Smoothness via smoothness on the lines and Marchaud's theorem in Banach spaces

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The talk is based on a recent joint work with M. Johanis (J. Math. Anal. Appl. 2015)

Marchaud's theorem (1927) gives (for bounded functions

$f : [a, b] \to \mathbb{R}$) .... a property of the $(k+1)$th modulus of smoothness $\omega_{k+1}(f; t)$ which implies that f is C^k-smooth.

We have proved a generalization of Marchaud's theorem for mappings $f : X \to Y$ between *real* Banach spaces.

Let X, Y be normed linear spaces, $U \subset X$. We define the kth modulus of smoothness of $f\colon U \to Y$ by

$$\omega_k(f;t) = \sup_{\substack{\|h\| \le t \\ [x, x+kh] \subset U}} \|\Delta_h^k f(x)\|, \quad t \in [0, +\infty),$$

where $[x, x+kh]$ denotes the segment with endpoints x and $x+kh$ and $\Delta_h^n f(x)$ is the n-th difference

$$\Delta_h^n f(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(x+kh).$$

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modulus.........a function $\omega : [0,\infty) \to [0,\infty)$ with $\omega(0) = 0$ which is non-decreasing and right continuous at 0.

Let X, Y be normed linear spaces, $U \subset X$ open.
$f\colon U \to Y$ is $C^{k,\omega}$-smooth .... f is C^k-smooth and $f^{(k)}$ is uniformly continuous with modulus ω.

It should me noted that, by a different frequently used terminology, the symbol $C^{k,\omega}$ denotes the larger class

$$\widetilde{C}^{k,\omega} := \bigcup_{m>0} C^{k,m\omega}.$$

Theorem M If f is a bounded function on $[a, b]$, $k \in \mathbb{N}$, and the $(k+1)$th modulus of smoothness $\omega_{k+1}(f; t)$ is so small that

$$\eta(t) = \int_0^t \frac{\omega_{k+1}(f; s)}{s^{k+1}} \, ds < +\infty \quad \text{for} \quad t > 0,$$

then $f \in C^{k, m_k \eta}$, where $m_k > 0$ depends only on k.

A version of Marchaud's theorem for locally bounded mappings $f : X \to Y$, where X, Y are Banach spaces follows immediately from Theorem M and the following result.

Theorem 1 Let X, Y be Banach spaces, $f : X \to Y$ locally bounded, ω a modulus. Then
f is $C^{k,\omega}$-smooth on each line $\implies$ f is $C^{k, m_k \omega}$-smooth.

Generalizations of Marchaud's theorem to functions in $\mathbb{R}^n$ are well-known.

H. Johnen and K. Scherer (1977) proved Marchaud's theorem for (apriori continuous) functions on "LG-domains" $G \subset \mathbb{R}^n$.

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a) The known proofs in $\mathbb{R}^n$ need relatively difficult finite-dimensional computation and cannot be used in infinite dimensional spaces.

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b) Our proof (for apriori continuous functions) uses only a little of analysis; it is essentially based on several non-trivial but well-known properties of polynomials in Banach spaces.

c) Our proof gives Marchaud's theorem also for functions $f : G \to \mathbb{R}$, where $G \subset \mathbb{R}^n$ is a domain ("UCC-domain") more general than LG-domain used by H. Johnen and K. Scherer.

Moreover, our proof works also for f which are apriori only locally bounded or Baire measurable.

Theorem 1 Let X, Y be Banach spaces, $f : X \to Y$ locally bounded, ω a modulus. Then
f is $C^{k,\omega}$-smooth on each line $\implies$ f is $C^{k,m_k\omega}$-smooth.

To our knowledge, Theorem 1 is new also for functions $f : \mathbb{R}^n \to \mathbb{R}$.

For continuous f, Theorem 1 can be rather easily proved using:

a) A version of the converse of Taylor theorem due to M. Johanis (2014) and

b) An old result by Mazur and Orlicz (1934) on polynomials in Banach spaces.

M. Johanis (slightly simplified) version of converse Taylor theorem:

Theorem J Let X, Y be normed linear spaces, $U \subset X$ an open convex bounded set, $f\colon U \to Y$, $k \in \mathbb{N}$, ω a modulus. Suppose that for each $x \in U$ there is a polynomial $P^x \in \mathcal{P}^k(X;Y)$ satisfying

$$(*) \qquad \|f(x+h) - P^x(h)\| \le \omega(\|h\|)\|h\|^k \quad \text{for } x+h \in U.$$

Then f is $C^{k,m\omega}$-smooth on U for some $m > 0$.

If f is $C^{k,\omega}$-smooth on U, then there exists P^x satisfying (*). However, Theorem J does not give a characterisation of the class $C^{k,\omega}$. But it yields a characterisation of the class $\widetilde{C}^{k,\omega} = \bigcup_{m>0} C^{k,m\omega}$.

The proof of Theorem J is similar to the proof of the well-known Converse Taylor theorem (L. A. Ljusternik and V. I. Sobolev (1961), F. Albrecht and H. G. Diamond (1971)) which characterize C^k mappings between normed linear spaces.
It is based on some nontrivial well-known properties of polynomials.

Theorem Let X, Y be normed linear spaces, $U \subset X$ an open set, $f\colon U \to Y$, and $k \in \mathbb{N}$. TFAE:

(i) $f \in C^k(U; Y)$

(ii) for each $x \in U$ there is a polynomial $P^x \in \mathcal{P}^k(X; Y)$ satisfying $P^x(0) = f(x)$ and

$$\lim_{\substack{(y,h)\to(x,0)\\ h\neq 0}} \frac{\|f(y+h) - P^y(h)\|}{\|h\|^k} = 0$$

The result by Mazur and Orlicz (1934):

Theorem MO Let X be Banach space, Y a normed linear space, $n \in \mathbb{N}$, and let $P\colon X \to Y$ be such that $\phi \circ P$ is Baire measurable for each $\phi \in Y^*$. TFAE:

(i) P is a continuous polynomial of degree $\leq n$,

(ii) P is a polynomial of degree $\leq n$ on each line,

(iii) $$\Delta_h^{n+1} P(x) = \sum_{k=0}^{n+1} (-1)^{n+1-k} \binom{n+1}{k} P(x+kh) = 0$$

for all $x, h \in X$.

Theorem 1 Let X be Banach space, Y a normed linear space, $f : X \to Y$ locally bounded, ω a modulus. Then f is $C^{k,\omega}$-smooth on each line $\implies$ f is $C^{k,m_k\omega}$-smooth.

Sketch of the proof for **continuous** f:

f is $C^{k,\omega}$-smooth on each line.......easily implies that for each $x \in U$ there exists $P^x : X \to Y$ which is a polynomial of degree $\leq k$ **on each line containing** 0, such that

$$(*) \quad \|f(x+h) - P^x(h)\| \leq \omega(\|h\|)\|h\|^k \quad \text{for each} \quad h \in X.$$

To be able to apply Theorem J, we need to prove that P^x is a polynomial of degree $\leq k$ on X. We prove this via Theorem MO.

We can suppose $x = 0$ and denote $P := P^x = P^0$.

Since f is continuous, it is easy to show that P is Baire measurable.

So, by Theorem MO, it is sufficient to prove that, for each $z \neq 0$ and $h \neq 0$,

$$\Delta_h^{n+1} P(z) = \sum_{k=0}^{n+1} (-1)^{n+1-k} \binom{n+1}{k} P(z + kh) = 0.$$

So fix $z \neq 0$ and $h \neq 0$ and denote $q(t) := \Delta_{th}^{n+1} P(tz)$.

So fix $z \neq 0$ and $h \neq 0$ and denote

$$q(t) := \Delta_{th}^{n+1} P(tz) = \sum_{j=0}^{k+1} (-1)^{k+1-j} \binom{k+1}{j} P(tx + jth).$$

By definition (and the fact that P is a polynomial of degree $\leq k$ on each line containing 0) ....
..............$q : \mathbb{R} \to Y$ is a polynomial of degree $\leq k$.

Using that

(a) f is $C^{k,\omega}$-smooth on each line $L_t := tz + \mathbb{R}h$ and

(b) the estimate $(*)$ $\|f(h) - P^x(h)\| \leq \omega(\|h\|)\|h\|^k$,

we easily obtain $q(t) = o(t^k)$, $t \to 0$, and so $q = 0$.

Now I present our main results (with weaker assumptions) which need some additional technical results.

Theorem 1* Let X, Y be normed linear spaces, $U \subset X$ an open set, $f\colon U \to Y$, $k \in \mathbb{N}$, and ω a modulus.
Suppose that U has the "UCC property" (e.g., U is convex bounded).
Suppose that f is $C^{k,\omega}$-smooth on every open segment in U and that either

(i) f is locally bounded, or

(ii) X is a Banach space and $\phi \circ f$ is Baire measurable for each $\phi \in Y^*$.

Then f is $C^{k,m\omega}$-smooth on U for some $m > 0$.

Theorem M* Let X, Y be normed linear spaces, $U \subset X$ an open set with the UCC property, $f\colon U \to Y$, and $k \in \mathbb{N}$. Suppose that either

(i) f is locally bounded, or

(ii) X is complete, f is bounded on every closed segment in U, and $\phi \circ f$ is Baire measurable for each $\phi \in Y^*$.

Let the $(k+1)$th modulus of smoothness $\omega_{k+1}(f;t)$ be so small that

$$\eta(t) := \int_0^t \frac{\omega_{k+1}(f;s)}{s^{k+1}}\,\mathrm{d}s < +\infty \quad \text{for} \quad t > 0.$$

Then f is $C^{k,m\eta}$-smooth on U for some $m > 0$.

a) An open $U \subset X$ has the UCC property, whenever it is convex and contains an unbounded cone.

b) Each LG domain $U \subset \mathbb{R}^n$ (used by Johnen and Scherer) has the UCC property.

For a convex bounded subset U of a normed linear space we define its "ellipticity" $e_U = \frac{\operatorname{diam} U}{\sup \{r\colon\ B(a,r) \subset U \text{ for some } a\}}$.

An open $U \subset X$ has the UCC (=uniform convex chain) property................if there exist $N \in \mathbb{N}$ and $e > 0$ such that for each $x, y \in U$ there is a polygonal path $[x_0, \dots, x_n]$, $n \leq N$, with $x = x_0$ and $y = x_n$ such that $\|x_j - x_{j-1}\| \leq \|x - y\|$ and the segment $[x_{j-1}, x_j]$ lies in an open convex bounded $V_j \subset U$ with $e_{V_j} \leq e$ for each $j = 1, \dots, n$.

Finally note that the proofs of

Theorem MO, Theorem J and a version of Theorem 1*

are contained in the recent monograph

P. Hájek and M. Johanis, *Smooth analysis in Banach spaces*, Walter de Gruyter, Berlin, 2014.