Γ-a.e. Differentiability of Convex and Quasiconvex Functions

(joint result with L. Zajíček)

Warwick, 2015

Theorem. Let X be a Banach space with X^* separable, $G \subset X$ an open convex set and $f \colon G \longrightarrow \mathbb{R}$ a continuous convex function. Then f is Fréchet differentiable Γ-almost everywhere in G.

Let X be a Banach space. $\Gamma(X)$ is the space of all continuous mappings $\gamma\colon [0,1]^{\mathbb{N}} \longrightarrow X$ which have continuous partial derivatives $D_k\gamma$. The topology on $\Gamma(X)$ is generated by the countable family of pseudonorms

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The space $\Gamma_n(X) := C^1([0,1]^n, X)$ is equipped with the norm

$$\|f\|_{C^1} = \max\big\{\|f\|_\infty, \|f'\|_\infty\big\}.$$

Definition. A Borel set $A \subset X$ is called Γ-null if

$$\mathscr{L}^{\mathbb{N}}\{t \in [0,1]^{\mathbb{N}} \mid \gamma(t) \in A\} = 0$$

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for residually many $\gamma \in \Gamma(X)$. Analogically, a Borel set $A \subset X$ is called Γ_n-null if

$$\mathscr{L}^{n}\{t \in [0,1]^{n} \mid \gamma(t) \in A\} = 0$$

for residually many $\gamma \in \Gamma_n(X)$.

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If A is an F_σ set which is Γ-null, then A is Γ_n-null for all $n \in \mathbb{N}$.

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(i) $r_k \searrow 0$;

(ii) $\|y_k - a\| = o(r_k)$, $k \to \infty$, and

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The closed set A is called an $\mathcal{P}^{dc}$-set if there is a subset $A_0 \subset X$ which is a countable union of $d.c.$-hypersurfaces and such that A is P-small at all points of $A \setminus A_0$.

Theorem. Let X be a Banach space with X^* separable, $G \subset X$ an open convex set and $f \colon G \longrightarrow \mathbb{R}$ a continuous convex function. Then the set of points where f is not Fréchet differentiable is a countable union of $\mathcal{P}^{dc}$-sets. Consequently, f is Fréchet differentiable Γ-almost everywhere in G.

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$$\mathscr{H}^m\big(B(y_k, r_k)\cap(y_k+V)\cap A\big) \le \lambda\, \mathscr{H}^m\big(B(y_k, r_k)\cap(y_k+V)\big),$$

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The closed set A is called an $\mathcal{P}^\Gamma_\lambda$-set, $\lambda \in [0,1)$, if there is a Borel subset $A_0 \subset X$ which is Γ-null and A is P_λ-small at all points of $A \setminus A_0$.

Criterion.
Let $A \subset X$ be a $\mathcal{P}_\lambda^\Gamma$-set, $\lambda \in [0, 1)$, in a separable Banach space X.
Then A is Γ-null.

A function $f\colon X \longrightarrow \mathbb{R}$ is quasiconvex if

$$f(\tau x + (1-\tau)y) \leq \max\{f(x),\ f(y)\}$$

for every $x, y \in X$ and $\tau \in [0,1]$.

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Proposition. Let $A \subset X$ be a closed convex subset of a separable Banach space X. Then the boundary ∂A of the set A is $P_{1/2}$-small at all of its points. Consequently, ∂A is Γ-null. In particular, a closed convex nowhere dense subset of X is Γ-null.

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Theorem. Let $f\colon X \longrightarrow \mathbb{R}$ be a continuous quasiconvex function on a separable Banach space X. Then f is Hadamard differentiable Γ-a.e.

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