Measuring sets with translation invariant Borel measures

András Máthé

University of Warwick

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David Preiss

David Preiss

Olga Maleva

David Preiss

Olga Maleva

Martin Rmoutil

David Preiss

Olga Maleva

Martin Rmoutil

Daniel Seco


David Preiss


Olga Maleva


Martin Rmoutil


Daniel Seco


Thomas Zürcher

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Nice examples of translation invariant measures:

- Hausdorff measures $\mathcal{H}^s$,
- generalised Hausdorff $\mathcal{H}^g$ and packing measures $\mathcal{P}^g$ with gauge function g.

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Lemma: If μ is translation invariant, $\mu(E) > 0$ and F contains uncountably many disjoint translates of E, then μ is not σ-finite on F.

Other examples

Theorem (Elekes–Keleti 2006)

- The set of Liouville numbers
  $L = \{x \in \mathbb{R} \setminus \mathbb{Q} \ : \text{ for every } n \text{ there are } p, q \text{ with } |x - p/q| < 1/q^n\}$
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Remark

σ-finite Borel measures on a Borel set $A \subset \mathbb{R}$ are inner regular.

Theorem/Observation (Elekes–Keleti)

Let ν be a measure on A. Then ν has a translation invariant extension to $\mathbb{R}$ if and only if

$$\nu(A' + t) = \nu(A') \quad \text{whenever} \quad A' \subset A \text{ and } A' + t \subset A$$

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Proof.

$$\mu(B) = \inf\left\{ \sum_{i=1}^{\infty} \nu(A_i) \;:\; A_i \subset A,\;\; B \subset \cup_i (A_i + t_i) \right\}.$$

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If A is such that $A\cap(A+t)$ is at most 1 point for all $t\in\mathbb{R}$, then any non-atomic measure on A is extendable.

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Corollary

If A is an uncountable Borel set such that $A\cap(A+t)$ is at most 1 point for all $t\in\mathbb{R}$, then A is measured.

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Aim: find a non-empty compact set in $\mathbb{R}$ which is not a union of countably many measured sets.

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Take $K = K_1 + K_2 + K_3 + \ldots$, and consider the infinite convolution of the normalised measures

$$\frac{\mu_i|_{K_i}}{\mu_i(K_i)}.$$

...

Haar null (shy) sets

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f is a (restriction of a) bounded linear functional.

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- By local injectivity, we can "pull back measures on $f(G)$ to measures on G". Therefore $A = f(G)$ is not a union of countably many measured sets.
- If (e_n) is boundedly complete, then $A = f(G)$ is σ-compact.

Union of measured sets in $\mathbb{R}$

The Hausdorff measure with gauge function g is

$$\mathcal{H}^g(A) = \lim_{\delta \to 0+} \inf \left\{ \sum_{i=1}^{\infty} g(\operatorname{diam} U_i) \ : A \subset \cup_{i=1}^{\infty} U_i \text{ and } \operatorname{diam} U_i < \delta \right\}$$

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The theorem holds even if $A = B$; or when B has Hausdorff dimension zero and A has Hausdorff dimension 1.

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$A_1 \approx \cup_{j \in S} I_j \qquad A_2 \approx \cup_{j \notin S} I_j$
(use limsup and liminf sets)

Technical part of the proof

Proposition (M)

Let $B \subset \mathbb{R}$ be a Borel set of the second Baire category and let $A \subset \mathbb{R}$ have Lebesgue measure zero. Then there are gauge functions g_1, g_2 such that $\mathcal{H}^{g_1}(B) > 0$, $\mathcal{H}^{g_2}(B) > 0$, and $\mathcal{H}^{\min(g_1,g_2)}(A) = 0$.

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Theorem (Balka–M)

For a typical compact set $K \subset \mathbb{R}$ there is a gauge function g with $\mathcal{H}^g(K) = 1$.

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So A is measured but not by Hausdorff measures.