Metric characterization of the Radon-Nikodým property

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  - Measure-theoretic definition (it gives the name to this property) $X \in \mathrm{RNP} \Leftrightarrow$ The following analogue of the Radon-Nikodým theorem holds for X-valued measures.
    - Let (Ω, Σ, μ) be a positive finite real-valued measure, and (Ω, Σ, τ) be an X-valued measure on the same σ-algebra which is absolutely continuous with respect to μ (this means $\mu(A) = 0 \Rightarrow \tau(A) = 0$) and satisfies the condition $\tau(A)/\mu(A)$ is a uniformly bounded set of vectors over all $A \in \Sigma$ with $\mu(A) \neq 0$. Then there is an $f \in L_1(\mu, X)$ such that

$$\forall A \in \Sigma \quad \tau(A) = \int_A f(\omega)d\mu(\omega).$$

Further equivalent definitions of the Radon-Nikodým property (RNP)

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  - In more detail: A Banach space X has the RNP if and only if each X-valued martingale $\{f_n\}$ on any probability space (Ω, Σ, μ), for which $\{||f_n(\omega)|| : \; n \in \mathbb{N}, \; \omega \in \Omega\}$ is a bounded set, converges in $L_1(\Omega, \Sigma, \mu, X)$.

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- And there are numerous others.

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- In 2009 Bill Johnson suggested the problem: Find a purely metric characterization of the Radon-Nikodým property (that is, find a characterization of the RNP which does not refer to the linear structure of the space).

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- In 2009 Bill Johnson suggested the problem: Find a purely metric characterization of the Radon-Nikodým property (that is, find a characterization of the RNP which does not refer to the linear structure of the space).
- My answer to this problem is based on the notion of a *thick family of geodesics*.

- **Definition.** Let u and v be two elements in a metric space (M, d_M). A *uv-geodesic* is a distance-preserving map $g : [0, d_M(u, v)] \to M$ such that $g(0) = u$ and $g(d_M(u, v)) = v$ (where $[0, d_M(u, v)]$ is an interval in $\mathbb{R}$).

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- A family T of uv-geodesics is called *thick* if there is $\alpha > 0$ such that for every $g \in T$ and for every finite collection of points $r_1, \ldots, r_n$ in the image of g, there is another uv-geodesic $\widetilde{g} \in T$ satisfying the conditions:

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  - (1) The image of $\widetilde{g}$ also contains $r_1, \ldots, r_n$.
  - (2) Possibly there are some more common points of g and $\widetilde{g}$.
  - (3) We can find a sequence $0 < s_1 < q_1 < s_2 < q_2 < \cdots < s_m < q_m < s_{m+1} < d_M(u, v)$, such that $g(q_i) = \widetilde{g}(q_i)$ $(i = 1, \ldots, m)$ are common points containing $r_1, \ldots, r_n$, and the images $g(s_i)$ and $\widetilde{g}(s_i)$ are such that the sum of deviations over them is nontrivially large in the sense that $\sum_{i=1}^{m+1} d_M(g(s_i), \widetilde{g}(s_i)) \geq \alpha$.

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- Furthermore, any geodesic obtained by combining finitely many pieces of g and $\widetilde{g}$ is in T.

Examples

- Infinite diamond; Laakso space.

We also need the following definitions

- Let $0 \leq C < \infty$. A map $f : (A, d_A) \to (Y, d_Y)$ between two metric spaces is called *C-Lipschitz* if

$$\forall u, v \in A \quad d_Y(f(u), f(v)) \leq C d_A(u, v).$$

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- Let $1 \leq C < \infty$. A map $f : A \to Y$ is called a *C-bilipschitz embedding* if there exists $r > 0$ such that

$$\forall u, v \in A \quad r d_A(u, v) \leq d_Y(f(u), f(v)) \leq r C d_A(u, v). \qquad (1)$$

A *bilipschitz embedding* is an embedding which is C-bilipschitz for some $1 \leq C < \infty$. The smallest constant C for which there exist $r > 0$ such that (1) is satisfied is called the *distortion* of f.

- **Theorem 1** (M.O. (2014)). A Banach space X does not have the RNP if and only if there exists a metric space M_X containing a thick family T_X of geodesics which admits a bilipschitz embedding into X.

RNP and thick family of geodesics

- **Theorem 1** (M.O. (2014)). A Banach space X does not have the RNP if and only if there exists a metric space M_X containing a thick family T_X of geodesics which admits a bilipschitz embedding into X.
- At the same time, it turns out that the metric space M_X cannot be chosen independently of X because the following result holds.

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- At the same time, it turns out that the metric space M_X cannot be chosen independently of X because the following result holds.
- **Theorem 2** (M.O. (2014)). For each thick family T of geodesics there exists a Banach space X which does not have the RNP and does not admit a bilipschitz embedding of T into X.

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- At the same time, it turns out that the metric space M_X cannot be chosen independently of X because the following result holds.
- **Theorem 2** (M.O. (2014)). For each thick family T of geodesics there exists a Banach space X which does not have the RNP and does not admit a bilipschitz embedding of T into X.
- The main goal of the rest of my talk is to describe the proof of Theorem 1.

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- In fact, it is well-known that Banach spaces without RNP can be such that their unit spheres do not contain line segments (for example, one can consider on $C(0,1)$ with the norm $|||x||| = ||x||_{C(0,1)} + ||x||_{L_2(0,1)}$).

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- In fact, it is well-known that Banach spaces without RNP can be such that their unit spheres do not contain line segments (for example, one can consider on $C(0,1)$ with the norm $|||x||| = ||x||_{C(0,1)} + ||x||_{L_2(0,1)}$).
- On the other hand, it is easy to check that such spaces have *uniqueness of geodesics* property (each pair of points is joined by only one geodesic), and therefore there are no thick families of geodesics in such space (and the words "bilipschitz embedding" in the statement of the Theorem 1 cannot be replaces by "isometric embedding").

- We are going to show that for any non-RNP space X there is an equivalent norm $|||\cdot|||$ on X such that $(X, |||\cdot|||)$ contains a thick family of geodesics. The new norm is very easy to construct: we pick any subspace Z of codimension one in X, pick a vector $x \in X$, $x \notin Z$ and let the unit ball of the new norm be the closure of convex hull of $(x + B_Z) \cup (-x + B_Z)$. It is clear that the new norm is equivalent to the original norm (choosing x and Z in a suitable way we can make it 2-equivalent to the original norm).

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- An important observation is that all vectors of the sets $(x + B_Z)$ and $(-x + B_Z)$ have norm 1 in the new norm $|||\cdot|||$.
- **Remark:** It is easy to check that line segments on the unit sphere of a Banach space imply the existence of infinite families of geodesics between some pairs of points in the space. Our goal is to show that the assumption $X \notin \mathrm{RNP}$ (and so also $Z \notin \mathrm{RNP}$) implies that in the present case we can find a thick family of geodesics.

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- **Definition:** Let Z be a Banach space and let $\delta > 0$. A set of vectors $\{x_{n,j}\}_{n=0,j=1}^{\infty\ \ m_n}$ in Z is called a δ-*bush* if $m_0 = 1$ and for every $n \geq 1$ there is a partition $\{A_k^n\}_{k=1}^{m_{n-1}}$ of $\{1, \dots, m_n\}$ such that

$$||x_{n,j} - x_{n-1,k}|| \geq \delta \qquad (2)$$

for every $n \geq 1$ and for every $j \in A_k^n$, and

$$x_{n-1,k} = \sum_{j \in A_k^n} \lambda_{n,j} x_{n,j} \qquad (3)$$

for some $\lambda_{n,j} \geq 0$, $\sum_{j \in A_k^n} \lambda_{n,j} = 1$.

- It is clear that in our situation we may assume that the bush $\{x_{n,j}\}_{n=0,j=1}^{\infty\ \ m_n}$ is contained in $x + B_Z$ and so that all elements of the bush satisfy $|||x_{n,j}||| = 1$. For simplicity of notation from now on we shall use $||\cdot||$ to denote the new norm.

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- We are going to use this δ-bush to construct a thick family T_X of geodesics in X joining 0 and $x_{0,1}$. First we construct a subset of the desired set of geodesics, this subset will be constructed as the set of limits of certain broken lines in X joining 0 and $x_{0,1}$. The constructed broken lines are also geodesics (but they do not necessarily belong to the family T_X).

- The mentioned above broken lines will be constructed using representations of the form $x_{0,1} = \sum_{i=1}^{m} z_i$, where z_i are such that $||x_{0,1}|| = \sum_{i=1}^{m} ||z_i||$. The broken line represented by such finite sequence $z_1, \ldots, z_m$ is obtained by letting $z_0 = 0$ and joining $\sum_{i=0}^{k} z_i$ with $\sum_{i=0}^{k+1} z_i$ with a line segment for $k = 0, 1, \ldots, m-1$. Vectors $\sum_{i=0}^{k} z_i$, $k = 0, 1, \ldots, m$ will be called *vertices* of the broken line.

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- The infinite set of broken lines which we construct is labelled by vertices of the infinite binary tree T_∞ in which each vertex is represented by a finite (possibly empty) sequence of 0 and 1.

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$$x_{0,1} = \lambda_{1,1}x_{1,1} + \cdots + \lambda_{1,m_1}x_{1,m_1},$$

where $||x_{1,j} - x_{0,1}|| \geq \delta$. We introduce the vectors

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- For these vectors we have
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$||y_{1,j} - x_{1,j}|| = ||y_{1,j} - x_{0,1}|| \geq \frac{\delta}{2}$, and $||y_{1,j}|| = 1$.
- We consider these auxiliary vectors $y_{1,j}$ because they help us to find a pair of geodesics with many intersections and many 'distant' pairs (like in the definition of the thick family).

- As a preliminary step to the construction of the broken lines corresponding to one-element sequences (0) and (1) we form a broken line represented by the points

$$\lambda_{1,1}y_{1,1},\dots,\lambda_{1,m_1}y_{1,m_1}. \tag{4}$$

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- The broken line corresponding to the one-element sequence (1) is represented by the sequence obtained from (4) if we replace each term $\lambda_{1,j}y_{1,j}$ by a two-element sequence

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- In the obtained broken lines each line segment corresponds either to a multiple of $x_{0,1}$ or to a multiple of some $x_{1,j}$. In the next step we replace each such line segment by a broken line. Now we describe how we do this.

- Broken lines corresponding to 2-element sequences are also formed in two steps. To get the broken lines labelled by $(0,0)$ and $(0,1)$ we apply the described procedure to the geodesic labelled (0), to get the broken lines labelled by $(1,0)$ and $(1,1)$ we apply the described procedure to the geodesic labelled (1).

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- In the preliminary step we replace each term of the form $\frac{\lambda_{1,k}}{2}x_{0,1}$ by a multiplied by $\frac{\lambda_{1,k}}{2}$ sequence $\lambda_{1,1}y_{1,1},\dots,\lambda_{1,m_1}y_{1,m_1}$, and we replace a term of the form $\frac{\lambda_{1,k}}{2}x_{1,k}$ by the multiplied by $\frac{\lambda_{1,k}}{2}$ sequence

$$\{\lambda_{2,j}y_{2,j}\}_{j\in A_k^2}, \tag{7}$$

ordered arbitrarily, where $y_{2,j}=\frac{x_{1,k}+x_{2,j}}{2}$ and $\lambda_{2,j}$, $x_{2,j}$, and A_k^2 are as in the definition of the δ-bush (it is easy to check that in the new norm we have $||y_{2,j}||=1$). We label the obtained broken lines by $\overline{(0)}$ and $\overline{(1)}$, respectively.

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- To get the sequence representing the broken line labelled by $(0, 1)$ we do the same but changing the order of terms in (8) and (9). To get the sequences representing the broken lines labelled by $(1, 0)$ and $(1, 1)$, we apply the same procedure to the broken line labelled $\overline{(1)}$.

- We continue in an "obvious" way and get broken lines for all vertices of the infinite binary tree T_∞. It is not difficult to see that vertices of a broken line corresponding to some vertex $(\theta_1, \ldots, \theta_n)$ are contained in the broken line corresponding to any extension $(\theta_1, \ldots, \theta_m)$ of $(\theta_1, \ldots, \theta_n)$ $(m > n)$

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- This implies that the sequence of broken lines corresponding to any ray (that is, a path which starts at the vertex corresponding to $\emptyset$ and is infinite in one direction) in T_∞ has a limit (which is not necessarily a broken line, but is a geodesic), and limits corresponding to two different rays have as common points at least vertices of the broken line corresponding to the common beginning $(\theta_1, \dots, \theta_n)$ of their labels.

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- The desired thick family of geodesics is obtained by pasting pieces of these geodesics in all "reasonable" ways. More precisely, it contains all geodesics which are obtained by pasting together finite number of geodesics corresponding to infinite rays of T_∞. (It is clear that this set is nonempty.)

- It remains only to show that it is a thick family of geodesics. So let g be one of such geodesics and let $\{r_i\}$ be a finite collection of points on it. We may assume that $\{r_i\}$ contains all points where pieces are pasted together. The geodesic g consists of finitely many pieces, consider one of them. Suppose that it corresponds to an infinite sequence $(\theta_i)_{i=1}^{\infty}$

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- We can find a finite piece $(\theta_1, \theta_2, \ldots, \theta_{n(\varepsilon)})$ of the sequence $(\theta_1, \theta_2, \ldots, \theta_n, \ldots)$ such that some of the vertices of the corresponding broken line are very close to those $\{r_i\}$ which are inside the piece which we consider (and are in the piece).

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- Now we change the pieces on intervals which do not contain $\{r_i\}$ (sketch a figure). Instead of geodesic corresponding to $(\theta_1, \theta_2, \ldots, \theta_n, \ldots)$ we put there the geodesic for which $\theta_{n(\varepsilon)+1}$ has the opposite sign (further signs will play no role and we can pick them arbitrarily).

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- The proof is completed by using the inequality which was described in the words "the sum of deviations on subintervals corresponding to vertices is $\geq \frac{\delta}{2} \times$(the length of the subinterval)."

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- We prove this by showing (Embeddability of a thick family into X) $\Rightarrow$ ($\exists$ bounded (in L_∞) divergent (in L_1) martingale in X)
- I hope that you will agree with me that this construction of a divergent martingale is very simple and natural.

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  - Let $\{x_{k,i}\}_{i=0}^{n_k}$ be an increasing sequence in $[0,1]$ defining the k-th partition; $x_{k,0}=0$, $x_{k,n_k}=1$. So $\{x_{k,i}\}_{i=0}^{n_k}$ is a subsequence of $\{x_{k+1,i}\}_{i=0}^{n_{k+1}}$.

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  - Martingale $M_k:([0,1],\mathcal{F}_k)\to X$ given by

$$M_k(t)=\frac{f(x_{k,i+1})-f(x_{k,i})}{x_{k,i+1}-x_{k,i}}\quad\text{for } t\in[x_{k,i+1},x_{k,i}),$$

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    where $\mathcal{F}_k$ is the (finite) σ-algebra defined by the k-th partition.
- This martingale is bounded because f is Lipschitz. So it remains to use the assumption on X in order to find a divergent martingale of this type.

Some useful observations

- We do not have to use the same map f for all partitions. We may use different maps $\{f_k\}$ for partitions $\{x_{k,i}\}_{i=0}^{n_k}$; $k = 1, \ldots,$ provided

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  - The maps f_{k+1} and f_k have the same restriction to $\{x_{k,i}\}_{i=0}^{n_k}$.
- This observation does not give anything new, we may consider the limit of f_k (which exists under natural assumptions) and use it as f; but in our work with thick family of geodesics it is convenient that we do not have to choose the limiting function ahead of time.

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- So let M be a metric space containing a thick family of geodesics which admits a bilipschitz embedding $F : M \to X$ satisfying

$$\delta d_M(x, y) \leq ||F(x) - F(y)|| \leq d_M(x, y).$$

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- All of the maps f_k will be compositions of the form $F \circ g : [0,1] \to X$, where g is a parametrization of one of the geodesics of the thick family.

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- **Note:** We do not need to have a similar estimate for $||M_{k+1} - M_k||_{L_1}$.
- It turns out that the definition of a thick family of geodesics provides a natural way of getting geodesics g_{k+1}, g_{k+2} and partitions $\{x_{k+1,i}\}_{i=0}^{n_{k+1}}$, $\{x_{k+2,i}\}_{i=0}^{n_{k+2}}$

Recall the definition of a thick family

- A family T of uv-geodesics is called *thick* if there is $\alpha > 0$ such that for every $g \in T$ and for every finite collection of points $r_1, \ldots, r_n$ in the image of g, there is another uv-geodesic $\widetilde{g}$ satisfying the conditions:

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  - (3) We can find a sequence $0 < s_1 < q_1 < s_2 < q_2 < \cdots < s_m < q_m < s_{m+1} < d_M(u, v)$, such that $g(q_i) = \widetilde{g}(q_i)$ $(i = 1, \ldots, m)$ are common points containing $r_1, \ldots, r_n$, and the images $g(s_i)$ and $\widetilde{g}(s_i)$ are such that the sum of deviations over them is nontrivially large in the sense that $\sum_{i=1}^{m+1} d_M(g(s_i), \widetilde{g}(s_i)) \geq \alpha$.

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- Furthermore, any geodesic obtained by combining finitely many pieces of g and $\widetilde{g}$ is in T.

Construction

- We use the definition of a thick family of geodesics for g_k and $\{r_i\} = \{g_k(x_{k,i})\}_{i=0}^{n_k}$ and get a geodesic $\widetilde{g}$ and a collection $0 < s_1 < q_1 < s_2 < q_2 < \cdots < s_m < q_m < s_{m+1} < 1$ such that $g_k(q_i) = \widetilde{g}(q_i)$ $(i = 1, \ldots, m)$ are common points containing $\{g_k(x_{k,i})\}_{i=0}^{n_k}$, and the images $g_k(s_i)$ and $\widetilde{g}(s_i)$ are such that the sum of deviations over them is nontrivially large in the sense that $\sum_{i=1}^{m+1} d_M(g_k(s_i), \widetilde{g}(s_i)) \geq \alpha$.

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- We use the definition of a thick family of geodesics for g_k and $\{r_i\} = \{g_k(x_{k,i})\}_{i=0}^{n_k}$ and get a geodesic $\widetilde{g}$ and a collection $0 < s_1 < q_1 < s_2 < q_2 < \cdots < s_m < q_m < s_{m+1} < 1$ such that $g_k(q_i) = \widetilde{g}(q_i)$ $(i = 1, \ldots, m)$ are common points containing $\{g_k(x_{k,i})\}_{i=0}^{n_k}$, and the images $g_k(s_i)$ and $\widetilde{g}(s_i)$ are such that the sum of deviations over them is nontrivially large in the sense that $\sum_{i=1}^{m+1} d_M(g_k(s_i), \widetilde{g}(s_i)) \geq \alpha$.
- We let $g_{k+1} = g_k$ and let $\{x_{k+1,i}\}_{i=0}^{n_{k+1}} = \{0, q_1, \ldots, q_m, 1\}$ and $\{x_{k+2,i}\}_{i=0}^{n_{k+2}} = \{0, s_1, q_1, s_2, \ldots, q_m, s_{m+1}, 1\}$
- The geodesic g_{k+2} will be picked to be the same as g_k on some of the intervals $[q_i, q_{i+1})$, and to be equal to $\widetilde{g}$ on the remaining intervals $[q_i, q_{i+1})$. The choice will be made according to our goal: to make $||M_{k+2} - M_{k+1}||_{L_1}$ nontrivially large. Observe that according to the definition the obtained geodesic will also be in the thick family, and so we can continue the induction.

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- The function M_{k+1} is constant on this interval. The function M_{k+2} (usually) has two values: one on the interval $[q_i, s_{i+1})$, and one on $[s_{i+1}, q_{i+1})$

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- This leads to a construction of a bounded (in L_∞) divergent (in L_1) martingale. Q. E. D.