Differentiability on L^p of a vector measure

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Relations between Banach Space Theory and Geometric Measure Theory
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Definition

Let $(X, \|\cdot\|)$ be real a Banach space. The space X, or its norm $\|\cdot\|$, is said to be:

- Gâteaux smooth (G) if for each $x \in S_X$ there is $f_x \in X^*$ (the derivative of $\|\cdot\|$ at x) such that

$$\lim_{t \to 0} \frac{\|x + th\| - 1}{t} = f_x(h) \quad \text{for all} \quad h \in X.$$

Properties of smoothness

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- Fréchet smooth (F) if $\|\cdot\|$ is Gâteaux smooth and, for all $x \in S_X$,

$$\lim_{t \to 0} \sup \left\{ \left| \frac{\|x + th\| - 1}{t} - f_x(h) \right| : \ h \in B_X \right\} = 0.$$

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- UF $\Rightarrow$ F $\Rightarrow$ G & UF $\Rightarrow$ UG $\Rightarrow$ G
- If μ is a probability measure and $p > 1$, then $L^p(\mu)$ is UF.

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A Banach space Y is Hilbert-generated if there exist a Hilbert H space and an operator $T : H \to Y$ such that $\overline{T(H)} = Y$.

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Proposition

If μ is a probability measure, then $L^2(\mu)$ generates $L^1(\mu)$. Thus, the space $L^1(\mu)$ is UG smooth renormable. However, in general $L^1(\mu)$ admits no equivalent F smooth norm.

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(i) for each $x^* \in X^*$, f is integrable with respect to the scalar measure $\langle m, x^* \rangle := x^* \circ m$, and

(ii) for each $A \in \Sigma$ there exists a vector $\int_A fdm \in X$ such that

$$\int_A fd\langle m, x^* \rangle = \left\langle \int_A fdm, x^* \right\rangle, \quad \text{for all} \quad x^* \in X^*.$$

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Remark

If X is a Banach lattice and m is positive, then for every $f \in L^p(m)$ we have

$$\|f\|_{L^p(m)} = \left\| \int_\Omega |f|^p dm \right\|^{1/p}.$$

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Remark

The natural norm of $L^p(m)$ is not necessarily Gâteaux smooth.

Let $\Omega = \{1, 2\}$, $\Sigma = \mathcal{P}(\Omega)$ and $m : \Sigma \to \ell_2^\infty$ be the (positive) vector measure defined by the formulae

$$m(\{1\}) = (1, 0) \text{ and } m(\{2\}) = (0, 1).$$

If $1 \le p < \infty$, then $\left(L^p(m), \|\cdot\|_{L^p(m)}\right)$ is isomoetric to ℓ_2^∞.

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Gâteaux and Fréchet smoothness of $L^p(m)$

Problem

Find conditions to ensure that $\|\cdot\|_{L^p(m)}$ is smooth ($p > 1$).

Theorem (Agud, Calabuig, Lajara, Sánchez, 2015)

If $p > 1$ and the norm of X is Gâteaux (Fréchet) smooth, then so is the norm $\|\cdot\|_{L^p(m)}$ on $L^p(m)$.

Problem

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Theorem (Agud, Calabuig, Lajara, Sánchez, 2015)

If $p > 1$ and the norm of X is Gâteaux (Fréchet) smooth, then so is the norm $\|\cdot\|_{L^p(m)}$ on $L^p(m)$. Moreover, for each $f \in S_{L^p(m)}$ and each $h \in L^p(m)$ we have

$$\|\cdot\|'_{L^p(m)}(f)(h) = \int_\Omega \mathrm{sign}(f)|f|^{p-1} h \, d\langle m, x_f^* \rangle,$$

where x_f^* stands for the (unique) norm-one functional in X^* such that

$$\left\langle x_f^*, \int_\Omega |f|^p dm \right\rangle = 1.$$

Sketch of the proof

Proposition

If $p > 1$ then the mapping $\varphi : L^p(m) \to X$ defined by the formula

$$\varphi(f) = \int_\Omega |f|^p \, dm$$

satisfies the following properties:

(i) φ is Gâteaux differentiable on $L^p(m)$ and for all $f, h \in L^p(m)$:

$$\varphi'(f)(h) = p \int_\Omega \mathrm{sign}(f)|f|^{p-1} h dm.$$

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(ii) For every $R > 0$ there exists a constant $C_{p,R} > 0$ such that

$$\|\varphi'(f)(h) - \varphi'(g)(h)\| \le C_{p,R} \|f - g\|_{L^p(m)}^{p-1} \|h\|_{L^p(m)}$$

whenever $f, g \in RB_{L^p(m)}$ and $h \in L^p(m)$.

Uniform smoothness of $L^p(m)$

Theorem (Agud, Calabuig, Lajara, Sánchez, 2015)

If $p > 1$ and X is uniformly Gâteaux (uniformly Fréchet) smooth, then so is the norm $\|\cdot\|_{L^p(m)}$ on $L^p(m)$.

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If $p > 1$ and X is uniformly Gâteaux (uniformly Fréchet) smooth, then so is the norm $\| \cdot \|_{L^p(m)}$ on $L^p(m)$.

Corollary

If $p > 1$ and X is super-reflexive, then so is the space $L^p(m)$.

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If $p > 1$ and X is simultaneously Fréchet and uniformly Gâteaux smooth, then so is the space $L^p(m)$.

Sketch of the proof

Lemma

Assume that X is smooth, and let $1 < p < \infty$. Let us write, for each $f, h \in L^p(m)$ with $f \neq 0$ and each $t \neq 0$,

$$\Delta(f,h,t) = \frac{\|f+th\|_{L^p(m)} - \|f\|_{L^p(m)}}{t} - \|\cdot\|'_{L^p(m)}(f)(h).$$

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Then, there exist constants $D_p, D'_p > 0$ such that

$$|\Delta(f,h,t)| \leq D_p \max\left\{|t|^{p-1}, |t|\right\} + D'_p \left| \frac{\|\varphi(f) + t\varphi'(f)(h)\| - 1}{t} - \langle \|\cdot\|'(\varphi(f)), \varphi'(f)(h)\rangle \right|$$

whenever $f, h \in S_{L^p(m)}$ and $0 < |t| < 1/2$.