Riemann integrability versus weak continuity

Gonzalo Martínez Cervantes

University of Murcia

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Definition

A function $f : [0,1] \to X$ *is said to be Riemann integrable with integral* $x \in X$ *if for every* $\varepsilon > 0$ *there is* $\delta > 0$ *such that*

$$\left\| \sum_{i=1}^{N} f(s_i)(t_i - t_{i-1}) - x \right\| < \varepsilon$$

whenever $0 = t_0 < t_1 < \ldots < t_N = 1$, $s_i \in (t_{i-1}, t_i)$ *and* $|t_i - t_{i-1}| < \delta$ *for every* $i = 1, \ldots, N$.

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$$f : [0,1] \to \ell_\infty([0,1])$$

$$f(r)(t) = \begin{cases} 0 & \text{for } 0 \leq t < r, \\ 1 & \text{for } r \leq t \leq 1, \end{cases} \qquad 0 \leq r \leq 1.$$

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(P2) Characterize those Banach spaces X such that every Riemann integrable function $f : [0, 1] \to X$ is continuous almost everywhere.

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- (R. Haydon, 1984) If a stable Banach space with uniformly separable types has the Schur property, then it has the LP. There is a separable stable space with the Schur property but not the LP.

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3. (Wang and Wan, 2001)$L^1([0,1])$ has the WLP.

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The LP is separably determined.

Theorem (M. Pizzoti, 1989)

If X is a Banach space without the LP and D is a countable dense subset of $[0,1]$, then there exists a Riemann integrable function $f : [0,1] \to X$ such that $f(t) = 0$ if $t \notin D$ and $\|f(t)\| = 1$ if $t \in D$.

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Main purpose

- **Characterize the spaces of the form $c_0(\Gamma)$, $\ell_p(\Gamma)$, $L^1(\mu)$, $\mathcal{C}(K)^*$ with the WLP.**

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Main purpose

- **Characterize the spaces of the form $c_0(\Gamma)$, $\ell_p(\Gamma)$, $L^1(\mu)$, $\mathcal{C}(K)^*$ with the WLP.**
- **What is the minimum cardinality of a set Γ such that $c_0(\Gamma)$ does not have the WLP?**

Let $\mathcal{F}$ be a family of null sets of $[0, 1]$. Does there exist a Lebesgue null set $N \subset \mathbb{R}$ such that for every $E \in \mathcal{F}$ there exists $x \in \mathbb{R}$ with $x + E \subset N$?

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  Does there exist N with the previous property in this case?

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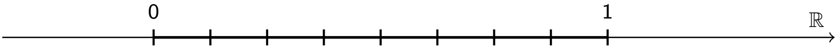

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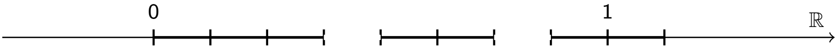

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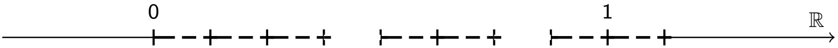

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Definition

X is asymptotic ℓ^1 with respect to its normalized basis $\{e_i\}$ if there is $C \geq 1$ such that for each $n \in \mathbb{N}$ there is a function $F_n : \mathbb{N} \cup \{0\} \to \mathbb{N}$ with $F_n(k) \geq k$ for all k so that

$$C^{-1} \sum_{i=1}^{n} |a_i| \leq \left\| \sum_{i=1}^{n} a_i x_i \right\|$$

for all normalized successive blocks $\{x_i\}_{i=1}^n$ with respect to $\{e_i\}$ that satisfy $F_n(0) \leq \operatorname{supp} x_1$ and $F_n(\max \operatorname{supp} x_i) < \min \operatorname{supp} x_{i+1}$, $i = 1, 2, ..., n-1$, and for all $\{a_i\}_{i=1}^n \subset \mathbb{R}$.