow[f]{} & X
\end{array}
$$

is commutative and it is a pushout in the category of Banach spaces with nonexpansive operators.

Amalgamation

Amalgamation Lemma

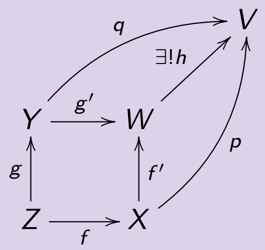
Let $f : Z \to X$, $g : Z \to Y$ be nonexpansive linear operators between Banach spaces and let $W(f,g) = (X \oplus_1 Y)/\Delta(f,g)$, $\Delta(f,g)$ is the closure of the linear space $\{(f(z), -g(z)) \in X \oplus_1 Y : z \in Z\}$.



Amalgamation

Amalgamation Lemma

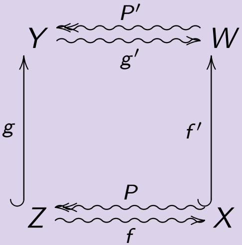
Let $f : Z \to X$, $g : Z \to Y$ be nonexpansive linear operators between Banach spaces and let $W(f,g) = (X \oplus_1 Y)/\Delta(f,g)$, $\Delta(f,g)$ is the closure of the linear space $\{(f(z), -g(z)) \in X \oplus_1 Y : z \in Z\}$.



Amalgamation

Amalgamation Lemma

Let $f : Z \to X$, $g : Z \to Y$ be nonexpansive linear operators between Banach spaces and let $W(f, g) = (X \oplus_1 Y)/\Delta(f, g)$, $\Delta(f, g)$ is the closure of the linear space $\{(f(z), -g(z)) \in X \oplus_1 Y : z \in Z\}$.



Lemma (Correcting ε-isometry), Kubiś, Solecki

Fix $\varepsilon \in (0,1)$, and let $f : X \to Y$ be a linear operator between Banach spaces, satisfying

$$(1-\varepsilon)\|x\|_X \leq \|f(x)\|_Y \leq (1+\varepsilon)\|x\|_X$$

for every $x \in X$. Let $i_0 : X \to X \oplus Y$, $j_0 : Y \to X \oplus Y$ be the canonical embeddings and consider the space $X \oplus_{(f,\varepsilon)} Y$, that is, $X \oplus Y$ endowed with the norm
$\|(x,y)\|_{f,\varepsilon} = \inf\{\|u\|_X + \|v\|_Y + \varepsilon\|w\|_X : x = u + w,\ y = v - f(w),\ u, w \in X, v \in Y\}$.
Then both i_0 and j_0 are isometric embeddings and

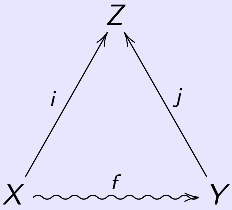
$$\|j_0 \circ f - i_0\| \leq \varepsilon.$$

We again fix $\varepsilon \in (0,1)$ and we fix a linear operator $f : X \to Y$ between Banach spaces such that

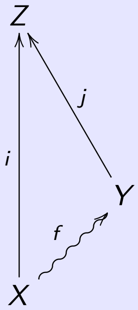
$$(1-\varepsilon)\cdot\|x\| \leq \|f(x)\| \leq \|x\|$$

for every $x \in X$. Consider the following category $\mathfrak{K}_f^\varepsilon$. The objects of $\mathfrak{K}_f^\varepsilon$ are pairs (i,j) of nonexpansive linear operators $i : X \to Z$, $j : Y \to Z$ between Banach spaces, satisfying

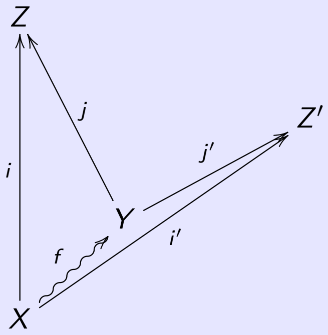
$$\|i - j\circ f\| \leq \varepsilon.$$



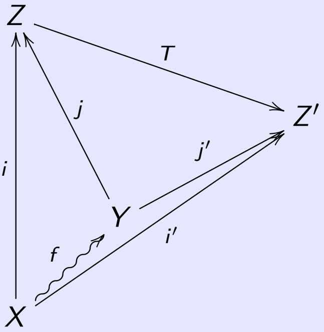
Given $A = (i, j)$ and $B = (i', j')$ in $\mathfrak{K}_f^{\varepsilon}$, where the common range of i, j is Z and the common range of i', j' is Z', an arrow from A to B is defined to be a nonexpansive linear operator $T : Z \to Z'$ satisfying $T \circ i = i'$ and $T \circ j = j'$.



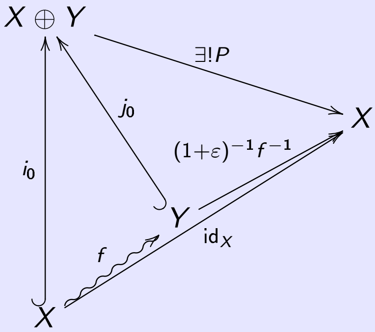
Given $A = (i, j)$ and $B = (i', j')$ in $\mathfrak{K}_f^\varepsilon$, where the common range of i, j is Z and the common range of i', j' is Z', an arrow from A to B is defined to be a nonexpansive linear operator $T : Z \to Z'$ satisfying $T \circ i = i'$ and $T \circ j = j'$.



Given $A = (i, j)$ and $B = (i', j')$ in $\mathfrak{K}_f^\varepsilon$, where the common range of i, j is Z and the common range of i', j' is Z', an arrow from A to B is defined to be a nonexpansive linear operator $T : Z \to Z'$ satisfying $T \circ i = i'$ and $T \circ j = j'$.



Suppose that f is a surjective operator, then there exist norm-one projections P, Q from $X \oplus_{(f,\varepsilon)} Y$ on X and Y, respectively.



In our construction we consider rational spaces, therefore to get Banach space (p-Banach space) we take completion of every obtained sequence. We use properties of a *Fraïssé sequence (generic)*. A sequence $\vec{U}$ in $\mathfrak{F}$ is a *Fraïssé sequence*, if it satisfies the following condition:

(A) Given $n \in \omega$ and an $\mathfrak{F}$-arrow $f : U_n \to Y$, then exist $m > n$ and an $\mathfrak{F}$-arrow $g : Y \to U_m$ such that $g \circ f$ is the bonding arrow from U_n to U_m.

The following theorem is crucial:

Theorem (Kubiś)

Every countable category has a Fraïssé sequence.

Theorem (Universality of FDD)

Let X be a Banach space with a monotone FDD. Then there exists an isometric embedding $e : X \to \mathbb{P}$ such that $e[X]$ is 1-complemented in $\mathbb{P}$.

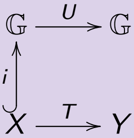
Theorem (Universality of the operator)

Given a nonexpansive linear operator $T : X \to Y$ between separable Banach spaces, there exist isometric embeddings $i : X \to \mathbb{G}$, $j : Y \to \mathbb{G}$ such that $\mathbf{U} \circ i = j \circ T$.

$$\begin{array}{ccc} \mathbb{G} & \xrightarrow{U} & \mathbb{G} \\ & & \\ X & \xrightarrow{T} & Y \end{array}$$

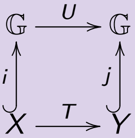
Theorem (Universality of the operator)

Given a nonexpansive linear operator $T : X \to Y$ between separable Banach spaces, there exist isometric embeddings $i : X \to \mathbb{G}$, $j : Y \to \mathbb{G}$ such that $\mathbf{U} \circ i = j \circ T$.



Theorem (Universality of the operator)

Given a nonexpansive linear operator $T : X \to Y$ between separable Banach spaces, there exist isometric embeddings $i : X \to \mathbb{G}$, $j : Y \to \mathbb{G}$ such that $\mathbf{U} \circ i = j \circ T$.



Theorem (Universality)

Every separable p-Banach space can be isometrically embedded into p-Gurariĭ space.

F. Cabello-Sanchez, J. Garbulińska-Węgrzyn, W. Kubiś, *Quasi-Banach spaceof almost universal disposition*, Journal of Functional Analysis 267 (2014), 744-771

J. Garbulińska-Węgrzyn, *Isometric uniqueness of a complementably universal Banach space for Schauder decompositions*, Banach J. Math. Anal. 8 (2014), no. 1, 211–220

J. Garbulińska-Węgrzyn, W. Kubiś, *A universal operator on the Gurariĭ space*, to appear in Journal of Operator Theory.

J. Garbulińska, W. Kubiś, *Remarks on Gurariĭ spaces*, Extracta Math. 26 (2011) 235-269