Metric X_p inequalities

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Joint work with

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Banach proved there that if L_q is isomorphic to a subspace of L_p then necessarily either $p \le q \le 2$ or $2 \le q \le p$, and that L_2 is isomorphic to a subspace of L_p for all p.

Banach also conjectured that L_q is isomorphic to a subspace of L_p if $p < q < 2$ or $2 < q < p$.

In the range $p < q < 2$, Banach's question was answered affirmatively by Kadec (1958), who showed that in this case L_q is linearly isometric to a subspace of L_p.

When $2 < q < p$, Banach's question was answered negatively by Paley (1936), i.e., L_q is not isomorphic to a subspace of L_p when $2 < q < p$.

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The cases $q < p < 2$ and $2 < p < q$ and also the cases when p and q are on opposite sides of 2 are best dealt with by Type and Cotype.

Since L_p, $p \leq 2$, has type p

$(\mathbb{E}_\pm \| \sum \pm x_i \|^p)^{1/p} \leq C(\sum \|x_i\|^p)^{1/p}$

clearly, for $q < p \leq 2$, the distance of ℓ_q^n from a subspace of L_p is of order $n^{\frac{1}{q}-\frac{1}{p}}$.

Similarly, for $q > p \geq 2$, using the fact that L_p has cotype p, the distance of ℓ_q^n from a subspace of L_p is of order $n^{\frac{1}{p}-\frac{1}{q}}$.

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The X_p inequality [JMST, '79]:

For $p > 2$, all n and all real numbers $a_1, \dots, a_n, \; x_1, \dots, x_n$

$$\mathbb{E}_{\pm,\pi} \Big| \sum_{i=1}^n \pm a_i x_{\pi(i)} \Big|^p \le$$

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$$\mathbb{E}_{\pm, S\subset\{1,\dots,n\}, |S|=k} |\sum_{i\in S} \pm x_i|^p \le$$

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Plugging for x_i the image of the ℓ_q^n canonical unit vector basis and optimizing over k, we get a lower estimate for the distortion of embedding ℓ_q^n into L_p. It is

$$\geq n^{\frac{(\frac{1}{2}-\frac{1}{q})(\frac{1}{q}-\frac{1}{p})}{\frac{1}{2}-\frac{1}{p}}}$$

and it matches the upper bound.

The linear background

One last linear remark:

The situation with the ℓ_p spaces is simpler:

For all $p \neq q$ ℓ_q does not embed into ℓ_p.

The non-linear background

A metric space (X, d_X) is said to admit a bi-Lipschitz embedding into a metric space (Y, d_Y) if there exist $s \in (0, \infty)$, $D \in [1, \infty)$ and a mapping $f : X \to Y$ such that

$$\forall\, x, y \in X, \qquad s d_X(x, y) \leq d_Y(f(x), f(y)) \leq D s d_X(x, y).$$

When this happens we say that that (X, d_X) embeds into (Y, d_Y) with distortion at most D. We denote by $c_Y(X)$ the infinum over such $D \in [1, \infty]$. When $Y = L_p$ we use the shorter notation $c_{L_p}(X) = c_p(X)$.

The non-linear background

It follows from general principles (mostly differentiation) that $c_p(L_q)$ and $c_p(\ell_q^n)$ are equal to their linear counterparts. But these principles no longer apply when dealing with $c_p(A)$ for a discrete set $A \subset L_q$

nor for $c_p(L_q^\alpha)$ where for $0 < \alpha < 1$ L_q^α denotes L_q with the metric $d_{q,\alpha}(x, y) = \|x - y\|_q^\alpha$.

It turns out however that the non-linear versions of Type and Cotype still apply in such situations when $q < p < 2$ and $2 < p < q$ or when p and q are on opposite sides of 2.

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A metric space (X, d_X) is said to have (Enflo) type $r \in [1, \infty)$ if for every $n \in \mathbb{N}$ and $f : \{-1,1\}^n \to X$,

$$\mathbb{E}\left[d_X(f(\varepsilon), f(-\varepsilon))^r\right] \lesssim_X \sum_{j=1}^{n} \mathbb{E}\left[d_X(f(\varepsilon), f(\varepsilon_1, \ldots, \varepsilon_{j-1}, -\varepsilon_j, \varepsilon_{j+1}, \ldots, \varepsilon_n))^r\right], \quad (1)$$

where the expectation is with respect to $\varepsilon \in \{-1,1\}^n$ chosen uniformly at random. Note that if X is a Banach space and f is the linear function given by $f(\varepsilon) = \sum_{j=1}^n \varepsilon_j x_j$ then this is the inequality defining type r.

For $p \in [1, \infty)$, L_p actually has Enflo type $r = \min\{p, 2\}$. i.e., $X = L_p$ satisfies (1) with $f : \{-1,1\}^n \to L_p$ allowed to be an arbitrary mapping rather than only a linear mapping. This statement was proved by Enflo in 1969 for $p \in [1, 2]$ (and by [NS, 2002] for $p \in (2, \infty)$).

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Here is an illustration how to use Enflo type to show that for $q < p \leq 2$ $c_p(\{-1,1\}^n, \|\cdot\|_q) \gtrsim n^{\frac{1}{q}-\frac{1}{p}}$ (that $c_p(\ell_q^n) \leq n^{\frac{1}{q}-\frac{1}{p}}$ is trivial).

Let $f : \{-1,1\}^n \to L_p$ be such that

$$\forall x, y \in \{-1,1\}^n, \qquad \|x-y\|_q \leq \|f(x)-f(y)\|_p \leq D\|x-y\|_q$$

Then

$$2^p n^{p/q} \leq \mathbb{E}\|f(\varepsilon)-f(-\varepsilon)\|_p^p \lesssim \sum_{j=1}^n \mathbb{E}\|f(\varepsilon)-f(\varepsilon_1,\dots,\varepsilon_{j-1},-\varepsilon_j,\varepsilon_{j+1},\dots,\varepsilon_n)\|_p^p \lesssim D^p n 2^p.$$

So $D \gtrsim n^{\frac{1}{q}-\frac{1}{p}}$.

Similarly one shows that for $\alpha > q/p$ $c_p(\{-1,1\}^n, \|\cdot\|_q^\alpha) \to \infty$.

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So $D \gtrsim n^{\frac{1}{q}-\frac{1}{p}}$.

Similarly one shows that for $\alpha > q/p$ $c_p(\{-1,1\}^n, \|\cdot\|_q^\alpha) \to \infty$.

The non-linear background

The definition of non-linear cotype is more problematic. Changing the direction of the inequality in the definition of type is no good if $f(\{-1,1\}^n)$ is a discrete set. A good definition was sought for a long time until the following:

A metric space (X, d_X) is said to have (Mendel-Naor) cotype $s \in [1, \infty)$ if for every $n \in \mathbb{N}$ there is an $m \in \mathbb{N}$ such that for all $f : \mathbb{Z}_{2m}^n \to X$,

$$\sum_{j=1}^{n} \frac{\mathbb{E}\left[d_X(f(x+me_j), f(x))^s\right]}{m^s} \lesssim_X \mathbb{E}\left[d_X(f(x+\varepsilon), f(x))^s\right],$$

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It was proved by Mendel and Naor (2006) that a Banach space $(X, \|\cdot\|_X)$ has Rademacher cotype s if and only if it has metric cotype s, in particular L_p has metric cotype $\max\{p, 2\}$.

Using this one can prove that for $2 < p < q$, for some, specific m depending on n and p, $c_p(|Z_m^n, \|\cdot\|_q) \to \infty$ when $n \to \infty$.

The cases when p and q are on different sides of 2 can also be dealt with.

$c_p(L_q^\alpha)$ can also be dealt with in these cases.

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One last non-linear remark:

The situation with the ℓ_p spaces is different:

If $1 \le q \le p < \infty$ and $\alpha \in (0,1]$ is such that $(\ell_q, \|x-y\|_q^\alpha)$ admits a bi-Lipschitz embedding into ℓ_p then necessarily $\alpha \le q/p$ [Baudier]. Also, for every $1 \le q \le p < \infty$, $(\ell_q, \|\cdot\|_q^{q/p})$ does admit a bi-Lipschitz embedding into ℓ_p [Albiac and Baudier].

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The inequality

Recall the linear X_p inequality:

$$\mathbb{E}_{\pm,S\subset\{1,\dots,n\},|S|=k}\|\sum_{i\in S}\pm x_i\|^p \le$$

$$C_p\left(\frac{k}{n}\sum_{i=1}^{n}\|x_i\|^p + \left(\frac{k}{n}\right)^{p/2}\mathbb{E}_{\pm}\|\sum_{i=1}^{n}\pm x_i\|^p\right)$$

The inequality

for $S \subset \{1,\dots,n\}$ and $\varepsilon \in \{-1,1\}^n$ we denote $\varepsilon_S = \sum_{j\in S} \varepsilon_j e_j$.

Theorem (Metric X_p inequality)

Fix $p \in [2,\infty)$. Suppose that $m,n \in \mathbb{N}$ and $k \in \{1,\dots,n\}$ satisfy $m \geq \frac{n^{3/2}\log p}{\sqrt{k}} + pn$. Then for every $f : \mathbb{Z}_{4m}^n \to L_p$ we have

$$\frac{1}{\binom{n}{k}} \sum_{\substack{S\subset\{1,\dots,n\}\\ |S|=k}} \frac{\mathbb{E}\left[\left\|f\left(x+2m\varepsilon_S\right)-f(x)\right\|_p^p\right]}{m^p}$$

$$\lesssim_p \frac{k}{n}\sum_{j=1}^n \mathbb{E}\left[\left\|f(x+e_j)-f(x)\right\|_p^p\right] + \left(\frac{k}{n}\right)^{\frac{p}{2}} \mathbb{E}\left[\left\|f\left(x+\varepsilon\right)-f(x)\right\|_p^p\right],$$

where the expectation is with respect to $(x,\varepsilon) \in \mathbb{Z}_{4m}^n \times \{-1,1\}^n$ chosen uniformly at random. The constant is $\left(\frac{Cp}{\log p}\right)^p$.

Corollaries

Theorem (L_p distortion of L_q grids)

For every $2 < p < \infty$ there exists $\alpha_p \in (0, \infty)$ such that for every $q \in (2, p)$ and $m, n \in \mathbb{N}$ we have

$$c_p(\mathbb{Z}_m^n, \|\cdot\|_q) \geq \alpha_p \left(\min\left\{ m^{\frac{q(p-2)}{q(p-2)+p-q}}, n \right\} \right)^{\frac{(\frac{1}{2}-\frac{1}{q})(\frac{1}{q}-\frac{1}{p})}{(\frac{1}{2}-\frac{1}{p})}}.$$

In particular, if $m \geq n^{1+\frac{p-q}{q(p-2)}}$, then

$$c_p(\mathbb{Z}_m^n, \|\cdot\|_q) \geq \alpha_p n^{\frac{(\frac{1}{2}-\frac{1}{q})(\frac{1}{q}-\frac{1}{p})}{(\frac{1}{2}-\frac{1}{p})}} \gtrsim \alpha_p c_p(\ell_q^n).$$

Corollaries

Some lower bound on m is needed:

$(\{-1,1\}^n, \|\cdot\|_q) = (\{-1,1\}^n, \|\cdot\|_2^{2/q})$ and the later (Lipschitz) isometrically embeds in L_2 which isometrically embeds in L_p.

This also shows that scaling (and using $\mathbb{Z}_m^n$ instead of just $\{-1,1\}^n$) is necessary in the metric X_p inequality.

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Corollaries

Theorem (L_q snowflakes in L_p)

For every $2 < q < p$ there exists $\delta(p, q) > 0$ such that if $\alpha \in (0, 1)$ is such that the metric space $(L_q, \|x - y\|_q^{\alpha})$ admits a bi-Lipschitz embedding into L_p then necessarily $\alpha \leq 1 - \delta(p, q)$. Specifically, α must satisfy $\alpha \leq 1 - \frac{(p-q)(q-2)}{2p^3}$.

Mendel and Naor (2004) showed that for $2 < q < p$, $L_q^{q/p}$, the (q/p)-snowflake of L_q, is isometric to a subset of L_p.
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We conjecture that this is sharp.

A conjecture

We conjecture that the metric X_p inequality holds whenever $m \geq C_p\sqrt{n/k}$. I.e.,

Conjecture

Fix $p \in [2,\infty)$. Suppose that $m, n \in \mathbb{N}$ and $k \in \{1,\dots,n\}$ satisfy $m \geq C_p\sqrt{n/k}$. Then for every $f : \mathbb{Z}_{4m}^n \to L_p$ we have

$$\frac{1}{\binom{n}{k}} \sum_{\substack{S\subset\{1,\dots,n\}\\|S|=k}} \frac{\mathbb{E}\left[\|f(x+2m\varepsilon_S)-f(x)\|_p^p\right]}{m^p}$$

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A conjecture

If the conjecture holds then

1. The snow flake conjecture holds: If $\alpha \in (0, 1)$ is such that the metric space $(L_q, \|x - y\|_q^\alpha)$ admits a bi-Lipschitz embedding into L_p, $2 < q < p$, then necessarily $\alpha \leq q/p$.

2. $c_p(\mathbb{Z}_m^n, \|\cdot\|_q)$ is given by the best of the two mentioned embeddings: The linear one (which works for all of ℓ_q^n) and the one given by thinking of $(\mathbb{Z}_m^n, \|\cdot\|_q)$ as $(\mathbb{Z}_m^n, \|\cdot\|_2^{2/q})$.

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