Calkin algebra of Banach spaces.

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Relations Between Banach Space Theory and Geometric Measure Theory
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The Calkin algebra

Let us start with some well known notion

As usual, for a Banach space X, we denote by

$\mathcal{L}(X)$ the space of all bounded linear operators defined on X

$\mathcal{K}(X)$ the spaces of all compact operators defined on X.

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It is named after J. W. Calkin,

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Two-sided ideals and congruences in the ring of bounded operators in Hilbert space.
Ann. of Math. **42** (1941), no. 2, 839–873.

who proved that the only non-trivial closed ideal of the bounded linear operators on ℓ_2 is the one of the compact operators.

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Question

Given a Banach algebra A, does there exist a Banach space X such that the Calkin algebra of X is isomorphic, as a Banach algebra, to A.

$$A = \mathcal{L}(X)/\mathcal{K}(X) \quad (= Cal(X)).$$

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A hereditarily indecomposable $\mathcal{L}^\infty$-space that solves the scalar-plus-compact problem.
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Indeed, the Argyros-Hydon space $\mathfrak{X}_{AH}$ is such that

$$Cal(\mathfrak{X}_{AH}) = \mathcal{L}(\mathfrak{X}_{AH})/\mathcal{K}(\mathfrak{X}_{AH}) = \mathbb{R}.$$

Similarly, for every $k \in \mathbb{N}$, one can carefully take $\mathfrak{X}_1, \dots, \mathfrak{X}_k$ versions of the Argyros-Haydon space to obtain

$$Cal((\mathfrak{X}_1 \oplus \cdots \oplus \mathfrak{X}_k)_\infty) = \mathbb{R}^k.$$

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In

M. Tarbard,
Operators on Banach Spaces of Bourgain-Delbaen Type.
arXiv:1309.7469v1 (2013).

exending the finite dimensional case, it was constructed a Banach space $\mathfrak{X}_T$ such that

$$Cal(\mathfrak{X}_T) = \ell_1.$$

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Or more generally, one may ask for what topological spaces K, the algebra $C(K)$ is isomorphic to the Calkin algebra of some Banach space.

Theorem (P. Motakis - D.P. - D. Zisimopoulou)

Let $\mathcal{T}$ be a well founded tree with a unique root such that every non maximal node of $\mathcal{T}$ has infinitely countable immediate successors.

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(iii) *There exists a bounded, one-to-one and onto algebra isomorphism $\Phi : Cal(X_{\mathcal{T}}) \longrightarrow C(\mathcal{T})$, where $C(\mathcal{T})$ denotes the algebra of all continuous functions defined on the compact topological space $\mathcal{T}$.*

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In other words, the Calkin of $X_\mathcal{T}$ is isomorphic, as a Banach algebra, to $C(\mathcal{T})$.

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Theorem (P. Motakis - D.P. - D. Zisimopoulou)

For every countable compact metric space K there exists a $\mathcal{L}_\infty$-space X, with X^* isomorphic to ℓ_1, so that its Calkin algebra is isomorphic, as a Banach algebra, to $C(K)$.

The construction

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The construction is based on two fixed strictly increasing sequences of natural numbers $(m_j, n_j)_{j\in\mathbb{N}}$ (which satisfy certain growth conditions) and it is a generalization of the Bourgain-Delbaen method for parameters $a = 1$ and using instead of b the sequence $(\frac{1}{m_j})_{j\in\mathbb{N}}$.

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We denote by $\mathfrak{X}_{AH}(L)$ the space constructed using the sequence $(m_j, n_j)_{j\in L}$.

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If $L \cap M$ is finite, then every $T : X_{AH}(L) \to X_{AH}(M)$ must be compact.

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The Argyros-Haydon sum of a sequence of separable Banach spaces $(X_n)_n$ was introduced in

D. Zisimopoulou
Bourgain-Delbaen $\mathcal{L}_\infty$-sums of Banach spaces.
arXiv:1402.6564 (2014).

The Argyros-Haydon sum

Given a sequence of separable Banach spaces $(X_n)_n$, the space $(\sum \oplus X_n)_{AH}$ is defined as a subspace of

$$\left(\left(\sum \oplus X_n\right)_\infty \oplus \ell_\infty(\Gamma)\right)_\infty, \text{ where } \Gamma = \cup_n \Delta_n.$$

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We define increasing spaces $Y_k = \sum_{n\leq k} \oplus Z_n$ which are the image of $\left(\sum_{n\leq k} \oplus X_k\right)_\infty \oplus \ell_\infty(\Gamma_k)$ through a bounded linear extension operator

$$i_n : \left(\sum_{k\leq n} \oplus X_k\right)_\infty \oplus \ell_\infty(\Gamma_k) \to \left(\left(\sum \oplus X_n\right)_\infty \oplus \ell_\infty(\Gamma)\right)_\infty$$

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where $\Gamma_k = \cup_{n\leq k}\Delta_n$.
The space $(\sum \oplus X_n)_{AH}$ is defined to be the closure of $\cup_k Y_k$.

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In a similar manner, as the Argyros-Haydon space, a sequence $(m_j, n_j)_j$ is used for constructing the Argyros-Haydon sum of a sequence of separable Banach spaces $(X_n)_n$.

The Argyros-Haydon sum

In a similar manner, as the Argyros-Haydon space, a sequence $(m_j, n_j)_j$ is used for constructing the Argyros-Haydon sum of a sequence of separable Banach spaces $(X_n)_n$.
Using an infinite subset of the natural numbers L and as parameters the sequence $(m_j, n_j)_{j\in L}$ we define the space $(\sum \oplus X_n)_{AH(L)}$

The space $X_{\mathcal{T}}$

Let $\mathcal{T}$ be well founded tree having unique root and for every non-maximal node t the set $succ(t)$ will be assumed to be infinitely countable. $\mathcal{T}$ is equiped with the compact Hausdorff topology having the sets $\mathcal{T}_t = \{s \in \mathcal{T} : \; s \geq t\}$, $t \in \mathcal{T}$, as a subbase.

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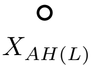
For infinite subset of the natural numbers L using the Argyros-Hydon sum we define $X_{(\mathcal{T},L)}$.

The space $X_{\mathcal{T}}$

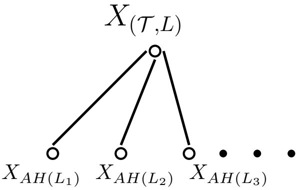
For $\mathcal{T}$ is a singleton and $L \subset \mathbb{N}$ we define $X_{(\mathcal{T},L)}$ to be the space $X_{AH}(L)$.
Tree of rank zero:



The space $X_{\mathcal{T}}$

For a tree of order one we define $X_{(\mathcal{T},L)} = \left(\sum \oplus X_{(\mathcal{T}_n,L_n)}\right)_{AH(L_0)}$.

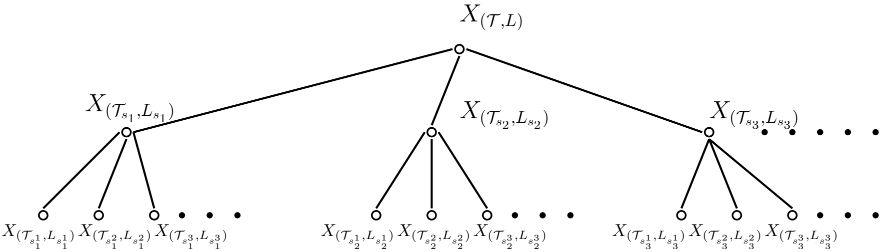
Tree of rank 1:



The space $X_{\mathcal{T}}$

For a tree of order two we define $X_{(\mathcal{T},L)} = \left(\sum \oplus X_{(\mathcal{T}_n, L_n)}\right)_{AH(L_0)}$ etc...

Tree of rank 2:



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(ii) Let $\mathcal{T}$ be a tree with $o(\mathcal{T}) = \alpha > 0$. Assume that for every tree $\mathcal{S}$ with $o(\mathcal{S}) < \alpha$, for every infinite subset of the natural numbers M, the space $X_{(\mathcal{S},M)}$ has been defined. Choose $\{s_n : n \in \mathbb{N}\}$ an enumeration of the set $succ(\emptyset_{\mathcal{T}})$. For a choice of L' an infinite subset of L and a partition of L' into infinite sets $(L_n)_{n=0}^{\infty}$.

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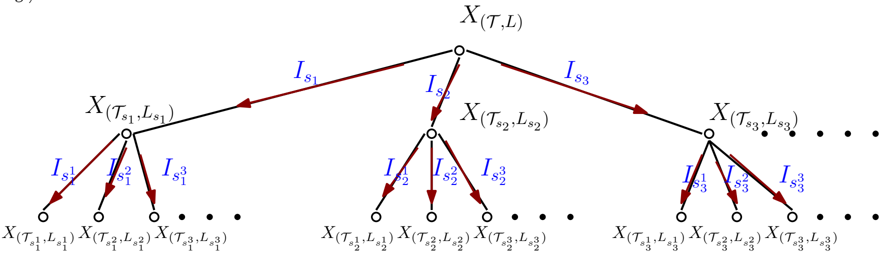
Then define

$$X_{(\mathcal{T},L)} = \left(\sum_{n=1}^{\infty} \oplus X_{(\mathcal{T}_{s_n},L_n)} \right)_{AH(L_0)}.$$

The space $X_{\mathcal{T}}$

The space $X_{(\mathcal{T},L)}$ is accompanied by a set of norm-one projections $I_s, s \in \mathcal{T}$.



Theorem

Let T be a bounded linear operator defined on $X_{(\mathcal{T},L)}$. Then there exist a unique continuous function $f : \mathcal{T} \to \mathbb{R}$ an increasing sequence $(\mathcal{S}_n)_n$ of finite downwards closed subtrees of $\mathcal{T}$ with $\mathcal{T} = \cup_n \mathcal{S}_n$ and a sequence of compact operators $(C_n)_n$ such that the following holds:

$$\lim_n \left\| T - \sum_{s \in \mathcal{S}_n} \left(f(s) - f(s^-) \right) I_s - C_n \right\| = 0$$

i.e. T is approximated by compact perturbations of linear combinations of the I_s, which are determined by the function f.

Thus, we can define the operator

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which assigns each T to the corresponding unique function f defined by the previous theorem.

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Then $\Phi_{(\mathcal{T},L)}$ is a norm-one algebra homomorphism with dense range and

$$\ker\Phi_{(\mathcal{T},L)} = \mathcal{K}(X_{(\mathcal{T},L)}).$$

Hence, the operator $\Phi_{(\mathcal{T},L)}$ induces an operator

$$\Phi_{(\mathcal{T},L)} : \mathcal{L}(X_{(\mathcal{T},L)})/\mathcal{K}(X_{(\mathcal{T},L)}) \longrightarrow C(\mathcal{T})$$

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linear span of $\{[I_s] : s \in \mathcal{T}\}$ is dense in $Cal(X_{(\mathcal{T},L)})$,

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Therefore,

$$\Phi_{(\mathcal{T},L)} : Cal(X_{(\mathcal{T},L)}) \longrightarrow C(\mathcal{T})$$

is an algebra isomorphism.

The Main Theorem

Let K be a countable compact metric space. Then there exists a $\mathcal{L}_\infty$-space X, with X^* isomorphic to ℓ_1, and a norm-one algebra isomorphism $\Phi : Cal(X) \longrightarrow C(K)$ that is one-to-one and onto. Even more, for every $\varepsilon > 0$ the space X can be chosen so that $\|\Phi\|\|\Phi^{-1}\| \leq 1 + \varepsilon$.

Remarks and Open Questions

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To avoid this, essentially we needed

(i) to modify slightly the construction of Argyros-Haydon sum of a sequence of separable Banach spaces defined by D. Zisimopoulou.

(ii) to estimate, in the sum $(\sum \oplus X_n)_{AH}$, the following

$$\|\sum_{k=1}^{n} \oplus T_k + \lambda P_{(n,\ +\infty)}\| \leq (1+\varepsilon)\max\{\max_{k\leq n}\|T_k\|,\ |\lambda|\},$$

where $P_{(n,\ +\infty)}$ is a projection with respect to the Schauder decomposition Z_k of Argyros-Haydon sum $(\sum \oplus X_n)_{AH}$, and T_k is a bounded linear operator on Z_k.

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Question (M. Feder (1982))

Do Banach spaces X and Y exist such that $\mathcal{K}(X,Y)$ is uncomplemented in $\mathcal{L}(X,Y)$ and such that c_0 does not embed in $\mathcal{K}(X,Y)$?

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as far as we know, $X_{(\mathcal{T},L)}$ is the first of such an example.

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2. Does there exists a Banach space whose Calkin algebra is reflexive and infinite dimensional?

Thank you for your attention.