Random unconditionality for bases in Banach spaces

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Relations Between Banach Space Theory and Geometric Measure Theory

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Outline

A basis $(x_n)_{n\in\mathbb{N}}$ of a Banach space X is unconditional provided for every $x \in X$ its expansion $\sum_{n\in\mathbb{N}} a_n x_n$ converges unconditionally.

TFAE:

- (x_n) is an unconditional basis.
- For every $A \subset \mathbb{N}$,

$$\sum_{n\in\mathbb{N}} a_n x_n \text{ converges} \Rightarrow \sum_{n\in A} a_n x_n \text{ converges.}$$

- For every choice of signs $(\epsilon_n)_{n\in\mathbb{N}}$,

$$\sum_{n\in\mathbb{N}} a_n x_n \text{ converges} \Rightarrow \sum_{n\in\mathbb{N}} \epsilon_n a_n x_n \text{ converges.}$$

- There is $C > 0$ such that for any scalars and signs

$$\|\sum_{n=1}^{m} \epsilon_n a_n x_n\| \leq C\|\sum_{n=1}^{m} a_n x_n\|.$$

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Definition (Billard-Kwapien-Pelczynski-Samuel, 1985)

A basis $(x_n)_{n\in\mathbb{N}}$ is of **Random unconditional convergence** (RUC) if

$$\sum_{n\in\mathbb{N}} a_n x_n \text{ converges } \Rightarrow \sum_{n\in\mathbb{N}} \epsilon_n a_n x_n \text{ converges a.s.}$$

$(x_n)_{n\in\mathbb{N}}$ is an RUC-basis iff there is $K \geq 1$ such that

$$\mathbb{E}_\epsilon\Big(\Big\|\sum_{n=1}^m \epsilon_n a_n x_n\Big\|\Big) = \frac{1}{2^m}\sum_{(\epsilon_n)\in\{-1,+1\}^m}\Big\|\sum_{n=1}^m \epsilon_n a_n x_n\Big\| \leq K\Big\|\sum_{n=1}^m a_n x_n\Big\|.$$

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Questions

- What properties of an unconditional basis work for RUC/RUD bases?
- Does every RUC/RUD basis have an unconditional subsequence (resp. blocks)?
- Is every block of an RUC/RUD basis, also RUC/RUD?
- Can reflexivity be characterized somehow? (in the spirit of James theorem)
- What are these bases good for?
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Example

The summing basis (s_n) in c_0 does not have any RUC nor RUD subsequence

$$\Big\|\sum_{n=1}^{m} a_n s_n\Big\| = \sup_{1\leq n\leq m}\Big|\sum_{j=1}^{n} a_j\Big|.$$

In particular,

$$\mathbb{E}_\epsilon\Big(\Big\|\sum_{n=1}^{m}\epsilon_n a_n s_n\Big\|\Big) = \int_0^1 \sup_{1\leq n\leq m}\Big|\sum_{j=1}^{n} a_j r_j(t)\Big| dt \approx \Big(\sum_{j=1}^{m} a_j^2\Big)^{\frac{1}{2}},$$

(by Levy's and Khintchine's inequalities).

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The unit basis (u_n) of James space is RUD.

$$\|\sum_{n\in\mathbb{N}} a_n u_n\|_J = \sup\Big\{\Big(\sum_{k=1}^{m}(a_{p_k} - a_{p_{k+1}})^2\Big)^{\frac{1}{2}} : p_1 < p_2 < \ldots < p_{m+1}\Big\}.$$

It holds that

$$\|\sum_{i=1}^{m} a_i u_i\|_J \leq \sqrt{2}\,\mathbb{E}_\epsilon\Big(\Big\|\sum_{i=1}^{m}\epsilon_i a_i u_i\Big\|_J\Big).$$

Example

The Haar basis in $L^1[0,1]$ is an RUD basis.

Recall, $L^1[0,1]$ has no unconditional basis. Actually, $L^1[0,1]$ does not embed in a space with unconditional basis [Pelczynski (1961)].

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Duality

Proposition

Let (x_n) be a basis, and (x_n^*) bi-orthogonal functionals.

- (x_n) *RUC* $\Rightarrow$ (x_n^*) *RUD.*
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However, if we take (s_n), the summing basis of c_0

$$s_n = (\overbrace{1, \ldots, 1}^{(n)}, 0 \ldots),$$

this is not RUC, although

$$s_n^* = (\overbrace{0, \ldots, 0}^{(n-1)}, 1, -1, 0, \ldots)$$

form an RUD basis in ℓ_1.

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Reflexivity

Theorem (James)

A Banach space X with unconditional basis which does not contain ℓ_1 nor c_0 subspaces, is reflexive.

Theorem

1. *Let (x_n) be a basis of a Banach space X such that every block is RUD. (x_n) is shrinking $\Leftrightarrow$ $\ell_1 \not\subset X$*
2. *Let (x_n) be a basis of a Banach space X such that every block is RUC. (x_n) is boundedly complete $\Leftrightarrow$ $c_0 \not\subset X$.*

Reflexivity

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Uniqueness

Theorem (Lindenstrauss-Zippin)

X has a unique unconditional basis iff $X \approx \ell_1, \ell_2$ or c_0.

Theorem (BKPS)

X has a unique RUC basis iff $X \approx \ell_1$.

Theorem

If X has an RUD basis, then there are non-equivalent RUD basis in X.

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Theorem

If X has an RUD basis, then there are non-equivalent RUD basis in X.

Things we know

Does every weakly null sequence have an RUD subsequence?
NO [e.g. Maurey-Rosenthal space (Studia 1977).]

Is every block sequence of an RUD basis also RUD?
NO [A modification of M-R.]

Given an RUD sequence, does it have an unconditional subsequence?
NO [e.g. a weakly null sequence in $L_1[0,1]$ without unconditional subsequences (Johnson-Maurey-Schechtman, JAMS 2007)]

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Every block sequence of the Haar basis in L_1 is RUD.

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Things we don't know

- Does every Banach space contain an RUD/RUC basic sequence?
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Thank you for your attention.