Ramsey properties of finite dimensional normed spaces

Brice Rodrigue Mbombo

Universidade de São Paulo
Joint work with
D. Bartošová and J. López-Abad
This work is support by a FAPESP grant Processo: 2012/20084-1

June 8, 2015

Definitions and Notations

- $\mathbb{U}$ is the Urysohn space (1925): the unique (up to isometry) Polish space that is both universal and ultrahomogeneous.
- $\mathbb{G}$ is The Gurarij space (1965): the unique (up to isometry (Lusky, 1976, Kubis and Solecki, 2011)) separable Banach space with the following property: Given finite-dimensional normed spaces $E \subseteq F$, given $\varepsilon > 0$, and given an isometric linear embedding $\gamma : E \longrightarrow \mathbb{G}$ there exists an injective linear operator $\psi : F \longrightarrow \mathbb{G}$ extending γ and satisfying that:

$$(1-\varepsilon)\|x\| \leq \|\psi(x)\| \leq (1+\varepsilon)\|x\|$$

- The Gurarij space $\mathbb{G}$ is in some way the analogue of the Urysohn space $\mathbb{U}$ in the category of Banach spaces

Definitions and Notations

- $\mathbb{P}$ is the *Poulsen simplex* (1961): the unique (up to affine homeomorphism) metrizable simplex whose extreme points $\partial_e \mathbb{P}$ are dense on it.
- A compact convex subset K of some locally convex space is called a (Choquet) *simplex* when every point $x \in K$ is the barycenter of a unique probability measure μ_x such that $\mu_x(\partial_e(K)) = 1$

Motivation 1

Groups	Universal for Polish groups	Extremely amenable?
$\mathrm{Homeo}([0,1]^{\aleph_0})$	Uspenkij, 1986	No
$\mathrm{Iso}(\mathbb{U})$	Uspenkij, 1990	Yes: Pestov, 2002
$\mathrm{Iso}_L(\mathbb{G})$	Ben Yaacov, 2012	motivation 1

Extreme amenability

Definition

A topological group G is extremely amenable if every continuous action of G on a compact set K has a fixed point. i.e there is $\xi \in K$ such that $g.\xi = \xi$ for every $g \in G$.

Remind

A topological group G is amenable iff every affine continuous action of G on a compact convex subset C of a locally convex space has a fixed point.

Motivation 2

Veech, 1977

No locally compact group is extremely amenable

Examples

1. The group $\mathcal{U}(\ell^2)$ is extremely amenable (Gromov and Milman, 1983). Using the concentration of measure phenomenon.
2. The group $Iso(\mathbb{U})$ is extremely amenable (Pestov, 2002). Using the concentration of measure phenomenon.
   Note: Extreme amenability of $Iso(\mathbb{U})$ can also be obtain via Ramsey result.

Structurale Ramsey Property

Given a family $\mathcal{K}$ of structures of the same sort, we say that $\mathcal{K}$ has the Ramsey Property when for every $A, B \in \mathcal{K}$ and $r \in \mathbb{N}$ there exists $C \in \mathcal{K}$ such that for every coloring

$$c : \binom{C}{A} := \{A' \subseteq C : A' \cong A\} \longrightarrow [r]$$

there is

$$B' \in \binom{C}{B}$$

such that

$$c \upharpoonright \binom{B'}{A}$$

is constant.

Examples of Ramsey class

1. The class of all finite ordered Graphs has the Ramsey property (Nešetřil and Rodl).
2. The class of finite-dimensional vector spaces over a finite field has the Ramsey property (Graham, Leeb and Rothschild)
3. The class of all finite ordered metric spaces has the Ramsey property (Nešetřil).

KPT

Kechris, Pestov and Todorcevic, 2005

Let $\mathbb{F}$ be a Fraïssé structure. The automorphism group of $\mathbb{F}$ is extremely amenable if and only if $Age(\mathbb{F}) :=$ Finitely generated substructures of $\mathbb{F}$ has the Ramsey property.

Note: This groups correspond to closed subgroups of S_∞

Examples from KPT

1. $\mathrm{Aut}(\mathbb{Q}, <)$ is EA (Pestov, 1998) (classical Ramsey property).
2. $\mathrm{Iso}(\mathbb{U})$ is EA (Ramsey property of finite ordered metric spaces).

Approximate Ramsey Properties

1. The characterization of Kechris, Pestov and Todorcevic cannot be applied to general Polish groups.
2. When dealing with Fraïssé metric structures (e.g. The Gurarij space), the right notion is the approximate Ramsey property for embeddings.This work was initiated recently by, among others, Ben Yaacov, Melleray and Tsankov.
3. In the case of the Gurarij space, we need to prove the Approximate Ramsey property for embedding between finite dimensional Banach space

Definition

1. Given two Banach spaces X and Y, by an *embedding* from X into Y we mean a linear operator $T : X \longrightarrow Y$ such that $\|T(x)\|_Y = \|x\|_X$ for all $x \in X$.
2. Let $\mathrm{Emb}(X, Y)$ be the collection of all embeddings from X into Y.
3. $\mathrm{Emb}(X, Y)$ is a metric space with the norm distance $d(T, U) := \|T - U\| := \sup_{x \in S_X} \|T(x) - U(x)\|$.

Bartošová , Lopez-Abad and M.

Given integers d, m and r, and given $\varepsilon > 0$, there exists $n \geqslant m$ such that for every coloring $c : \mathrm{Emb}(\ell_\infty^d, \ell_\infty^n) \longrightarrow [r]$ there are $T \in \mathrm{Emb}(\ell_\infty^m, \ell_\infty^n)$ and $\widetilde{r} < r$ such that

$$T \circ \mathrm{Emb}(\ell_\infty^d, \ell_\infty^m) \subseteq (c^{-1}\{\widetilde{r}\})_\varepsilon.$$

We denote $\mathbf{n}_\infty(d, m, r, \varepsilon)$ the minimal such n.

Given $m \leq n$ let

$$\mathbb{E}_{m,n} = \{A \in \mathcal{M}_{n,m} : A \text{ is the matrix of } \alpha \in Emb(\ell_\infty^m, \ell_\infty^n)\}$$

.

Reformulation

$n = \mathbf{n}_\infty(d, m, r, \varepsilon)$ is the minimal n (if exists) such that for every coloring $c : \mathbb{E}_{d,n} \longrightarrow [r]$ there are $A \in \mathbb{E}_{m,n}$ and $\widetilde{r} < r$ such that: $A \cdot \mathbb{E}_{d,m} \subseteq (c^{-1}\{\widetilde{r}\})_\varepsilon$.

Remind

$A \in \mathbb{E}_{m,n}$ if and only if for every column vector c of A one has that $\|c\|_\infty = 1$ and for every row vector r of A one has that $\|r\|_1 \leq 1$

The main ingredient of the proof

Definition

1. Given two integers d and n, let $\mathcal{P}_n^d$ be the set of all partitions of $[n]$ into d many pieces.
2. $\mathcal{P}$ is *coarser* than $\mathcal{Q}$, denoted as $\mathcal{P} \prec \mathcal{Q}$ if for every $P \in \mathcal{P}$ there is $A \subseteq \mathcal{Q}$ such that $P = \bigcup_{Q \in A} Q$.
3. Given $\mathcal{Q} \in \mathcal{P}_n^m$, and $d \leq m$, let $\langle \mathcal{Q} \rangle^d := \{\mathcal{P} \in \mathcal{P}_n^d : \mathcal{P} \prec \mathcal{Q}\}$

The following is the fundamental pigeonhole principle of partitions.

Graham-Rothschild

For every d, m and r there is n such that for every coloring $c : \mathcal{P}_n^d \longrightarrow [r]$, there exists $\mathcal{Q} \in \mathcal{P}_n^m$ such that $c \restriction \langle Q \rangle^d$ is constant.

Denote $GR(d, m, r)$ the minimal such n. A bound of $\mathbf{n}_\infty(d, m, r, \varepsilon)$ is of the form $GR(\alpha, \beta, r)$

Ramsey Property for Polyedral spaces

Definition

1. A finite dimensional space F is called *polyhedral* when the set of extremal points of its unit ball $\partial_e(B_F)$ is finite.
2. Given an integer d, let Pol_d be the class of all polyhedral spaces F such that $\#\partial_e(B_{F^*}) = 2d$.

Bartošová , Lopez-Abad and M.

Given $d, m \in \mathbb{N}$, $r \in \mathbb{N}$ and $\varepsilon > 0$, there is $n = \mathbf{n_{pol}}(d, m, r, \varepsilon)$ such that for every $F \in \mathrm{Pol}_d$ every $G \in \mathrm{Pol}_m$ and every coloring $c : \mathrm{Emb}(F, \ell_\infty^n) \longrightarrow r$, there is $T \in \mathrm{Emb}(G, \ell_\infty^n)$and $\widetilde{r} < r$ such that

$$T \circ \mathrm{Emb}(F, G) \subseteq (c^{-1}\{\widetilde{r}\})_\varepsilon.$$

In fact $\mathbf{n_{pol}}(d, m, r, \varepsilon) = \mathbf{n}_\infty(d, m, r, \varepsilon)$

Ramsey Property for finite dimensional

Given two finite dimensional spaces F and G and given $\theta \geq 1$. Let $\mathrm{Emb}_\theta(F,G) := \{T : F \longrightarrow G : T \text{ is 1-1 and } \|T\|, \|T^{-1}\| \leq \theta\}$.

Bartošová , Lopez-Abad and M.

Given finite normed dimensional spaces F and G, an integer r, numbers $\theta \geqslant 1$ and $\varepsilon > 0$, there exists a finite dimensional space H such that for every coloring $c : \mathrm{Emb}_\theta(F,H) \longrightarrow r$, there are $\widetilde{T} \in \mathrm{Emb}(G,H)$ and $\widetilde{r} < r$ such that

$$\widetilde{T} \circ \mathrm{Emb}_\theta(F,G) \subseteq (c^{-1}\{\widetilde{r}\})_{\theta-1+\varepsilon}.$$

Lipschitz colouring

Remark

In general the approximate Ramsey theorem can also be state in term of Lipschitz colouring.

For example:

Lipschitz colouring vs discrete colouring

The following are equivalent.

(a) Approximate Ramsey property for finite dimensional normed spaces.

(b) For every finite dimensional spaces F and G, $r \in \mathbb{N}$, $\theta \geqslant 1$ and $\varepsilon > 0$ there exists a finite dimensional space H such that $\mathrm{Emb}(G, H) \neq \emptyset$ and for every Lipschitz mapping $f : \mathrm{Emb}_\theta(F, H) \longrightarrow \mathbb{R}$ there exists $\gamma \in \mathrm{Emb}(G, H)$ such that $\mathrm{osc}(f \restriction \gamma \circ \mathrm{Emb}_\theta(F, G)) < \theta - 1 + \varepsilon$.

Extreme amenability of $Iso_L(\mathbb{G})$

Modulo

W. Kubis and S. Solecki

Let $X \subseteq \mathbb{G}$ be a subspace of finite dimension, $\theta > 1$ and let $\gamma \in \mathrm{Emb}_\theta(X, \mathbb{G})$. Then there exists $g \in Iso_L(\mathbb{G})$ such that $\|g \restriction X - \gamma\| \leq \theta - 1$.

We obtain:

Bartošová , Lopez-Abad and M.

The group $Iso_L(\mathbb{G})$ is extremely amenable.

Remark

The Extreme amenability of the Polish group $Iso_L(\mathbb{G})$ is in fact, via the approximate ultra homogeneity of $\mathbb{G}$, equivalent to the Ramsey Property for finite dimensional normed spaces.

Topological dynamic: Universal minimal flow

Definition

Let G be a Hausdorff topological group.

- A G-flow is a continuous action of G on a compact Hausdorff space X. Notation: $G \curvearrowright X$.
- $G \curvearrowright X$ is *minimal* if it contains no proper subflows, i.e., there is no (non-$\emptyset$) compact G-invariant set other than X

Universal minimal flow

Definition

$G \curvearrowright X$ is universal when:

$$\forall\, G \curvearrowright Y \text{ minimal } \exists\, \pi : X \longrightarrow Y$$

continuous, onto, and so that

$$\forall g \in G\ \forall x \in X,\ \pi(g.x) = g.\pi(x)$$

"Every minimal G-flow is a continuous image of $G \curvearrowright X$"

Folklore

Let G be a Hausdorff topological group. Then there is a unique G-flow that is both minimal and universal.

Notation: $G \curvearrowright M(G)$

Universal minimal flow

General question

Describe $G \curvearrowright M(G)$ explicitly when G is a "concrete" group.

Remark

G is extremely amenable iff $M(G)$ is a singleton.

Example: Pestov, 98

$Homeo_+(\mathbb{S}^1) \curvearrowright M(Homeo_+(\mathbb{S}^1))$ is the natural action $Homeo_+(\mathbb{S}^1) \curvearrowright \mathbb{S}^1$.

The Poulsen simplex $\mathbb{P}$

Bartošová , Lopez-Abad and M.

The universal minimal flow of the group $\mathrm{Aut}(\mathbb{P})$ of affine homeomorphisms on $\mathbb{P}$ with the compact-open topology is $\mathbb{P}$.

More consequences: Ramsey property for finite metric spaces

1. Let $\mathcal{F}(M,p)$ be the *Lipschitz Free space* over the pointed metric space (M,p).
2. If M is a finite metric space, then $\mathcal{F}(M)$ is a polyhedral space.

Pestov, 2002

For every finite metric spaces M and N, $r \in \mathbb{N}$ and $\varepsilon > 0$ there exists a finite metric space P such that for every coloring $c : \mathrm{Emb}(M,P) \longrightarrow r$, there exists $\sigma \in \mathrm{Emb}(N,P)$ and $\bar{r} < r$ such that

$$\sigma \circ \mathrm{Emb}(M,N) \subseteq (c^{-1}(\bar{r}))_{\varepsilon}.$$

Consequence of the extreme amenability of the group $Iso(\mathbb{U})$

Question

Question

Can the extreme amenability of the group $Iso_L(\mathbb{G})$ be obtain by using the technique of concentration of measure developped by Gromov and Milman? More precisely is the group $Iso_L(\mathbb{G})$ a Lévy group?

Gromov and Milman

1. An increasing sequence (G_n) of compact subgroups of G, equipped with their Haar probability measures μ_n, is a Lévy sequence if for every open $V \ni e$ and every sequence $A_n \subseteq K_n$ of measurable sets such that $\liminf_{n\to\infty} \mu_n(A_n) > 0$, we have that $\lim_{n\to\infty} \mu_n(VA_n) = 1$.
2. G is a Lévy group if it has a Lévy sequence of compact subgroups whose union is dense in G.