Low distortion embeddings of uniformly discrete spaces

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The constant 2 is optimal as every separable metric space $\underset{2}{\hookrightarrow} c_0$ (Kalton-Lancien).

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- $\ell_1 \underset{1}{\hookrightarrow} X$ (Godefroy, Kalton)

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Remark

- The Hamming cube $C_1^\infty = \{0,1\}^{<\omega}$ equipped with the distance $d(x,y) = \sum_{i=1}^\infty |x_i - y_i|$ does not help either as $C_1^\infty \underset{1+\varepsilon}{\hookrightarrow} C([0,\omega^\omega])$ (Baudier, Freeman, Schlumprecht, Zsak, 2014).

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- On the other hand $C_1^\infty \underset{1}{\hookrightarrow} X \implies \ell_1 \underset{1}{\subseteq} X$.

Consequences of the Theorem

- Given a separable metric space N, we know that $N \underset{2}{\hookrightarrow} c_0$ but does there exist an equivalent norm $|\cdot|$ on c_0 such that $N \underset{D}{\hookrightarrow} (c_0, |\cdot|)$ for some $D < 2$?

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Let $M =\uparrow \bigcup M_k$ for some finite sets (M_k). Then $\forall\, D \in [1, 2)$, $\varepsilon > 0$ and $n \in M$ $\exists\, k \in \mathbb{N}$ such that $M_k \underset{D}{\hookrightarrow} X$ implies $\ell_1^n \underset{1+\varepsilon}{\subseteq} X$.

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We are going to give a direct proof with estimates of the constants for a particular choice of (M_n).

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- Finally, we equip M_n with the shortest path metric.

Theorem (A)

Let $D \in [1, \frac{4}{3})$ *and* $n \in \mathbb{N}$. *Then* $M_n \underset{D}{\hookrightarrow} X$ *implies that* $\ell_1^n \underset{D'}{\subseteq} X$ *where* $D' = \frac{D}{4-3D}$.

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- Reduce D' at the cost of augmenting the n of M_n using

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We get

If $D < \frac{4}{3}$, $\varepsilon > 0$ *and* $w \geq -\log_2(\frac{\log(1+\varepsilon)}{\log(\frac{D}{4-3D})})$,
then $M_{n^{2^w}} \underset{D}{\hookrightarrow} X$ *implies that* $\ell_1^n \underset{1+\varepsilon}{\subseteq} X$.

Theorem (B)

Let $D \in [1,2)$. $\forall\, \alpha \in (0,1)\, \exists\, \eta = \eta(\alpha, D) \in (0,1)$ *such that* $M_k \underset{D}{\hookrightarrow} X$ *implies that* $\ell_1^{\eta k} \underset{D'}{\subseteq} X$ *(with* $D' = \frac{2D}{2-D}$*) whenever* $k > \frac{\log_2(\frac{2D}{2-D})+1}{1-\alpha}$.

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- Lemma. *Let* Γ *be a set,* $(f_i)_{i=1}^n \subset KB_{\ell_\infty(\Gamma)}$.
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- Find r and δ?!

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- $$|\mathcal{S}| \geq \frac{2^k - 1}{c} > 2^{\alpha k} \geq \left(\frac{ek}{\eta k}\right)^{\eta k} > \sum_{i=0}^{\lceil \eta k\rceil - 1} \binom{k}{i}$$
- $\Longrightarrow \exists H$ of cardinality $\lceil \eta k \rceil$ such that $(f(i))_{i\in H}$ is $\frac{2D}{2-D}$-equivalent to the u.v.b. of $\ell_1^{\lceil \eta k\rceil}$ Q.E.D.

Final remarks

- $\forall\, D \geq 1, \varepsilon > 0, Y, \dim Y < \infty,\ \exists\, F \subset Y$ finite s.t. $F \underset{D}{\hookrightarrow} X \implies Y \underset{D+\varepsilon}{\subseteq} X$.

Final remarks

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Thank you for your attention!