The Bohnenblust–Hille and Hardy–Littlewood inequalities

Daniel Pellegrino

Universidade Federal da Paraiba, Brazil

Relations Between Banach Space Theory and Geometric Measure Theory
Mathematics Research Centre - UK, June 8-13, 2015

Goals and motivation

In this talk we are mainly concerned in presenting better estimates for certain inequalities....

Goals and motivation

In this talk we are mainly concerned in presenting better estimates for certain inequalities....

Why?????

Goals and motivation

In this talk we are mainly concerned in presenting better estimates for certain inequalities....

Why?????

My preferred answer is: Mathematical curiosity but I am afraid this is not an entirely good answer....

Goals and motivation

In this talk we are mainly concerned in presenting better estimates for certain inequalities....

Why?????

My preferred answer is: Mathematical curiosity but I am afraid this is not an entirely good answer....

...fortunately, I can also say that these estimates (both for the real and complex cases) are important in different concrete applications, although this is not our main goal.

Goals and motivation

In this talk we are mainly concerned in presenting better estimates for certain inequalities....

Why?????

My preferred answer is: Mathematical curiosity but I am afraid this is not an entirely good answer....

...fortunately, I can also say that these estimates (both for the real and complex cases) are important in different concrete applications, although this is not our main goal.

Real case: Quantum Inf. Theory (A. Montanaro, J. Math. Phys., 2012).

Complex Case: Asymptotic behaviour of the Bohr radius (Bayart, P., Seoane, Advances in Mathematics, 2014), among others.

An application for the case of complex scalars

Before presenting the Bohnenblust–Hille and Hardy–Littlewood inequalities let us present an old problem that can be finally solved using the estimates we that will talk about:

An application for the case of complex scalars

Before presenting the Bohnenblust–Hille and Hardy–Littlewood inequalities let us present an old problem that can be finally solved using the estimates we that will talk about:

PROBLEM. What is the exact asymptotic behaviour of the (Harald) Bohr radius?

An application: the (Harald) Bohr radius

The Bohr radius K_n of the n-dimensional polydisk is the largest positive number r such that all polynomials $\sum_\alpha a_\alpha z^\alpha$ on $\mathbb{C}^n$ satisfy

$$\sup_{z \in r\mathbb{D}^n} \sum_\alpha |a_\alpha z^\alpha| \leq \sup_{z \in \mathbb{D}^n} \left| \sum_\alpha a_\alpha z^\alpha \right|,$$

where

$$\mathbb{D}^n = \{z = (z_1, ..., z_n) \in \mathbb{C}^n : max\, |z_i| \leq 1\}.$$

An application: the (Harald) Bohr radius

The Bohr radius K_n of the n-dimensional polydisk is the largest positive number r such that all polynomials $\sum_\alpha a_\alpha z^\alpha$ on $\mathbb{C}^n$ satisfy

$$\sup_{z \in r\mathbb{D}^n} \sum_\alpha |a_\alpha z^\alpha| \leq \sup_{z \in \mathbb{D}^n} \left| \sum_\alpha a_\alpha z^\alpha \right|,$$

where

$$\mathbb{D}^n = \{z = (z_1, ..., z_n) \in \mathbb{C}^n : max\,|z_i| \leq 1\}.$$

The Bohr radius K_1 was studied and estimated by H. Bohr in 1913-1914, and it was shown independently by M. Riesz, I. Schur and F. Wiener that

$$K_1 = 1/3.$$

An application: the (Harald) Bohr radius

The Bohr radius K_n of the n-dimensional polydisk is the largest positive number r such that all polynomials $\sum_\alpha a_\alpha z^\alpha$ on $\mathbb{C}^n$ satisfy

$$\sup_{z\in r\mathbb{D}^n} \sum_\alpha |a_\alpha z^\alpha| \leq \sup_{z\in\mathbb{D}^n} \left| \sum_\alpha a_\alpha z^\alpha \right|,$$

where

$$\mathbb{D}^n = \{z = (z_1, ..., z_n) \in \mathbb{C}^n : max\,|z_i| \leq 1\}.$$

The Bohr radius K_1 was studied and estimated by H. Bohr in 1913-1914, and it was shown independently by M. Riesz, I. Schur and F. Wiener that

$$K_1 = 1/3.$$

For $n \geq 2$, exact values or K_n are unknown.

Known estimates for the Bohr radius

Known estimates for the Bohr radius

Boas and Khavinson (Proc. AMS, 1997):

$$\frac{1}{3}\sqrt{\frac{1}{n}} \leq K_n \leq 2\sqrt{\frac{\log n}{n}}.$$

Known estimates for the Bohr radius

Boas and Khavinson (Proc. AMS, 1997):

$$\frac{1}{3}\sqrt{\frac{1}{n}} \leq K_n \leq 2\sqrt{\frac{\log n}{n}}.$$

Defant and Frerick (Israel J. Math, 2006):

$$K_n \geq c\sqrt{\log n/(n \log \log n)}.$$

Known estimates for the Bohr radius

Boas and Khavinson (Proc. AMS, 1997):

$$\frac{1}{3}\sqrt{\frac{1}{n}} \leq K_n \leq 2\sqrt{\frac{\log n}{n}}.$$

Defant and Frerick (Israel J. Math, 2006):

$$K_n \geq c\sqrt{\log n/(n \log\log n)}.$$

Defant, Frerick, Cerda, Ouaines, Seip (Annals, 2011):

$$K_n = b_n\sqrt{\frac{\log n}{n}} \text{ with } \frac{1}{\sqrt{2}} + o(1) \leq b_n \leq 2.$$

The exact asymptotic growth of the Bohr radius

Using the subexponentiality (that we will see in this talk) of the Bohnenblust–Hille inequality we obtain the exact asymptotic growth of the Bohr radius:

The exact asymptotic growth of the Bohr radius

Using the subexponentiality (that we will see in this talk) of the Bohnenblust–Hille inequality we obtain the exact asymptotic growth of the Bohr radius:

Bayart, P., Seoane (Advances in Math, 2014):

$$K_n \sim_{+\infty} \sqrt{\frac{\log n}{n}},$$

i.e.,

$$\lim_{n\to\infty} \frac{K_n}{\sqrt{\frac{\log n}{n}}} = 1.$$

The exact asymptotic growth of the Bohr radius

Using the subexponentiality (that we will see in this talk) of the Bohnenblust–Hille inequality we obtain the exact asymptotic growth of the Bohr radius:

Bayart, P., Seoane (Advances in Math, 2014):

$$K_n \sim_{+\infty} \sqrt{\frac{\log n}{n}},$$

i.e.,

$$\lim_{n\to\infty} \frac{K_n}{\sqrt{\frac{\log n}{n}}} = 1.$$

The main ingredient of the proof is to repeat the proof of Defant, Frerick, Cerda, Ouaines, Seip (Annals, 2011) with our more accurate estimates.

The multilinear Bohnenblust–Hille inequality

Now that we have tried to convince you that what we will talk about has applications besides our sole curiosity, let us (formally) begin the talk.

The multilinear Bohnenblust–Hille inequality

Now that we have tried to convince you that what we will talk about has applications besides our sole curiosity, let us (formally) begin the talk.

Bohnenblust and Hille (Annals, 1931):

The multilinear Bohnenblust–Hille inequality

Now that we have tried to convince you that what we will talk about has applications besides our sole curiosity, let us (formally) begin the talk.

Bohnenblust and Hille (Annals, 1931):

There exists a sequence of positive scalars $(C_{\mathbb{K},m})_{m=1}^{\infty} \geq 1$ such that

$$\left(\sum_{i_1,\ldots,i_m=1}^{N} \left|U(e_{i_1},\ldots,e_{i_m})\right|^{\frac{2m}{m+1}}\right)^{\frac{m+1}{2m}} \leq C_{\mathbb{K},m}\left\|U\right\| \qquad (1)$$

The multilinear Bohnenblust–Hille inequality

Now that we have tried to convince you that what we will talk about has applications besides our sole curiosity, let us (formally) begin the talk.

Bohnenblust and Hille (Annals, 1931):

There exists a sequence of positive scalars $(C_{\mathbb{K},m})_{m=1}^{\infty} \geq 1$ such that

$$\left(\sum_{i_1,\ldots,i_m=1}^{N} \left|U(e_{i_1},\ldots,e_{i_m})\right|^{\frac{2m}{m+1}}\right)^{\frac{m+1}{2m}} \leq C_{\mathbb{K},m}\,\|U\| \qquad (1)$$

for all m-linear forms $U : \ell_\infty^N \times \ldots \times \ell_\infty^N \to \mathbb{K}$ and every positive integer N.

The multilinear Bohnenblust–Hille inequality

Now that we have tried to convince you that what we will talk about has applications besides our sole curiosity, let us (formally) begin the talk.

Bohnenblust and Hille (Annals, 1931):

There exists a sequence of positive scalars $(C_{\mathbb{K},m})_{m=1}^{\infty} \geq 1$ such that

$$\left(\sum_{i_1,\ldots,i_m=1}^{N} \left| U(e_{i_1}, \ldots, e_{i_m}) \right|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq C_{\mathbb{K},m} \left\| U \right\| \qquad (1)$$

for all m-linear forms $U : \ell_\infty^N \times \ldots \times \ell_\infty^N \to \mathbb{K}$ and every positive integer N.

The best (and unknown) constant $C_{\mathbb{K},m}$ in this inequality will be denoted by $\mathrm{B}_{\mathbb{K},m}^{\mathrm{mult}}$.

The polynomial Bohnenblust–Hille inequality

The polynomial Bohnenblust–Hille inequality

Bohnenblust and Hille (Annals, 1931):

The polynomial Bohnenblust–Hille inequality

Bohnenblust and Hille (Annals, 1931):

For any $m \geq 1$, there exists a constant $D_{\mathbb{K},m} \geq 1$ such that, for any $n \geq 1$, for any m-homogeneous polynomial $P(z) = \sum_{|\alpha|=m} a_\alpha z^\alpha$ on ℓ_∞^n,

$$\left(\sum_{|\alpha|=m} |a_\alpha|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq D_{\mathbb{K},m} \|P\|_\infty,$$

where $\|P\|_\infty = \sup_{z \in \mathbb{D}^n} |P(z)|$.

The polynomial Bohnenblust–Hille inequality

Bohnenblust and Hille (Annals, 1931):

For any $m \geq 1$, there exists a constant $D_{\mathbb{K},m} \geq 1$ such that, for any $n \geq 1$, for any m-homogeneous polynomial $P(z) = \sum_{|\alpha|=m} a_\alpha z^\alpha$ on ℓ_∞^n,

$$\left(\sum_{|\alpha|=m} |a_\alpha|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq D_{\mathbb{K},m} \|P\|_\infty,$$

where $\|P\|_\infty = \sup_{z \in \mathbb{D}^n} |P(z)|$.

The best (and unknown) constant $D_{\mathbb{K},m}$ in this inequality will be denoted by $\mathrm{B}_{\mathbb{K},m}^{\mathrm{pol}}$.

The polynomial Bohnenblust–Hille inequality

Bohnenblust and Hille (Annals, 1931):

For any $m \geq 1$, there exists a constant $D_{\mathbb{K},m} \geq 1$ such that, for any $n \geq 1$, for any m-homogeneous polynomial $P(z) = \sum_{|\alpha|=m} a_\alpha z^\alpha$ on ℓ_∞^n,

$$\left(\sum_{|\alpha|=m} |a_\alpha|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq D_{\mathbb{K},m} \|P\|_\infty,$$

where $\|P\|_\infty = \sup_{z \in \mathbb{D}^n} |P(z)|$.

The best (and unknown) constant $D_{\mathbb{K},m}$ in this inequality will be denoted by $\mathrm{B}_{\mathbb{K},m}^{\mathrm{pol}}$.

In both inequalities the exponent is sharp (this can be proved using the Kahane–Salem–Zygmund inequality).

Estimates for the complex BH constants along the history

Estimates for the complex BH constants along the history

H. F. Bohnenblust and E. Hille (Annals, 1931):

$\mathrm{B}_{\mathbb{C},m}^{\mathrm{mult}} \leq m^{\frac{m+1}{2m}} \left(\sqrt{2}\right)^{m-1}$

Estimates for the complex BH constants along the history

H. F. Bohnenblust and E. Hille (Annals, 1931):

$$\mathrm{B}_{\mathbb{C},m}^{\mathrm{mult}} \leq m^{\frac{m+1}{2m}} \left(\sqrt{2}\right)^{m-1}$$

A.M. Davie (J. London Math. Soc., 1973)

$$\mathrm{B}_{\mathbb{C},m}^{\mathrm{mult}} \leq \left(\sqrt{2}\right)^{m-1}$$

Estimates for the complex BH constants along the history

H. F. Bohnenblust and E. Hille (Annals, 1931):

$$\mathrm{B}_{\mathbb{C},m}^{\mathrm{mult}} \leq m^{\frac{m+1}{2m}} \left(\sqrt{2}\right)^{m-1}$$

A.M. Davie (J. London Math. Soc., 1973)

$$\mathrm{B}_{\mathbb{C},m}^{\mathrm{mult}} \leq \left(\sqrt{2}\right)^{m-1}$$

H. Queffélec (J. Analyse, 1995)

$$\mathrm{B}_{\mathbb{C},m}^{\mathrm{mult}} \leq \left(\frac{2}{\sqrt{\pi}}\right)^{m-1}$$

New estimates: multilinear case

The estimates that we obtained for the multilinear Bohnenblust–Hille inequalities are:

Theorem (Bayart, P., Seoane, Advances in Math, 2014)

$$\mathrm{B}^{\mathrm{mult}}_{\mathbb{C},m} \leq m^{\frac{1-\gamma}{2}}.$$

Numerically, $\frac{1-\gamma}{2} \simeq 0.211392$ *(here* γ *denotes the Euler-Mascheroni constant).*

Theorem (Bayart, P., Seoane, Advances in Math, 2014)

$$\mathrm{B}^{\mathrm{mult}}_{\mathbb{R},m} \leq 1.3 m^{\frac{2-\log 2-\gamma}{2}}.$$

Numerically, $\frac{2-\log 2-\gamma}{2} \simeq 0.36482$ *(here* γ *denotes the Euler-Mascheroni constant).*

Ingredients of the proof

- Khintchine's inequality (in its 'multiple index' version);

Ingredients of the proof

- Khintchine's inequality (in its 'multiple index' version);

- Best constants of the Khintchine inequality (obtained by Uffe Haagerup (Studia Math, 1981);

Ingredients of the proof

- Khintchine's inequality (in its 'multiple index' version);

- Best constants of the Khintchine inequality (obtained by Uffe Haagerup (Studia Math, 1981);

- A 'new' interpolation approach (that after some time, thanks to an anonymous referee, we noticed that it was consequence of a mixed L_p Holder inequality, due to A. Benedek and R. Panzone (Duke, 1961));

The polynomial Bohnenblust–Hille inequality: estimates for the constants

The polynomial Bohnenblust–Hille inequality: estimates for the constants

Bohnenblust and Hille (Annals, 1931):

$$\mathrm{B}_{\mathbb{C},m}^{\mathrm{pol}} \leq \left(\sqrt{2}\right)^{m-1} \frac{m^{\frac{m}{2}}(m+1)^{\frac{m+1}{2}}}{2^m (m!)^{\frac{m+1}{2m}}}$$

The polynomial Bohnenblust–Hille inequality: estimates for the constants

Bohnenblust and Hille (Annals, 1931):

$$\mathrm{B}_{\mathbb{C},m}^{\mathrm{pol}} \leq \left(\sqrt{2}\right)^{m-1} \frac{m^{\frac{m}{2}}(m+1)^{\frac{m+1}{2}}}{2^m (m!)^{\frac{m+1}{2m}}}$$

Defant, Frerick, Cerda, Ouaines, Seip (Annals, 2011):

The polynomial Bohnenblust–Hille inequality: estimates for the constants

Bohnenblust and Hille (Annals, 1931):

$$\mathrm{B}^{\mathrm{pol}}_{\mathbb{C},m} \leq \left(\sqrt{2}\right)^{m-1} \frac{m^{\frac{m}{2}}(m+1)^{\frac{m+1}{2}}}{2^m (m!)^{\frac{m+1}{2m}}}$$

Defant, Frerick, Cerda, Ouaines, Seip (Annals, 2011): The polynomial BH inequality is hypercontractive.

$$\mathrm{B}^{\mathrm{pol}}_{\mathbb{C},m} \leq \left(1 + \frac{1}{m-1}\right)^{m-1} \sqrt{m}(\sqrt{2})^{m-1}$$

New estimates: polynomial case

The estimates that we obtained for polynomial Bohnenblust–Hille inequalities are:

Theorem (Bayart, P., Seoane, Advances in Math, 2014)

For any $\epsilon > 0$, *there exists* $\kappa > 0$ *such that, for any* $m \geq 1$,

$$\mathrm{B}^{\mathrm{pol}}_{\mathbb{C},m} \leq \kappa(1+\epsilon)^m.$$

New estimates: polynomial case

The estimates that we obtained for polynomial Bohnenblust–Hille inequalities are:

Theorem (Bayart, P., Seoane, Advances in Math, 2014)

For any $\epsilon > 0$*, there exists* $\kappa > 0$ *such that, for any* $m \geq 1$*,*

$$\mathrm{B}_{\mathbb{C},m}^{\mathrm{pol}} \leq \kappa(1+\epsilon)^{m}.$$

Ingredients of the proof:

New estimates: polynomial case

The estimates that we obtained for polynomial Bohnenblust–Hille inequalities are:

Theorem (Bayart, P., Seoane, Advances in Math, 2014)

For any $\epsilon > 0$, there exists $\kappa > 0$ such that, for any $m \geq 1$,

$$\mathrm{B}_{\mathbb{C},m}^{\mathrm{pol}} \leq \kappa(1+\epsilon)^m.$$

Ingredients of the proof:

- The multilinear approach.

New estimates: polynomial case

The estimates that we obtained for polynomial Bohnenblust–Hille inequalities are:

Theorem (Bayart, P., Seoane, Advances in Math, 2014)

For any $\epsilon > 0$, *there exists* $\kappa > 0$ *such that, for any* $m \geq 1$,

$$\mathrm{B}_{\mathbb{C},m}^{\mathrm{pol}} \leq \kappa(1+\epsilon)^m.$$

Ingredients of the proof:

- The multilinear approach.

- A Lemma of L. Harris.

New estimates: polynomial case

The estimates that we obtained for polynomial Bohnenblust–Hille inequalities are:

Theorem (Bayart, P., Seoane, Advances in Math, 2014)

For any $\epsilon > 0$*, there exists* $\kappa > 0$ *such that, for any* $m \geq 1$*,*

$$\mathrm{B}_{\mathbb{C},m}^{\mathrm{pol}} \leq \kappa(1+\epsilon)^m.$$

Ingredients of the proof:

- The multilinear approach.

- A Lemma of L. Harris.

- An inequality due to Weissler/Bayart.

New estimates: polynomial case

The estimates that we obtained for polynomial Bohnenblust–Hille inequalities are:

Theorem (Bayart, P., Seoane, Advances in Math, 2014)

For any $\epsilon > 0$*, there exists* $\kappa > 0$ *such that, for any* $m \geq 1$*,*

$$\mathrm{B}^{\mathrm{pol}}_{\mathbb{C},m} \leq \kappa(1+\epsilon)^m.$$

Ingredients of the proof:

- The multilinear approach.

- A Lemma of L. Harris.

- An inequality due to Weissler/Bayart.

- A generalization of an inequality due to Blei that we obtain via interpolation (or the mixed Höler inequality).

Lower estimates for BH multilinear constants - Real case

...if we look for lower estimates then, by finding adequate n-linear forms, as we will see next, we get lower bounds for the BH constants (case of real scalars):

Lower estimates for BH multilinear constants - Real case

...if we look for lower estimates then, by finding adequate n-linear forms, as we will see next, we get lower bounds for the BH constants (case of real scalars):

$$\begin{array}{rcl} \sqrt{2} \leq & \mathrm{B}_{\mathbb{R},2}^{\mathrm{mult}} & \leq \sqrt{2} \\ 1.587 \leq & \mathrm{B}_{\mathbb{R},3}^{\mathrm{mult}} & \leq 1.782 \\ 1.681 \leq & \mathrm{B}_{\mathbb{R},4}^{\mathrm{mult}} & \leq 2 \\ 1.741 \leq & \mathrm{B}_{\mathbb{R},5}^{\mathrm{mult}} & \leq 2.298 \\ 2^{1-\frac{1}{n}} \leq & \mathrm{B}_{\mathbb{R},n}^{\mathrm{mult}} & < n^{\frac{2-\log 2-\gamma}{2}}. \end{array}$$

(D. Diniz, G. Munoz, P, J. Seoane, Proc. Amer. Math. Soc., 2014)

How did we get these lower bounds?

How did we get these lower bounds?

Case $m = 2$:

Let

$$T_2 : \ell_\infty^2 \times \ell_\infty^2 \to \mathbb{R}$$

be defined by

$$T_2(x, y) = x_1 y_1 + x_1 y_2 + x_2 y_1 - x_2 y_2.$$

How did we get these lower bounds?

Case $m = 2$:

Let

$$T_2 : \ell_\infty^2 \times \ell_\infty^2 \to \mathbb{R}$$

be defined by

$$T_2(x, y) = x_1 y_1 + x_1 y_2 + x_2 y_1 - x_2 y_2.$$

Since the norm of T_2 is 2, from

$$\left(\sum_{i,j} \left| T_2 \left(e_i, e_j \right) \right|^{\frac{4}{3}} \right)^{\frac{3}{4}} \leq \mathrm{B}_{\mathbb{R},2}^{\mathrm{mult}} \left\| T_2 \right\|$$

How did we get these lower bounds?

Case $m = 2$:

Let

$$T_2 : \ell_\infty^2 \times \ell_\infty^2 \to \mathbb{R}$$

be defined by

$$T_2(x, y) = x_1 y_1 + x_1 y_2 + x_2 y_1 - x_2 y_2.$$

Since the norm of T_2 is 2, from

$$\left(\sum_{i,j} \left| T_2 \left(e_i, e_j \right) \right|^{\frac{4}{3}} \right)^{\frac{3}{4}} \leq \mathrm{B}_{\mathbb{R},2}^{\mathrm{mult}} \left\| T_2 \right\|$$

we get

$$\mathrm{B}_{\mathbb{R},2}^{\mathrm{mult}} \geq 2^{1-\frac{1}{2}} = \sqrt{2}$$

Case $m = 3$:

Case $m = 3$:

Consider $T_3 : \ell_\infty^4 \times \ell_\infty^4 \times \ell_\infty^4 \to \mathbb{R}$ given by

$$T_3(x, y, z) =$$

$$(z_1+z_2)\left(x_1y_1 + x_1y_2 + x_2y_1 - x_2y_2\right)+(z_1-z_2)\left(x_3y_3 + x_3y_4 + x_4y_3 - x_4y_4\right).$$

and so on.

Case $m = 3$:

Consider $T_3 : \ell_\infty^4 \times \ell_\infty^4 \times \ell_\infty^4 \to \mathbb{R}$ given by

$$T_3(x, y, z) =$$

$$(z_1+z_2)\left(x_1y_1 + x_1y_2 + x_2y_1 - x_2y_2\right)+(z_1-z_2)\left(x_3y_3 + x_3y_4 + x_4y_3 - x_4y_4\right).$$

and so on.

This procedure seems to be useless for the complex case....

Upper and lower bounds far from each other. Which one is the villain?

Upper and lower bounds far from each other. Which one is the villain?

$$2^{1-\frac{1}{n}} \leq \mathrm{B}_{\mathbb{R},n}^{\mathrm{mult}} < n^{0.36482}$$

.

Upper and lower bounds far from each other. Which one is the villain?

The upper and lower bounds are far from each other..... Why is that? Which one is the villain? Are the lower estimates bad? Are the upper estimates way too rough? Both?

Upper and lower bounds far from each other. Which one is the villain?

The upper and lower bounds are far from each other..... Why is that? Which one is the villain? Are the lower estimates bad? Are the upper estimates way too rough? Both?

A favourable evidence to the approach that we used to obtain the lower bounds is that the very same multilinear forms used in it have recently (P. - 2015) provided the optimal constants for the mixed (ℓ_1, ℓ_2)-Littlewood inequality for real scalars, as we will see next.

An argument favourable to the lower bounds

The mixed (ℓ_1, ℓ_2)-Littlewood inequality is a very important result in this framework and reads as follows:

Theorem (Mixed (ℓ_1, ℓ_2)-Littlewood inequality)

For all real m-linear forms $U : c_0 \times \cdots \times c_0 \to \mathbb{R}$ we have

$$\sum_{j_1=1}^{N} \left(\sum_{j_2,\ldots,j_m=1}^{N} \left| U(e_{j_1}, \ldots, e_{j_m}) \right|^2 \right)^{\frac{1}{2}} \leq \left(\sqrt{2}\right)^{m-1} \|U\|$$

for all positive integers N.

Theorem (P., preprint in arXiv, 2015)

The optimal constants of the mixed (ℓ_1, ℓ_2)-Littlewood inequality are $\left(\sqrt{2}\right)^{m-1}$.

Proof. Use the multilinear forms we just defined a couple of slides ago. □

The villain?

So maybe the villain are the upper bounds....

Figura : Upper bounds?

The villain?

So maybe the villain are the upper bounds....

Figura : Upper bounds?

...this would be a very nice surprise (at least for me)....the optimal BH constants for real multilinear forms would be bounded by 2....but up to now this is just speculation...

Polynomials + Real scalars: a different panorama

We have just seen that the constants of the complex polynomial BH inequality have a subexponential growth. It is interesting to remark that for real scalars a similar result does not hold:

Polynomials + Real scalars: a different panorama

We have just seen that the constants of the complex polynomial BH inequality have a subexponential growth. It is interesting to remark that for real scalars a similar result does not hold:

Theorem (Campos, Jimenez, Munoz, D.P and Seoane, Lin. Algebra Appl., 2015)

$$\mathrm{B}_{\mathbb{R},m}^{\mathrm{pol}} > \left(\frac{2\sqrt[4]{3}}{\sqrt{5}}\right)^{m} > (1.17)^{m}$$

for all positive integers $m > 1$.

The proof is simple. Just to find a suitable polynomial....
We also have:

Theorem (Campos, Jimenez, Munoz, D.P and Seoane + Bayart, D.P., Seoane, Lin. Algebra Appl., 2015)

$$\limsup_{m} \sqrt[m]{\mathrm{B}_{\mathbb{R},m}^{\mathrm{pol}}} = 2.$$

The Hardy–Littlewood inequalities

The Hardy–Littlewood inequality are essentially what happens when we replace the ∞ by p in the Bohnenblust–Hille inequalities:

The Hardy–Littlewood inequalities

The Hardy–Littlewood inequality are essentially what happens when we replace the ∞ by p in the Bohnenblust–Hille inequalities:

- (Hardy–Littlewood/Praciano-Pereira) (1934/1980) For $2m \leq p \leq \infty$ there exists a constant $C_{m,p}^{\mathbb{K}} \geq 1$ such that, for all continuous m–linear forms $T : \ell_p^n \times \cdots \times \ell_p^n \to \mathbb{K}$ and all positive integers n,

$$\left(\sum_{j_1,\ldots,j_m=1}^{n} \left| T(e_{j_1}, \ldots, e_{j_m}) \right|^{\frac{2mp}{mp+p-2m}} \right)^{\frac{mp+p-2m}{2mp}} \leq C_{m,p}^{\mathbb{K}} \left\| T \right\| . \qquad (2)$$

The Hardy–Littlewood inequalities

The Hardy–Littlewood inequality are essentially what happens when we replace the ∞ by p in the Bohnenblust–Hille inequalities:

- (Hardy–Littlewood/Praciano-Pereira) (1934/1980) For $2m \leq p \leq \infty$ there exists a constant $C_{m,p}^{\mathbb{K}} \geq 1$ such that, for all continuous m–linear forms $T : \ell_p^n \times \cdots \times \ell_p^n \to \mathbb{K}$ and all positive integers n,

$$\left(\sum_{j_1,\ldots,j_m=1}^{n} \left| T(e_{j_1}, \ldots, e_{j_m}) \right|^{\frac{2mp}{mp+p-2m}} \right)^{\frac{mp+p-2m}{2mp}} \leq C_{m,p}^{\mathbb{K}} \left\| T \right\|. \qquad (2)$$

- (Hardy–Littlewood/Dimant–Sevilla-Peris)(1934/2013) For $m < p \leq 2m$ there exists a constant $C_{m,p}^{\mathbb{K}} \geq 1$ such that, for all continuous m–linear forms $T : \ell_p^n \times \cdots \times \ell_p^n \to \mathbb{K}$ and all positive integers n,

$$\left(\sum_{j_1,\ldots,j_m=1}^{n} \left| T(e_{j_1}, \ldots, e_{j_m}) \right|^{\frac{p}{p-m}} \right)^{\frac{p-m}{p}} \leq C_{m,p}^{\mathbb{K}} \left\| T \right\|. \qquad (3)$$

The exponents in both inequalities are optimal.

The Hardy–Littlewood inequalities

The original estimates for $C_{m,p}^{\mathbb{K}}$ are

- $C_{m,p}^{\mathbb{K}} = \left(\sqrt{2}\right)^{m-1}$.

The Hardy–Littlewood inequalities

The original estimates for $C_{m,p}^{\mathbb{K}}$ are

- $C_{m,p}^{\mathbb{K}} = \left(\sqrt{2}\right)^{m-1}$.

When $2m \leq p \leq \infty$ we (Araujo, P., Diniz, Journal of Functional Analysis 2014) improved these constants to:

- $C_{m,p}^{\mathbb{R}} \leq \left(\sqrt{2}\right)^{\frac{2m(m-1)}{p}} \left(B_{\mathbb{R},m}^{\mathrm{mult}}\right)^{\frac{p-2m}{p}}$
- $C_{m,p}^{\mathbb{C}} \leq \left(\frac{2}{\sqrt{\pi}}\right)^{\frac{2m(m-1)}{p}} \left(B_{\mathbb{C},m}^{\mathrm{mult}}\right)^{\frac{p-2m}{p}}$,

  where we recall that $B_{\mathbb{K},m}^{\mathrm{mult}}$ are the optimal constants of the Bohnenblust–Hille inequality over $\mathbb{K}$ (at the level m).

Tools for the proof

The tools for the proof:

Tools for the proof

The tools for the proof:

- The multiple Khinchine inequality.

Tools for the proof

The tools for the proof:

- The multiple Khinchine inequality.

- A Hölder-type inequality for mixed sums (that we use to call interpolative approach)

Tools for the proof

The tools for the proof:

- The multiple Khinchine inequality.

- A Hölder-type inequality for mixed sums (that we use to call interpolative approach)

- Some tricks that seem to go back to Hardy–Littlewood.

Tools for the proof

The tools for the proof:

- The multiple Khinchine inequality.

- A Hölder-type inequality for mixed sums (that we use to call interpolative approach)

- Some tricks that seem to go back to Hardy–Littlewood.

In a preprint (2014) with G. Araujo we have shown that for

$$p > 2m^3 + 4m^2 + 2m$$

the above estimates can be improved for big values of p. For instance, for complex scalars,

$$C_{m,p}^{\mathbb{C}} < m^{0.211392}$$

Lower bounds for the Hardy–Littlewood inequalities

Lower bounds for the Hardy–Littlewood inequalities

The analytical search of lower bounds for the Hardy–Littlewood inequality seems much harder than the case of the Bohnenblust–Hille inequality and we had to look for computational-assistance....

Lower bounds for the Hardy–Littlewood inequalities

The analytical search of lower bounds for the Hardy–Littlewood inequality seems much harder than the case of the Bohnenblust–Hille inequality and we had to look for computational-assistance....

..... and in this direction we have some results with W. Cavalcante, J. Campos, V. Fávaro, D. Nunez-Alarcon (and we are still working), although my knowledge of programming is zero!

References

This talk contains (or at least is related) to results from the several recent papers from 2012-2015 in collaboration with:

N. Albuquerque (Joao Pessoa, Brazil);

G. Araujo (Joao Pessoa, Brazil);

F. Bayart (Clermont-Ferrand, France);

J.R. Campos (Rio Tinto, Brazil);

W. Cavalcante (Joao Pessoa, Brazil)

D. Diniz (Campina Grande, Brazil);

V. Favaro (Uberlandia, Brazil);

P. Jimenez (Madrid, Spain);

G.A. Muñoz (Madrid, Spain);

D. Nunez-Alarcon (Recife, Brazil);

P. Rueda (Valencia, Spain);

J.B. Seoane (Madrid, Spain);

D. M. Serrano-Rodriguez (Recife, Brazil)

THANK YOU!