Some unrelated results in non separable Banach space theory

Bill Johnson

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Relations Between Banach Space Theory and Geometric Measure Theory

Subspaces of L_p that embed into $L_p(\mu)$ with μ finite

Bill Johnson & Gideon Schechtman

[ER 73] Enflo, Per; Rosenthal, Haskell P., Some results concerning $L_p(\mu)$-spaces. JFA 14 (1973), 325–348.

If $1 < p \neq 2 < \infty$, μ is finite, and $L_p(\mu)$ is non separable, can $L_p(\mu)$ have an unconditional basis?

If the density character of $L_p(\mu)$ is at least $\aleph_\omega$ the answer is no; in fact, $L_p(\mu)$ does not embed isomorphically into any Banach space that has an unconditional basis [ER 73].

So it is consistent that $L_p\{-1,1\}^{2^{\aleph_0}}$ does not embed into a space with unconditional basis.
Is it consistent that $L_p\{-1,1\}^{2^{\aleph_0}}$ has an unconditional basis?
Does $L_p\{-1,1\}^{\aleph_1}$ have an unconditional basis?

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Another topic considered in [ER 73] was (isomorphic) embeddings of $\ell_p(\aleph_1)$ into $L_p(\mu)$ with μ finite. When $2 < p < \infty$ there is no embedding because the formal identity $I_{p,2}$ from $L_p(\mu)$ into $L_2(\mu)$ is a one to one bounded linear operator and every bounded linear operator from $\ell_p(\aleph)$ into a Hilbert space is a compact linear operator and hence cannot be one to one if $\aleph$ is uncountable.

[ER 73] For $1 < p < 2$ there is no isomorphic embedding of $\ell_p(\aleph_1)$ into $L_p(\mu)$ with μ finite. For $p = 1$ essentially everything is known and due to Rosenthal [Ros 70]. So if X is any subspace of $L_p(\mu)$ with μ finite and, as usual, $1 < p \neq 2 < \infty$, $\ell_p(\aleph_1)$ does not embed into X.

What can you say about a subspace X of some completely general L_p space which has the property that $\ell_p(\aleph_1)$ does not embed into X?

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Conjecture: X must embed into $L_p(\mu)$ with μ finite.

[JS 13] The answer is yes for $1 < p < 2$.

The “easier" case, when $2 < p < \infty$, remains open. However, [JS 13] does contain a characterization of subspaces X of L_p that do not contain a subspace isomorphic to $\ell_p(\aleph_1)$. This characterization is relatively easy to prove.

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Proposition

Let X be a subspace of some L_p space, $2 < p < \infty$. The following are equivalent:

1. $\ell_p(\aleph_1)$ *isometrically embeds into* X.
2. *There is a subspace of* X *that is isomorphic to* $\ell_p(\aleph_1)$ *and is complemented in* L_p.
3. $\ell_p(\aleph_1)$ *isomorphically embeds into* X.
4. *There is no one to one (bounded, linear) operator from* X *into a Hilbert space.*

(1) $\Rightarrow$ (2) EVERY isometric copy of an L_p space in an L_p space is norm one complemented.

(2) $\Rightarrow$ (3) is obvious; (3) $\Rightarrow$ (4) was already mentioned. This leaves only (4) $\Rightarrow$ (1), but even (3) $\Rightarrow$ (1) or (2) $\Rightarrow$ (1) requires some thought.

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Let X be a subspace of some L_p space, $2 < p < \infty$. The following are equivalent:

(1) *$\ell_p(\aleph_1)$ isometrically embeds into X.*

(4) *There is no one to one (bounded, linear) operator from X into a Hilbert space.*

[Maharam 42] gives that X is a subspace of $L_p := (\sum_{\gamma\in\Gamma} L_p\{-1,1\}^{\aleph_\gamma})_p$ for some set Γ of ordinal numbers, where $\{-1,1\}$ is endowed with the uniform probability measure.
(4) $\Rightarrow$ (1): For countable $\Gamma' \subset \Gamma$, the projection $P_{\Gamma'} : L_p \to (\sum_{\gamma\in\Gamma'} L_p\{-1,1\}^{\aleph_\gamma})_p$ is not one to one on X, because $(\sum_{\gamma\in\Gamma'} L_p\{-1,1\}^{\aleph_\gamma})_p$ maps one to one into the Hilbert space $(\sum_{\gamma\in\Gamma'} L_2\{-1,1\}^{\aleph_\gamma})_2$ in an obvious way when Γ' is countable.

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On the other hand, given any x in X, there is a countable subset $x(\Gamma)$ of Γ so that $P_\gamma x = 0$ for all γ not in $x(\Gamma)$. Thus if one takes a collection of unit vectors x in X maximal with respect to the property that $x(\Gamma) \cap y(\Gamma) = \emptyset$ when $x \neq y$, then the collection must have cardinality at least $\aleph_1$ and hence $\ell_p(\aleph_1)$ embeds isometrically into X.

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Let X be a subspace of some L_p space, $1 < p < 2$. Then X embeds into $L_p(\mu)$ for some finite measure μ if and only if $\ell_p(\aleph_1)$ does not embed (isomorphically) into X.

Assume $X \subset L_p := (\sum_{\gamma \in \Gamma} L_p\{-1,1\}^{\aleph_\gamma})_p$ but $X \not\hookrightarrow L_p(\mu)$ with μ a finite measure.

Since disjoint unit vectors $(x_\alpha)_{\alpha \in A}$ act just like the unit vector basis of $\ell_p(A)$, one would like to find such with $|A| = \aleph_1$. Examples show that this cannot be done. However, if we just wanted to find a copy of ℓ_p in X, it would be enough to get unit vectors $(x_n)_{n=1}^\infty$ which are "almost disjoint"– a perturbation argument would then get an isomorphic copy of ℓ_p in X. For $p = 1$, the unit vector basis for $\ell_1(A)$ is very stable under perturbations; this is what Rosenthal used in proving the theorem for $p = 1$. When $p > 1$, something more is needed.

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Idea: Build a long unconditionally basic sequence $(x_\alpha)_{\alpha < \aleph_1}$ of unit vectors in X that have "big disjoint pieces". The type p property of L_p and unconditionality give $\|\sum_\alpha t_\alpha x_a\| \le C(\sum_\alpha |t_\alpha|^p)^{1/p}$ and the "diagonal principle" gives the corresponding lower estimate.

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$X \not\hookrightarrow L_p(\mu)$ with μ a finite measure.
Call a set S of vectors in $L_p = (\sum_{\gamma\in\Gamma} L_p\{-1,1\}^{\aleph_\gamma})_p$ a *generalized martingale difference set* (GMD set, in short) provided that for every finite subset F of S and every γ in Γ, the sequence $(P_\gamma x)_{x\in F}$ can be ordered to be a martingale difference sequence. We allow 0 to appear in a martingale difference sequence, but the definition requires that $P_\gamma x \neq P_\gamma y$ if $P_\gamma x \neq 0$. Since a martingale difference sequence is unconditional in $L_p(\mu)$ for any probability μ, any $1 < p < \infty$, and with the unconditional constant depending only on p [Burkholder 73], a GMD set in L_p is unconditionally basic for our range of p.

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GMD set: For every finite subset F of S and every γ in Γ, the sequence $(P_\gamma x)_{x \in F}$ can be ordered to be a martingale difference sequence.

Take a set V of pairs $(x, \gamma(x))_{x \in M}$ in $X \times \Gamma$ maximal with respect to the properties that $\|x\| = 1$, $P_{\gamma(x)} x \neq 0$, the $\gamma(x)$ are all distinct, and M is a GMD set.

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Higher cardinals

A Banach space X is an $L_p(\aleph)$ space, where $\aleph$ is an infinite cardinal, provided X is isometric to $(\sum_{\alpha\in\Gamma} L_p(\mu_\alpha))_p$ with $|\Gamma| \le \aleph$ and each μ_α a finite measure.

Proposition

Let X be a subspace of some L_p space, $2 < p < \infty$, and let $\aleph$ be an uncountable cardinal. The following are equivalent:

(1) *$\ell_p(\aleph)$ isometrically embeds into X.*

(2) *There is a subspace of X that is isomorphic to $\ell_p(\aleph)$ and is complemented in L_p.*

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X is an $L_p(\aleph)$ space if X is isometric to $(\sum_{\alpha\in\Gamma} L_p(\mu_\alpha))_p$ with $|\Gamma| \le \aleph$ and μ_a finite.

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Let X be a subspace of some L_p space, $1 < p < 2$, and let $\aleph$ be an uncountable cardinal. The following are equivalent.

(1) *For all $\epsilon > 0$, $\ell_p(\aleph)$ is $1+\epsilon$-isomorphic to a subspace of X.*

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Lemma.

Let $1 < p < 2$ and let $\aleph$ be an uncountable cardinal. If $\aleph' < \aleph$, then $\ell_p(\aleph)$ is not isomorphic to a subspace of any $L_p(\aleph')$ space.

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Open Problems

1. Does $L_p\{-1,1\}^{\aleph_1}$, $1 < p \neq 2 < \infty$, have an unconditional basis or at least embed into a space that has an unconditional basis?

2. If X is a subspace of some L_p space, $2 < p < \infty$, and $\ell_p(\aleph_1)$ does not embed into X, must X embed into $L_p(\mu)$ for some finite measure μ?

3. Can $L_p\{-1,1\}^{\aleph_1}$, $2 < p < \infty$, be written as an unconditional sum of subspaces each of which is isomorphic to a Hilbert space?

If $L_p\{-1,1\}^{\aleph_1}$ has an unconditional basis, then (3) has an affirmative answer by an old result of Kadec and Pełczyński. But we do not know how to prove even that $L_p\{-1,1\}^{\aleph_1}$ cannot be written as an unconditional sum of subspaces that are *uniformly* isomorphic to Hilbert spaces.

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What is the connection between Tauberian operators on L_1 and dense range, non surjective operators on ℓ_∞?

If T is 1-1 Tauberian, T^{**} is 1-1.

Thus, if T is a Tauberian operator on L_1 that is 1-1 but does not have closed range, then T^* is a dense range operator on L_∞ that is not surjective.

Since L_∞ is isomorphic to ℓ_∞ [Pełcziński, '58], this would answer Nasseri's question.

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If T satisfies this condition, then you can play around and get a perturbation S of T that is Tauberian, 1-1, and has dense, non closed range (so is not surjective). Taking the adjoint of S and replacing L_∞ by its isomorphic ℓ_∞, you would have a 1-1, dense range, non surjective operator on ℓ_∞. To get S from T, take a 1-1 nuclear operator from the kernel on T that has dense range in L_1, extend it to a nuclear operator on L_1, and add it to T. This does not quite work, but some fiddling produces the desired S.

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Say an operator $T : X \to Y$ (X an L_1 space) is (r, N)-Tauberian provided whenever $(x_n)_{n=1}^N$ are disjoint unit vectors in X, then $\max_{1 \le n \le N} \|Tx_n\| \ge r$.

Lemma $T : X \to Y$ is Tauberian iff $\exists\ r > 0$ and N s.t. T is (r, N)-Tauberian.

Proof: T being (r, N)-Tauberian implies that if (x_n) is a disjoint sequence of unit vectors in X, then $\liminf_n \|Tx_n\| > 0$, so T is Tauberian [G,M-A (1)]. Conversely, suppose there are disjoint collections $(x_k^n)_{k=1}^n$, $n = 1, 2, \ldots$ with $\max_{1 \le k \le n} \|Tx_k^n\| \to 0$ as $n \to \infty$. Then The closed sublattice generated by $\cup_{n=1}^\infty (x_k^n)_{k=1}^n$ is a separable L_1 space, hence is order isometric to $L_1(\mu)$ for some probability measure μ by Kakutani's theorem. Choose $1 \le k(n) \le n$ s.t. the support of $x_{k(n)}^n$ in $L_1(\mu)$ has measure at most $1/n$. Since T is Tauberian, necessarily $\liminf_n \|Tx_{k(n)}^n\| > 0$ [G, A-M] , a contradiction.

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Conclusion from the proof: Computer science is connected to non separable Banach space theory!

T. Kania, WBJ & G. Schechtman

$\ell_\infty^c(\lambda)$ is the set of bounded functions on λ that have countable support.

General problem: Classify the complemented subspaces of $C(K)$ for K compact.

For metrizable K, this has been done only for $c_0 \approx C(\mathbb{N} \cup \{\infty\})$ and $C(\omega^\omega)$.

For non separable $C(K)$, this has been done for only a few spaces: $\ell_\infty = C(\beta\mathbb{N})$; $c_0(\lambda)$ for any uncountable set λ; direct sums of some of the above examples.

New examples: $\ell_\infty^c(\lambda)$ for any set λ and direct sums of $\ell_\infty^c(\lambda)$ with some of the above.

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Why $\ell_\infty^c(\lambda)$? For example, $\ell_\infty(\lambda)$ is more interesting.

Classifying complemented subspaces of $\ell_\infty(\lambda)$ for all λ is the same as classifying the injective Banach space. This is a great problem and nothing has been done on it for more than 40 years. We have no new information on this problem.

Spaces of the form $\ell_\infty^c(\lambda)$ are not injective when λ is uncountable, but they are separably injective (X is separably injective provided every operator from a subspace of a separable space into X extends to the whole space); in fact, they form the simplest class of separably injective spaces that have no separable, infinite dimensional complemented separable subspaces.

$\ell_\infty^c(\lambda)$ for $\lambda \geq \aleph_\omega$ provide the first examples of $C(K)$ for which the complemented subspaces are classified and there are infinitely many isomorphic types of complemented subspaces.

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Theorem.

Let λ be an infinite cardinal number. Then every infinite dimensional, complemented subspace of $\ell_\infty^c(\lambda)$ is isomorphic either to ℓ_∞ or to $\ell_\infty^c(\kappa)$ for some cardinal $\kappa \leqslant \lambda$. In particular, $\ell_\infty^c(\lambda)$ is a primary Banach space.

There are three main steps in the proof of the classification theorem, the first being:

Proposition.

Let λ be a cardinal number and let $T\colon \ell_\infty^c(\lambda) \to \ell_\infty^c(\lambda)$ be an operator that is not an isomorphism on any sublattice isometric to $c_0(\lambda)$. Then for every $\varepsilon > 0$ there is subset Λ of λ so that $|\Lambda| < \lambda$ and

$$\|TR_{\lambda\setminus\Lambda}\| \leqslant \varepsilon.$$

Consequently, if also T is a projection onto a subspace X, then X is isomorphic to a complemented subspace of $\ell_\infty^c(\kappa)$ for some $\kappa < \lambda$.

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Proposition.

$T\colon \ell^c_\infty(\lambda) \to \ell^c_\infty(\lambda)$ not an isomorphism on any sublattice isometric to $c_0(\lambda)$. Then $\forall \varepsilon > 0 \;\; \exists \Lambda \subset \lambda$ s.t. $|\Lambda| < \lambda$ and $\|TR_{\lambda \setminus \Lambda}\| \leqslant \varepsilon$. Consequently, if T is a projection onto a subspace X, then X is isomorphic to a complemented subspace of $\ell^c_\infty(\kappa)$ for some cardinal number $\kappa < \lambda$.

For the "consequently" statement, suppose that T is a projection onto a subspace X. Then

$$I_X = (TR_{\lambda \setminus \Lambda} + TR_\Lambda)|_X,$$

so

$$\|(I_X - TR_\Lambda)|_X\| \leqslant \varepsilon$$

hence if $\varepsilon < 1$, there is an operator U on X so that

$$I_X = (UTR_\Lambda)|_X.$$

Thus I_X factors through $\ell^c_\infty(|\Lambda|)$.

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The preceding proposition and the next two propositions for the case $\Lambda = \lambda$, prove, via transfinite induction, the classification theorem.

Proposition.

If X is complemented in $\ell_\infty^c(\lambda)$ and $c_0(\Lambda)$ embeds into X, then $\ell_\infty^c(\Lambda)$ embeds into X.

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If X is a subspace of $\ell_\infty^c(\lambda)$ that is isomorphic to $\ell_\infty^c(\Lambda)$, then there is a subspace Y of X s.t. Y is isomorphic to $\ell_\infty^c(\Lambda)$ and Y is complemented in $\ell_\infty^c(\lambda)$.

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Thanks for your attention!