Separable elastic Banach spaces are universal

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joint with Dale Alspach

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Theorem [Alspach, S.] If X is separable elastic infinite dimensional, then $C[0,1] \hookrightarrow X$.

The context this notion arises

Given X, the diameter of the isomorphism class

$$D(X) = \sup\{d(X_1, X_2) : X_1 \approx X \approx X_2\}$$

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Observe: $D(X) < \infty \implies X'$ is $D(X)$-elastic for all $X' \approx X$.

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2. For every $C < \infty$ construct weakly null normalized $(x_n) \in X$ and equivalent norm $|.|_C$ on X so that every subsequence has length $n = n(C)$ blocks $(y_i)_1^n$ which are badly ($\geq C$) unconditional.
   (*Start with* c_0*-sequence, construct a family of bad renormings and embed back using elastic, and use [AOST] to construct a dominating one away from* c_0 *and* ℓ_1 *and do Maurey-Rosenthal.*)

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3. But every X with a normalized basis can be renormed $|.|_n$ so that all blocks of length n are 3 unconditional while the basis is still normalized in $|\cdot|_n$.
   (*Easy.*)

The new proof of the diameter theorem

Once we prove

$$X \text{ separable elastic} \implies C[0,1] \hookrightarrow X,$$

the proof of [JO] becomes an easy observation:

Let (e_i) be a monotone normalized basis in $C[0,1]$. Let $n \in \mathbb{N}$, and let $|\cdot|_n$ be a renorming of $C[0,1]$ so that every normalized block sequence $(x_i)_1^n$ of (e_i) of length n is 3-unconditional in $|\cdot|_n$ (The easy Step 3 above). By assumption, $C[0,1]$ K-embeds into $(C[0,1], |\cdot|_n)$. Since the basis (e_i) is reproducible, there exists a block sequence (u_i) of the basis in $(C[0,1], |\cdot|_n)$ that is $K+\varepsilon$ equivalent to (e_i). So then (e_i) must be block n unconditional with constant $3(K+\varepsilon)$. Since n is arbitrary and (e_i) is not unconditional, this is a contradiction.

Embedding $C[0,1]$

The Main Theorem. Let X be a separable elastic Banach space. If a sequence of $C_0(\alpha_n)$ spaces embed into X where each $\alpha_n < \omega_1$, then $\left(\sum_{n=1}^{\infty} C_0(\alpha_n)\right)_{c_0}$ embeds into X.

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- $C_0(\omega^{\omega^\alpha}) \approx C_0(\omega^{\omega^\alpha n})$ for all n, and $\left(\sum_{n=1}^{n} C_0(\omega^{\omega^\alpha n})\right)_{c_0} \approx C_0(\omega^{\omega^{\alpha+1}})$. Similar for limit ordinals α.

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- So the Main theorem yields $C(\alpha) \hookrightarrow X$ for all $\alpha < \omega_1$.
- **Bourgain.** $C(\alpha) \hookrightarrow X$ for all $\alpha < \omega_1 \implies C[0,1] \hookrightarrow X$.

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- the reproducibility property of their canonical bases (**Lindesntrauss-Pelczynski**)

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Let Z be a space with a 1-unconditional basis (z_n), $(\sum Y_n)_Z$ be the direct sum of spaces Y_n with norm $\|\cdot\|_n$ with respect to (z_n). Consider a tree $\mathcal{T}$ of tuples consisting of pairs of subspaces and isomorphisms

$$((X_1, T_1), (X_2, T_2), \ldots, (X_k, T_k)),$$

where $X_n \subseteq X$ and $T_n : X_n \to Y_n$ such that $\|T_n\| \leq C$, $\|T_n^{-1}\| \leq 1$, and for all $x_n \in X_n$, we have

$$\left\|\sum_{n=1}^{k} x_n\right\| \leq \|(T_n x_n)\|_Z \leq D \left\|\sum_{n=1}^{k} x_n\right\|, \; 1 \leq n \leq k.$$

Partially order $\mathcal{T}$ by extension.

An ordinal index for direct sums

Theorem. Let Z be a Banach space with a normalized 1-unconditional basis (z_n), and let X and $Y_n, n \in \mathbb{N}$ be separable Banach spaces. If $\mathcal{T}$ is a $(\sum Y_n)_Z$-tree in X with index ω_1 and constants C, D, then X contains a subspace which is D-isomorphic to $(\sum Y_n)_Z$.

A glimpse into the proof

Want to show for all $\epsilon > 0$ and limit $\alpha < \omega_1$, X contains a $\left(\sum_{n=1}^{\infty} Y_n\right)_{c_0}$-tree of order at least α with both constants $K(1+\epsilon)$.

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For this construct Banach spaces V^α, $\alpha < \omega_1$ which are isomorphic to subspaces of X and contain $\left(\sum_{n=N}^{\infty} Y_n\right)_{c_0}$-trees of order α for each $N \geq 1$.

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We will show each Y_n has a **good basis** (*complementably reproducible*) $(y_{n,k})_{k=1}^{\infty}$. This means that for every embeddings $T_n : Y_n \to X$ with $\|T_n\| \leq K$ and $\|T_n^{-1}\| \leq 1$, we can find subsequences $(y_{n,k})_{k \in K_n}$ such that

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- $(T_n y_{n,k})_{k \in K_n, n \in \mathbb{N}}$ is (equivalent to) a block basis,
- for each m there is a projection $\|P_m\| \lesssim K$ from $V = [T_n y_{n,k} : k \in K_n, n \in \mathbb{N}]$ onto $[T_m y_{m,k} : k \in K_m]$ with $P_m y \approx 0$ for all $y \in [T_n y_{n,k} : k \in K_n, n \in \mathbb{N}, n \neq m]$.

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For each $i \in \mathbb{N}$ define a norm $\|\cdot\|_i$ on $V = [T_n y_{n,k} : k \in K_n, n \in \mathbb{N}]$ by (**reseting constants**)

$$\|y\|_i = \sup\left\{\|R_m T_m^{-1} P_m y\| : m \in \mathbb{N}\right\} \vee \frac{\|y\|}{iC},$$

where C is good basis constant, R_n be the basis equivalence from $(y_{n,k})_{k \in K_n}$ to $(y_{n,k})_{k=1}^{\infty}$.

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This is an equivalent norm on V.
For each i, let $V^i = (V, \|.\|_i)$. V^i's have good bases. Embed back into X using K-elastic, reset the constants to get V^ω.

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- If the basis $(x_n^{\gamma-1})_n$ for $C(\omega^{\gamma-1})$ is defined for each $k < \omega$ let $x_{k,n}^{\gamma-1}$ have support in $(\omega^\gamma(k-1), \omega^\gamma k]$ and satisfy

$$x_{k,n}^{\gamma-1}(\rho) = x_n^{\gamma-1}(\rho - \omega^{\gamma-1}(k-1)) \text{ for } \omega^{\gamma-1}(k-1) < \rho \leq \omega^{\gamma-1}k.$$

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- If γ is a limit fix $(\gamma_k) \nearrow \gamma$, and let $x_{k,n}^\gamma$ have support in $(\omega^{\gamma_{k-1}}, \omega^{\gamma_k}]$ and satisfy

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This means that for all $\epsilon_k > 0$ and embedding $T : C_0(\omega^\alpha) \to Y$, there is a winning strategy for the second player in a two-player game in Y for picking a subsequence $(Tx_{n_k})_{k=1}^\infty$ and blocks $(w_k)_{k=1}^\infty$ of the basis (y_n) of Y such that

1. $\|Tx_{n_k} - w_k\| < \epsilon_k$ for each $k \in \mathbb{N}$,
2. (x_{n_k}) is 1-equivalent to (x_n).

Pelczynski's weak injectivity

Let $T : C_0(\omega^\alpha) \to X$ be an isomorphic embedding, and for $\rho \leq \omega^\alpha$ let δ_ρ Dirac functional. Let $(y_\rho^*)_{\rho \leq \omega^\alpha} \subset 2\|(T^*)^{-1}\| B_{X^*}$ satisfy $T^* y_\rho^* = \delta_\rho$ for all $\rho \leq \omega^\alpha$. Then there is a compact subset Γ of $[1, \omega^\alpha]$ homeomorphic to $[1, \omega^\alpha]$ and a (weak*) compact subset $(w_\rho^*)_{\rho \in \Gamma}$ of Y^* such that

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- $S^* w^*_\rho = \delta_\rho$ for all $\rho \in \Gamma$, the map $\rho \to w^*_\rho$ is a homeomorphism,
- there is a subsequence of $(x^\alpha_m)_{m \in M}$ equivalent to (x^α_n), with contractively complemented closed linear span such that the restriction to Γ induces an isomorphism R from the span of the subsequence onto $C_0(\Gamma)$ and $R^* \delta_\rho = S^* w^*_\rho$ for all $\rho \in \Gamma$.

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The mapping $S : [x_m : m \in M] \to C_0(\Gamma)$ where $\Gamma = \{\gamma(m) : m \in M\}$ satisfying $(Sx_m)(\gamma(k)) = x_m(\gamma(k))$ is a surjective isometry,

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and the projection P is of the form TEV where $V : X \to C_0(\Gamma)$ is defined by $(Vz)(\gamma(m)) = w_m^*(z)$ for all $z \in X$, E is the extension operator which maps $C_0(\Gamma)$ into $C_0(\omega^\alpha)$ with range in $[x_m : m \in M]$ with $SE = I$.

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Explicitly

$$E_f(\beta) = \begin{cases} f(\beta) & \text{if } \beta \in \Gamma, \\ f(\gamma(m)) & \text{if } x_m(\beta) = 1, x_{m'}(\beta) = 0 \text{ for all } m' > m, \\ 0 & \text{else.} \end{cases}$$

Pelczynski's weak injectivity

The mapping $S : [x_m : m \in M] \to C_0(\Gamma)$ where $\Gamma = \{\gamma(m) : m \in M\}$ satisfying $(Sx_m)(\gamma(k)) = x_m(\gamma(k))$ is a surjective isometry,

and the projection P is of the form TEV where $V : X \to C_0(\Gamma)$ is defined by $(Vz)(\gamma(m)) = w_m^*(z)$ for all $z \in X$, E is the extension operator which maps $C_0(\Gamma)$ into $C_0(\omega^\alpha)$ with range in $[x_m : m \in M]$ with $SE = I$.

Explicitly

$$E_f(\beta) = \begin{cases} f(\beta) & \text{if } \beta \in \Gamma, \\ f(\gamma(m)) & \text{if } x_m(\beta) = 1, x_{m'}(\beta) = 0 \text{ for all } m' > m, \\ 0 & \text{else.} \end{cases}$$

The norm of the projection P is at most $\|T\| \sup_{m \in M} \|w_m^*\|$.

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(iii) there is a projection P from Y onto $[w_k]$ with $\|P\| \leq 2\|T\|\|T^{-1}\|$, and $\|Pz\| < \epsilon\|z\|$ for all $z \in [u_{j_k}]$.