Constructing Banach ideals using upper ℓ_p-estimates

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Relations between Banach space theory
and geometric measure theory
at the University of Warwick,
Coventry, UK

Banach ideals and Schreier families

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For convenience, denote by $\mathcal{S}_{\omega_1}$ the family of all finite subsets of $\mathbb{N}$. (This is nonstandard notation, but it greatly simplifies the writing.)

The classes $\mathcal{WD}_{\ell_p}^{(\infty,\xi)}$

Let us fix some $1 < p \leq \infty$ and $1 \leq \xi \leq \omega_1$.

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We denote by $\mathcal{WD}_{\ell_p}^{(\infty,\xi)}(X, Y)$ the set of all operators $T \in \mathcal{L}(X, Y)$ for which there exists a uniform constant $C > 0$ with the following property: Each normalized weakly null sequence $(x_n) \subset X$ admits a subsequence (x_{n_k}) such that for all $(\alpha_k) \in c_{00}$ with support in $\mathcal{S}_\xi$ we get

$$\|\sum \alpha_k T x_{n_k}\| \leq C\|(\alpha_k)\|_{\ell_p}.$$

Using Schreier families

An operator $T : X \to Y$ is called **strictly singular** ($\mathcal{SS}$) just in case for every normalized basic sequence $(x_n) \subseteq X$ and every $\epsilon > 0$ there exists $(\alpha_n) \in c_{00}$ such that $\|\sum \alpha_n Tx_n\| < \epsilon \|\sum \alpha_n x_n\|$.

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An operator $T : X \to Y$ is **weakly compact** ($\mathcal{W}$) just in case for every normalized basic sequence $(x_n) \subseteq X$ and every $\epsilon > 0$ there is $(\alpha_n) \in c_{00}$ such that $\|\sum \alpha_n Tx_n\| < \epsilon \|\sum \alpha_n s_n\|$, where (s_n) is the summing basis for c_0.

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The **Rosenthal** ($\mathcal{R}$) and **$\mathcal{S}_\xi$-Rosenthal** ($\mathcal{R}_\xi$) operators can be similarly defined using the canonical basis for ℓ_1 instead of the summing basis for c_0.

Using Schreier families

Theorem (Odell-Teixeira, 2010)

There exists X such that $\mathcal{SS}_1(X)$ is not closed under addition. In particular, $\mathcal{SS}_1$ is not an operator ideal.

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Proposition (*Causey-Freeman-anonymous*)

$$\bigcap_{1\leq\xi<\omega_1} \mathcal{WD}_{\ell_p}^{(\infty,\xi)}(X,Y) = \mathcal{WD}_{\ell_p}^{(\infty,\omega_1)}(X,Y).$$

Operator ideal families with ℓ_p parameters

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The new family of ideals sits here:

$$\mathcal{N}_p \subsetneq \mathcal{I}_p \subsetneq \Pi_p \subsetneq \mathcal{V} \subsetneq \mathcal{WD}_{\ell_p}^{(\infty,\xi)}.$$

A seminorm for the spaces $\mathcal{WD}_{\ell_p}^{(\infty,\xi)}(X, Y)$

For each T in class $\mathcal{WD}_{\ell_p}^{(\infty,\xi)}$, we denote

$$C_{(p,\xi)}(T) := \inf C,$$

where the "inf" ranges over all possible (uniform) domination constants in the definition of $\mathcal{WD}_{\ell_p}^{(\infty,\xi)}$.

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Proposition.

$C_{(p,\xi)}$ is a seminorm on the space $\mathcal{WD}_{\ell_p}^{(\infty,\xi)}(X, Y)$.

Classes $\mathcal{WD}_{\ell_p}^{(C,\xi)}$

For each $0 \leq C < \infty$, define

$$\mathcal{WD}_{\ell_p}^{(C,\xi)}(X,Y) = \left\{ T \in \mathcal{WD}_{\ell_p}^{(\infty,\xi)}(X,Y) : C_{(p,\xi)}(T) \leq C \right\}.$$

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It is evident that

$$\mathcal{WD}_{\ell_p}^{(\infty,\xi)}(X,Y) = \bigcup_{C\geq 0} \mathcal{WD}_{\ell_p}^{(C,\xi)}(X,Y) = \bigcup_{n=1}^{\infty} \mathcal{WD}_{\ell_p}^{(n,\xi)}(X,Y).$$

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We also have

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However, when $C \in (0,\infty)$, the components $\mathcal{WD}_{\ell_p}^{(C,\xi)}(X,Y)$ do not form linear spaces.

$\mathcal{WD}_{\ell_p}^{(\infty,\xi)}$ fails to be norm-closed but $\mathcal{WD}_{\ell_p}^{(\infty,1)}$ is F_σ

In the special case $\xi = 1$, the sets $\mathcal{WD}_{\ell_p}^{(C,1)}(X, Y)$, $0 \leq C < \infty$, are always norm-closed in $\mathcal{L}(X, Y)$.

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Proposition.

For any $1 < p \leq \infty$ and any $1 \leq \xi \leq \omega_1$, there exists a space X such that $\mathcal{WD}_{\ell_p}^{(\infty,\xi)}(X)$ is not norm-closed.

$\mathcal{WD}_{\ell_p}^{(\infty,1)}$ is a Banach ideal

$$\text{Define} \quad \|T\|_{(p,\xi)} := C_{(p,\xi)}(T) + \|T\|_{op}.$$

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Proposition.

In the special case $\xi = 1$, the space $\mathcal{WD}_{\ell_p}^{(\infty,1)}(X, Y)$ is complete under the norm $\|\cdot\|_{(p,1)}$. In particular, class $\mathcal{WD}_{\ell_p}^{(\infty,1)}$ forms a Banach ideal.

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We do not yet know if the classes $\mathcal{WD}_{\ell_p}^{(\infty,\xi)}$ form Banach ideals when $\xi \neq 1$.

Significance of p parameter

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When ξ is fixed, the norm-closed operator ideals $\overline{\mathcal{WD}}_{\ell_p}^{(\infty,\xi)}$ are all distinct as p ranges over $1 \leq p \leq \infty$.

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In particular, for $1 < q < p < \infty$,

$$\mathcal{WD}_{\ell_p}^{(\infty,\xi)}(\ell_q) = \mathcal{K}(\ell_q) \neq \mathcal{L}(\ell_q) = \mathcal{WD}_{\ell_q}^{(\infty,\xi)}(\ell_q).$$

Significance of ξ parameter

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Recall again that $\bigcap_{1\leq\xi<\omega_1} \mathcal{WD}_{\ell_p}^{(\infty,\xi)}(X,Y) = \mathcal{WD}_{\ell_p}^{(\infty,\omega_1)}(X,Y)$.

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Proposition.

There exists a strictly increasing sequence (ξ_n) of countable ordinals $1 \leq \xi_n < \xi_{n+1} < \omega_1$, $n \in \mathbb{N}$, and a sequence (X_n) of Banach spaces, such that for all $m, n \in \mathbb{N}$ with $m < n$ we have

$$\overline{\mathcal{WD}}_{\ell_p}^{(\infty,\xi_n)}(X_m) \subsetneq \overline{\mathcal{WD}}_{\ell_p}^{(\infty,\xi_m)}(X_m)$$

That's all folks!

Thank you for listening!