On the structure of the Almost Overcomplete and Almost Overtotal sequences in Banach spaces

C. Zanco (Università degli Studi - Milano (Italy), talk based on a joint work with V.P. Fonf (Ben-Gurion University - Beer-Sheva, Israel), J. Somaglia (Università degli Studi - Milano, Italy), S. Troyanski (Universidad de Murcia - Spain and Bulgarian Academy of Science - Sofia, Bulgaria)

Relations between Banach Space Theory and Geometric Measure Theory, University of Warwick, 8 - 12 June, 2015

Basic concept

Definition

A sequence in a Banach space X is called *overcomplete* in X whenever each of its subsequences is complete in X. A sequence in the dual space X^* is called *overtotal on* X whenever each of its subsequences is total on X.

Basic concept

Definition

A sequence in a Banach space X is called *overcomplete* in X whenever each of its subsequences is complete in X. A sequence in the dual space X^* is called *overtotal on* X whenever each of its subsequences is total on X.

J. Lyubich (1958)

Let $\{e_k\}_{k\in\mathbb{N}}$ *be any bounded sequence such that* $[\{e_k\}_{k\in\mathbb{N}}] = X$*. Then the sequence*

$$\{y_m\}_{m=2}^{\infty} = \{\sum_{k=1}^{\infty} e_k m^{-k}\}_{m=2}^{\infty}$$

is OC in X*.*

Proof

$\{y_{m_j}\}_{j=1}^{\infty}$ any subsequence of $\{y_m\}_{m=2}^{\infty} = \{\sum_{k=1}^{\infty} e_k m^{-k}\}_{m=2}^{\infty}$

$$f \in X^* \cap \{y_{m_j}\}^{\perp}$$

D the open unit disk in the complex field

$$\phi : D \to \mathbb{C},\ \ \phi(t) = \sum_{k=1}^{\infty} f(e_k) t^k$$

$$f(y_{m_j}) = \phi(1/m_j) = 0, \forall j \in \mathbb{N} \ \Rightarrow \phi \equiv 0 \Rightarrow f(e_k) = 0 \ \forall k \in \mathbb{N}$$

f arbitrarily chosen $\Rightarrow [\{y_{m_j}\}] = X$

Definition

A sequence in a Banach space X is called *almost overcomplete* in X whenever the closed linear span of each of its subsequences has finite codimension in X. A sequence in the dual space X^* is called *almost overtotal on* X whenever the annihilator (in X) of each of its subsequences has finite dimension.

definition

Definition

A sequence in a Banach space X is called *almost overcomplete* in X whenever the closed linear span of each of its subsequences has finite codimension in X. A sequence in the dual space X^* is called *almost overtotal on* X whenever the annihilator (in X) of each of its subsequences has finite dimension.

Clearly, any overcomplete $\langle$resp. overtotal$\rangle$ sequence is almost overcomplete $\langle$resp. almost overtotal$\rangle$ and the converse is not true.

Remarks

- It is easy to see that, if $\{(x_n, x_n^*)\}$ is a countable biorthogonal system, then neither $\{x_n\}$ can be almost overcomplete in $[\{x_n\}]$, nor $\{x_n^*\}$ can be almost overtotal on $[\{x_n\}]$. In particular, no almost overcomplete sequence admits basic subsequences.

Remarks

- It is easy to see that, if $\{(x_n, x_n^*)\}$ is a countable biorthogonal system, then neither $\{x_n\}$ can be almost overcomplete in $[\{x_n\}]$, nor $\{x_n^*\}$ can be almost overtotal on $[\{x_n\}]$. In particular, no almost overcomplete sequence admits basic subsequences.
- If X admits a total sequence $\{x_n^*\} \subset X^*$, then there is an overtotal sequence on X. Indeed, set $Y = [\{x_n^*\}]$: Y is a separable Banach space, so it admits an overcomplete sequence $\{y_n^*\}$. It is easy to see that $\{y_n^*\}$ is overtotal on X.

Remarks

- It is easy to see that, if $\{(x_n, x_n^*)\}$ is a countable biorthogonal system, then neither $\{x_n\}$ can be almost overcomplete in $[\{x_n\}]$, nor $\{x_n^*\}$ can be almost overtotal on $[\{x_n\}]$. In particular, no almost overcomplete sequence admits basic subsequences.
- If X admits a total sequence $\{x_n^*\} \subset X^*$, then there is an overtotal sequence on X. Indeed, set $Y = [\{x_n^*\}]$: Y is a separable Banach space, so it admits an overcomplete sequence $\{y_n^*\}$. It is easy to see that $\{y_n^*\}$ is overtotal on X.
- If X is reflexive, a sequence is almost overcomplete in X if and only if it is almost overtotal on X^*.

Compactness result (V. Fonf, C.Z., 2014)

Theorem

Each almost overcomplete bounded sequence in a Banach space as well as any sequence in a dual space that is almost overtotal on a predual space is relatively norm-compact.

On the structure of AOC sequences (V. Fonf, J. Somaglia, S. Troyanski, C.Z., 2015)

Theorem

Any (infinite-dimensional) separable Banach space X contains an AOC sequence $\{x_n\}_{n\in\mathbb{N}}$ with the following property: for each $i \in \mathbb{N}$, $\{x_n\}_{n\in\mathbb{N}}$ admits a subsequence $\{x_{n_j^{(i)}}\}_{j\in\mathbb{N}}$ such that both the following conditions are satisfied

On the structure of AOC sequences (V. Fonf, J. Somaglia, S. Troyanski, C.Z., 2015)

Theorem

Any (infinite-dimensional) separable Banach space X contains an AOC sequence $\{x_n\}_{n\in\mathbb{N}}$ with the following property: for each $i \in \mathbb{N}$, $\{x_n\}_{n\in\mathbb{N}}$ admits a subsequence $\{x_{n_j^{(i)}}\}_{j\in\mathbb{N}}$ such that both the following conditions are satisfied

a) $\operatorname{codim}_X[\{x_{n_j^{(i)}}\}_{j\in\mathbb{N}}] = i$;

On the structure of AOC sequences (V. Fonf, J. Somaglia, S. Troyanski, C.Z., 2015)

Theorem

Any (infinite-dimensional) separable Banach space X contains an AOC sequence $\{x_n\}_{n\in\mathbb{N}}$ with the following property: for each $i \in \mathbb{N}$, $\{x_n\}_{n\in\mathbb{N}}$ admits a subsequence $\{x_{n_j^{(i)}}\}_{j\in\mathbb{N}}$ such that both the following conditions are satisfied

a) $\mathrm{codim}_X[\{x_{n_j^{(i)}}\}_{j\in\mathbb{N}}] = i$;

b) $\bigcap_{i\in\mathbb{N}}[\{x_{n_j^{(i)}}\}_{j\in\mathbb{N}}] = \{0\}$.

Idea for the construction.

$\{e_k, e_k^*\}_{k\in\mathbb{N}} \subset X \times X^*$, biorthogonal system, a normalized M-basis for X. For $i = 1, 2, \ldots$ put

$$Y_i = [\{e_k\}_{k\notin\{i,i+1,i+2,\ldots,2i-1\}}]$$

Idea for the construction.

$\{e_k, e_k^*\}_{k\in\mathbb{N}} \subset X \times X^*$, biorthogonal system, a normalized M-basis for X. For $i = 1, 2, \dots$ put

$$Y_i = [\{e_k\}_{k \notin \{i, i+1, i+2, \dots, 2i-1\}}]$$

$\mathrm{codim}_X Y_i = i$

Idea for the construction.

$\{e_k, e_k^*\}_{k\in\mathbb{N}} \subset X \times X^*$, biorthogonal system, a normalized M-basis for X. For $i = 1, 2, \ldots$ put

$$Y_i = [\{e_k\}_{k\notin\{i,i+1,i+2,\ldots,2i-1\}}]$$

$\operatorname{codim}_X Y_i = i$

$$\{y_m^{(i)}\}_{m\geq 2} = \{\sum_{k=1,k\notin\{i,i+1,i+2,\ldots,2i-1\}}^{\infty} m^{-ik} e_k\}_{m\geq 2}$$

provides an OC sequence in Y_i, $i = 1, 2, \ldots$..

Idea for the construction.

$\{e_k, e_k^*\}_{k\in\mathbb{N}} \subset X \times X^*$, biorthogonal system, a normalized M-basis for X. For $i = 1, 2, ...$ put

$$Y_i = [\{e_k\}_{k\notin\{i,i+1,i+2,...,2i-1\}}]$$

$\mathrm{codim}_X Y_i = i$

$$\{y_m^{(i)}\}_{m\geq 2} = \{\sum_{k=1,k\notin\{i,i+1,i+2,...,2i-1\}}^{\infty} m^{-ik} e_k\}_{m\geq 2}$$

provides an OC sequence in Y_i, $i = 1, 2, ...$.

Order in any way the countable set $\cup_{i\in\mathbb{N},m\geq 2}\{y_m^{(i)}\}$ as a sequence $\{x_n\}_{n\in\mathbb{N}}$.

For each i, select a subsequence $\{x_{n_p^{(i)}}\}_{p\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ whose terms belong to $\{y_m^{(i)}\}_{m\geq 2}$: this last sequence being *OC* in Y_i, we have $\mathrm{codim}_X[\{x_{n_p^{(i)}}\}_{p\in\mathbb{N}}] = \mathrm{codim}_X Y_i = i$.

For each i, select a subsequence $\{x_{n_p^{(i)}}\}_{p\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ whose terms belong to $\{y_m^{(i)}\}_{m\geq 2}$: this last sequence being OC in Y_i, we have $\mathrm{codim}_X[\{x_{n_p^{(i)}}\}_{p\in\mathbb{N}}] = \mathrm{codim}_X Y_i = i$.

$$\cap_{i=1}^{\infty} Y_i = \{0\} \Rightarrow \cap_{i=1}^{\infty}[\{x_{n_p^{(i)}}\}_{p\in\mathbb{N}}] = \{0\}$$

For each i, select a subsequence $\{x_{n_p^{(i)}}\}_{p\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ whose terms belong to $\{y_m^{(i)}\}_{m\geq 2}$: this last sequence being OC in Y_i, we have $\mathrm{codim}_X[\{x_{n_p^{(i)}}\}_{p\in\mathbb{N}}] = \mathrm{codim}_X Y_i = i$.

$$\cap_{i=1}^{\infty} Y_i = \{0\} \ \Rightarrow \ \cap_{i=1}^{\infty}[\{x_{n_p^{(i)}}\}_{p\in\mathbb{N}}] = \{0\}$$

A) For some $\bar{i}$, $\{x_{n_j}\}_{j\in\mathbb{N}}$ contains infinitely many terms from $\{y_m^{(\bar{i})}\}_{m\geq 2}$: being $\{y_m^{(\bar{i})}\}_{m\geq 2}$ OC in $Y_{\bar{i}}$, we have $\mathrm{codim}_X[\{x_{n_j}\}_{j\in\mathbb{N}}] \leq \mathrm{codim}_X Y_{\bar{i}} = \bar{i}$.

B) For each i, $\{x_{n_j}\}_{j\in\mathbb{N}}$ contains at most finitely many terms from $\{y_m^{(i)}\}_{m\geq 2}$.

B) For each i, $\{x_{n_j}\}_{j\in\mathbb{N}}$ contains at most finitely many terms from $\{y_m^{(i)}\}_{m\geq 2}$.

$$f \in \{x_{n_j}\}_{j\in\mathbb{N}}^{\perp}$$

$$f(e_{\overline{k}}) \neq 0 \text{ for some index } \overline{k}$$

For $j \in \mathbb{N}$, let

$$y_{m(j)}^{(i(j))} = x_{n_j}$$

B) For each i, $\{x_{n_j}\}_{j\in\mathbb{N}}$ contains at most finitely many terms from $\{y_m^{(i)}\}_{m\geq 2}$.

$$f \in \{x_{n_j}\}_{j\in\mathbb{N}}^{\perp}$$

$$f(e_{\overline{k}}) \neq 0 \text{ for some index } \overline{k}$$

For $j \in \mathbb{N}$, let

$$y_{m(j)}^{(i(j))} = x_{n_j}$$

$$A = \{i : i = i(j),\ j \in \mathbb{N},\ i(j) > \overline{k}\}.$$

$i(j)$ goes to infinity with j, so A is infinite and we have $e_{\overline{k}} \in Y_i$ for every $i \in A$.

B) For each i, $\{x_{n_j}\}_{j\in\mathbb{N}}$ contains at most finitely many terms from $\{y_m^{(i)}\}_{m\geq 2}$.

$$f \in \{x_{n_j}\}_{j\in\mathbb{N}}^{\perp}$$

$$f(e_{\overline{k}}) \neq 0 \text{ for some index } \overline{k}$$

For $j \in \mathbb{N}$, let

$$y_{m(j)}^{(i(j))} = x_{n_j}$$

$$A = \{i : i = i(j),\ j \in \mathbb{N},\ i(j) > \overline{k}\}.$$

$i(j)$ goes to infinity with j, so A is infinite and we have $e_{\overline{k}} \in Y_i$ for every $i \in A$.
For $i \in A$, put

$$m_i = \min\{m(j) : i(j) = i,\ y_{m(j)}^{(i(j))} \in \{y_m^{(i)}\}_{m\geq 2}\}$$

From $f(x_{n_j}) = 0 \ \forall j \in \mathbb{N}$ it follows that, for each $i \in A$,

$$f(e_{\overline{k}}) = -m_i^{i\overline{k}} \sum_{k>\overline{k},\, k \notin \{i, i+1, i+2, \dots, 2i-1\}}^{\infty} m_i^{-ik} f(e_k)$$

From $f(x_{n_j}) = 0 \ \forall j \in \mathbb{N}$ it follows that, for each $i \in A$,

$$f(e_{\overline{k}}) = -m_i^{i\overline{k}} \sum_{k>\overline{k},\, k\notin\{i,i+1,i+2,\ldots,2i-1\}}^{\infty} m_i^{-ik} f(e_k)$$

$$|f(e_{\overline{k}})| \leq m_i^{i\overline{k}} \|f\| \sum_{k>\overline{k},\, k\notin\{i,i+1,i+2,\ldots,2-1\}}^{\infty} m_i^{-ik} \leq$$

$$\leq \|f\| \sum_{k=\overline{k}+1}^{\infty} m_i^{i(\overline{k}-k)} \leq 2\|f\| m_i^{-i} \ \rightarrow 0 \text{ as } i \rightarrow \infty$$

Theorem

Any (infinite-dimensional) separable Banach space X contains an AOC sequence $\{x_n\}_{n\in\mathbb{N}}$ with the following property: $\{x_n\}_{n\in\mathbb{N}}$ admits countably many subsequences $\{x_{n_j^{(i)}}\}_{j\in\mathbb{N}}$, $i = 1, 2, ...,$ such that both the following conditions are satisfied

a) $\operatorname{codim}_X[\{x_{n_j^{(i)}}\}_{j\in\mathbb{N}}] = 1$;

b) $\bigcap_{i\in\mathbb{N}}[\{x_{n_j^{(i)}}\}_{j\in\mathbb{N}}] = \{0\}$.

Put $Y_i = [\{e_k\}_{k\neq i}]$ in the previous construction.

Theorem

Let $\{x_n\}_{n\in\mathbb{N}}$ be any AOC sequence in any (infinite-dimensional) separable Banach space X and let $\{x_{n_j^{(1)}}\} \supset \{x_{n_j^{(2)}}\} \supset \{x_{n_j^{(3)}}\} \supset \ldots$ any countable family of nested subsequences of $\{x_n\}_{n\in\mathbb{N}}$. Then the increasing sequence of integers $\{\mathrm{codim}_X[\{x_{n_j^{(i)}}\}]\}_{i\in\mathbb{N}}$ is finite (so eventually constant).

Proof

$\{x_n\}_{n\in\mathbb{N}}$ an *AOC* not *OC* sequence

$\{x_{n_j^{(1)}}\}_{j\in\mathbb{N}}$ any of its subsequences whose linear span is not dense in X

Proof

$\{x_n\}_{n\in\mathbb{N}}$ an *AOC* not *OC* sequence

$\{x_{n_j^{(1)}}\}_{j\in\mathbb{N}}$ any of its subsequences whose linear span is not dense in X

Put

$$X_1 = [\{x_{n_j^{(1)}}\}_{j\in\mathbb{N}}], \qquad p_1 = \mathrm{codim}_X X_1 \geq 1.$$

Proof

$\{x_n\}_{n\in\mathbb{N}}$ an *AOC* not *OC* sequence

$\{x_{n_j^{(1)}}\}_{j\in\mathbb{N}}$ any of its subsequences whose linear span is not dense in X

Put

$$X_1 = [\{x_{n_j^{(1)}}\}_{j\in\mathbb{N}}], \qquad p_1 = \mathrm{codim}_X X_1 \geq 1.$$

If $\{x_{n_j^{(1)}}\}_{j\in\mathbb{N}}$ is *OC* in X_1 we are done; otherwise, let $\{x_{n_{j_k}^{(1)}}\}_{k\in\mathbb{N}}$ be any of its subsequences whose linear span is not dense in X_1.

Proof

$\{x_n\}_{n\in\mathbb{N}}$ an *AOC* not *OC* sequence

$\{x_{n_j^{(1)}}\}_{j\in\mathbb{N}}$ any of its subsequences whose linear span is not dense in X

Put

$$X_1 = [\{x_{n_j^{(1)}}\}_{j\in\mathbb{N}}], \qquad p_1 = \operatorname{codim}_X X_1 \geq 1.$$

If $\{x_{n_j^{(1)}}\}_{j\in\mathbb{N}}$ is *OC* in X_1 we are done; otherwise, let $\{x_{n_{j_k}^{(1)}}\}_{k\in\mathbb{N}}$ be any of its subsequences whose linear span is not dense in X_1.

Put

$$\{x_{n_{j_k}^{(1)}}\}_{k\in\mathbb{N}} = \{x_{n_j^{(2)}}\}_{j\in\mathbb{N}}, \quad X_2 = [\{x_{n_j^{(2)}}\}_{j\in\mathbb{N}}], \quad p_2 = \operatorname{codim}_X X_2 > p_1.$$

Proof

$\{x_n\}_{n\in\mathbb{N}}$ an *AOC* not *OC* sequence

$\{x_{n_j^{(1)}}\}_{j\in\mathbb{N}}$ any of its subsequences whose linear span is not dense in X

Put

$$X_1 = [\{x_{n_j^{(1)}}\}_{j\in\mathbb{N}}], \qquad p_1 = \mathrm{codim}_X X_1 \geq 1.$$

If $\{x_{n_j^{(1)}}\}_{j\in\mathbb{N}}$ is *OC* in X_1 we are done; otherwise, let $\{x_{n_{j_k}^{(1)}}\}_{k\in\mathbb{N}}$ be any of its subsequences whose linear span is not dense in X_1.

Put

$$\{x_{n_{j_k}^{(1)}}\}_{k\in\mathbb{N}} = \{x_{n_j^{(2)}}\}_{j\in\mathbb{N}}, \quad X_2 = [\{x_{n_j^{(2)}}\}_{j\in\mathbb{N}}], \quad p_2 = \mathrm{codim}_X X_2 > p_1.$$

Let

$$\{x_{n_j^{(1)}}\}_{j\in\mathbb{N}} \supset \{x_{n_j^{(2)}}\}_{j\in\mathbb{N}} \supset \dots \supset \{x_{n_j^{(i)}}\}_{j\in\mathbb{N}} \supset \dots$$

be subsequences of $\{x_n\}_{n\in\mathbb{N}}$ such that $p_i \uparrow \infty$ as $i \uparrow \infty$, where $p_i = \mathrm{codim}_X X_i$ with $X_i = [\{x_{n_j^{(i)}}\}_{j\in\mathbb{N}}]$.

$\{f_i\}_{i=1}^{\infty} \subset X^*$ such that, for each i, $f_i \in X_{i+1}^{\top} \setminus X_i^{\top}$.

$\{f_i\}_{i=1}^{\infty} \subset X^*$ such that, for each i, $f_i \in X_{i+1}^{\top} \setminus X_i^{\top}$.

For each i, let y_i be an element of the sequence $\{x_{n_j^{(i)}}\}_{j\in\mathbb{N}}$ not belonging to the sequence $\{x_{n_j^{(i+1)}}\}_{j\in\mathbb{N}}$ such that $f_i(y_i) \neq 0$.

$f_k(y_i) = 0 \ \forall k \leq i$.
WLOG we may assume $f_i(y_i) = 1$.

$\{f_i\}_{i=1}^{\infty} \subset X^*$ such that, for each i, $f_i \in X_{i+1}^{\top} \setminus X_i^{\top}$.

For each i, let y_i be an element of the sequence $\{x_{n_j^{(i)}}\}_{j\in\mathbb{N}}$ not belonging to the sequence $\{x_{n_j^{(i+1)}}\}_{j\in\mathbb{N}}$ such that $f_i(y_i) \neq 0$.

$f_k(y_i) = 0 \ \forall k \leq i$.
WLOG we may assume $f_i(y_i) = 1$.

$$g_1 = f_1, \quad g_2 = f_2 - f_2(y_1)g_1, \quad g_3 = f_3 - f_3(y_1)g_1 - f_3(y_2)g_2, \ldots$$

$$\ldots, \quad g_k = f_k - \sum_{i=1}^{k-1} f_k(y_i)g_i, \ldots.$$

$g_k(y_i) = \delta_{k,i}$ for each $k, i \in \mathbb{N}$, so actually $\{y_k, g_k\}_{k\in\mathbb{N}}$ is a biorthogonal system with $\{y_k\}_{k\in\mathbb{N}} \subset \{x_n\}_{n\in\mathbb{N}}$. This is a contradiction since $\{x_n\}_{n\in\mathbb{N}}$ was an *AOC* sequence.

Corollary

Any AOC sequence $\{x_n\}_{n\in\mathbb{N}}$ *in a separable Banach space* X *contains some subsequence* $\{x_{n_j}\}_{j\in\mathbb{N}}$ *that is OC in* $[\{x_{n_j}\}_{j\in\mathbb{N}}]$ *(with, of course,* $[\{x_{n_j}\}_{j\in\mathbb{N}}]$ *of finite codimension in* X*).*

On the structure of AOT sequences (V. Fonf, J. Somaglia, S. Troyanski, C.Z., 2015)

Theorem

Let X be any (infinite-dimensional) separable Banach space. Then there is a sequence $\{f_n\}_{n\in\mathbb{N}} \subset X^*$ that is AOT on X and, for each $i \in \mathbb{N}$, admits a subsequence $\{f_{n_j^{(i)}}\}_{j\in\mathbb{N}}$ such that both the following conditions are satisfied

a) $\dim\{f_{n_j^{(i)}}\}_{j\in\mathbb{N}}^{\top} = i$;

b) $[\bigcup_{i\in\mathbb{N}}\{f_{n_j^{(i)}}\}_{j\in\mathbb{N}}^{\top}] = X$.

Theorem

Let $\{f_n\}_{n\in\mathbb{N}}$ be any sequence AOT on any (infinite-dimensional) Banach space X and let $\{f_{n_j^{(1)}}\} \supset \{f_{n_j^{(2)}}\} \supset \{f_{n_j^{(3)}}\} \supset \ldots$ any countable family of nested subsequences of $\{f_n\}_{n\in\mathbb{N}}$. Then the increasing sequence of integers $\{\dim\{f_{n_j^{(i)}}\}^\top\}_{i\in\mathbb{N}}$ is finite (so eventually constant).

Corollary

Any AOT sequence $\{f_n\}_{n\in\mathbb{N}}$ on a Banach space X contains some subsequence $\{f_{n_j}\}_{j\in\mathbb{N}}$ that is OT on any subspace of X complemented to $\{f_{n_j}\}_{j\in\mathbb{N}}^\top$ (with, of course, $\{f_{n_j}\}_{j\in\mathbb{N}}^\top$ of finite dimension).

Compactness result for AOC sequences

Theorem

Each almost overcomplete bounded sequence in a Banach space is relatively norm-compact.

Theorem

Each almost overcomplete bounded sequence in a Banach space is relatively norm-compact.

Application

Let X be a Banach space and $\{x_n\} \subset B_X$ be a sequence that is not relatively norm-compact. Then there exists an infinite-dimensional subspace Y of X^* such that $|\{x_n\} \cap Y^\top| = \infty$. (For instance this is true for any δ-separated sequence $\{x_n\} \subset B_X$ $(\delta > 0)$.)

Proof

Let $\{x_n\}$ be an almost overcomplete bounded sequence in a (separable) Banach space $(X, \|\cdot\|)$. Without loss of generality we may assume, possibly passing to an equivalent norm, that the norm $\|\cdot\|$ is locally uniformly rotund (LUR) and that $\{x_n\}$ is normalized under that norm.

Proof

Let $\{x_n\}$ be an almost overcomplete bounded sequence in a (separable) Banach space $(X, \|\cdot\|)$. Without loss of generality we may assume, possibly passing to an equivalent norm, that the norm $\|\cdot\|$ is locally uniformly rotund (LUR) and that $\{x_n\}$ is normalized under that norm.

First note that $\{x_n\}$ is relatively weakly compact: otherwise, it is known that it should admit some subsequence that is a basic sequence, a contradiction. Hence, by the Eberlein-Šmulyan theorem, $\{x_n\}$ admits some subsequence $\{x_{n_k}\}$ that weakly converges to some point $x_0 \in B_X$.

Proof

Let $\{x_n\}$ be an almost overcomplete bounded sequence in a (separable) Banach space $(X, \|\cdot\|)$. Without loss of generality we may assume, possibly passing to an equivalent norm, that the norm $\|\cdot\|$ is locally uniformly rotund (LUR) and that $\{x_n\}$ is normalized under that norm.

First note that $\{x_n\}$ is relatively weakly compact: otherwise, it is known that it should admit some subsequence that is a basic sequence, a contradiction. Hence, by the Eberlein-Šmulyan theorem, $\{x_n\}$ admits some subsequence $\{x_{n_k}\}$ that weakly converges to some point $x_0 \in B_X$.

Two cases must now be considered.

1) $\|x_0\| < 1$. From $\|x_{n_k} - x_0\| \geq 1 - \|x_0\| > 0$, according to a well known fact, it follows that some subsequence $\{x_{n_{k_i}} - x_0\}$ is a basic sequence: hence
$\mathrm{codim}[\{x_{n_{k_{2i}}} - x_0\}] = \mathrm{codim}[\{x_{n_{k_{2i}}}\}, x_0] = \mathrm{codim}[\{x_{n_{k_{2i}}}\}] = \infty$, a contradiction.

1) $\|x_0\| < 1$. From $\|x_{n_k} - x_0\| \geq 1 - \|x_0\| > 0$, according to a well known fact, it follows that some subsequence $\{x_{n_{k_i}} - x_0\}$ is a basic sequence: hence
$\mathrm{codim}[\{x_{n_{k_{2i}}} - x_0\}] = \mathrm{codim}[\{x_{n_{k_{2i}}}\}, x_0] = \mathrm{codim}[\{x_{n_{k_{2i}}}\}] = \infty$, a contradiction.

2) $\|x_0\| = 1$. Since we are working with a LUR norm, the subsequence $\{x_{n_k}\}$ actually converges to x_0 in the norm too and we are done.

Compactness result for AOT sequences

Theorem

Le X be a separable Banach space. Any bounded sequence that is almost overtotal on X is relatively norm-compact.

Theorem

Le X be a separable Banach space. Any bounded sequence that is almost overtotal on X is relatively norm-compact.

Sketch of the proof

- Let $\{f_n\}_{n=1}^{\infty} \subset X^*$ be a bounded sequence almost overtotal on X. WLOG we may assume $\{f_n\} \subset S_{X^*}$. Let $\{f_{n_k}\}$ be any subsequence of $\{f_n\}$: since X is separable, WLOG we may assume that $\{f_{n_k}\}$ weakly* converges, say to f_0.
  Let Z be a separable subspace of X^* that is 1-norming for X. Set $Y = [\{f_n\}_{n=0}^{\infty}, Z]$. Clearly X isometrically embeds into Y^* and X is 1-norming for Y.

- There is an equivalent norm $|||\cdot|||$ on Y such that, for any sequence $\{h_k\}$ and h_0 in Y,

$$h_k(x) \to h_0(x) \ \ \forall x \in X \quad \text{implies} \quad |||h_0||| \leq \liminf |||h_k|||$$

and, in addition,

$$|||h_k||| \to |||h_0||| \quad \text{implies} \quad |||h_k - h_0||| \to 0.$$

- There is an equivalent norm $|||\cdot|||$ on Y such that, for any sequence $\{h_k\}$ and h_0 in Y,

$$h_k(x) \to h_0(x) \ \ \forall x \in X \quad \text{implies} \quad |||h_0||| \leq \liminf |||h_k|||$$

and, in addition,

$$|||h_k||| \to |||h_0||| \quad \text{implies} \quad |||h_k - h_0||| \to 0.$$

- Take such an equivalent norm on Y and set $h_k = f_{n_k}$ and $h_0 = f_0$. By (**??**), we are done if we prove that $|||h_k||| \to |||h_0|||$.
Suppose to the contrary that

$$|||f_{n_k}||| \not\to |||f_0|||.$$

- $\{n_{k_i}\}$ and $\delta > 0$ exist such that $|||f_{n_{k_i}}||| - |||f_0||| > \delta$, which forces $|||f_{n_{k_i}} - f_0||| > \delta$ for i big enough.

- $\{n_{k_i}\}$ and $\delta > 0$ exist such that $|||f_{n_{k_i}}||| - |||f_0||| > \delta$, which forces $|||f_{n_{k_i}} - f_0||| > \delta$ for i big enough.
- By W.B. Johnson - H. Rosenthal (1972) some subsequence $\{f_{n_{k_{i_m}}} - f_0\}_{m=1}^{\infty}$ is a w^*-basic sequence.

- $\{n_{k_i}\}$ and $\delta > 0$ exist such that $|||f_{n_{k_i}}||| - |||f_0||| > \delta$, which forces $|||f_{n_{k_i}} - f_0||| > \delta$ for i big enough.
- By W.B. Johnson - H. Rosenthal (1972) some subsequence $\{f_{n_{k_{i_m}}} - f_0\}_{m=1}^{\infty}$ is a w^*-basic sequence.
- For $m = 1, 2, \ldots$ put $g_m = f_{n_{k_{i_m}}}$. For some sequence $\{x_m\}_{m=1}^{\infty}$ in X,

$$\{(g_m - f_0, x_m)\}_{m=1}^{\infty} \quad \text{is a biorthogonal system.}$$

- $\{n_{k_i}\}$ and $\delta > 0$ exist such that $|||f_{n_{k_i}}||| - |||f_0||| > \delta$, which forces $|||f_{n_{k_i}} - f_0||| > \delta$ for i big enough.
- By W.B. Johnson - H. Rosenthal (1972) some subsequence $\{f_{n_{k_{i_m}}} - f_0\}_{m=1}^{\infty}$ is a w^*-basic sequence.
- For $m = 1, 2, ...$ put $g_m = f_{n_{k_{i_m}}}$. For some sequence $\{x_m\}_{m=1}^{\infty}$ in X,

$$\{(g_m - f_0, x_m)\}_{m=1}^{\infty} \quad \text{is a biorthogonal system.}$$

- Only two cases must now be considered.

- $\{n_{k_i}\}$ and $\delta > 0$ exist such that $|||f_{n_{k_i}}||| - |||f_0||| > \delta$, which forces $|||f_{n_{k_i}} - f_0||| > \delta$ for i big enough.
- By W.B. Johnson - H. Rosenthal (1972) some subsequence $\{f_{n_{k_{i_m}}} - f_0\}_{m=1}^{\infty}$ is a w^*-basic sequence.
- For $m = 1, 2, \ldots$ put $g_m = f_{n_{k_{i_m}}}$. For some sequence $\{x_m\}_{m=1}^{\infty}$ in X,

$$\{(g_m - f_0, x_m)\}_{m=1}^{\infty} \quad \text{is a biorthogonal system.}$$

- Only two cases must now be considered.
- 1) For some sequence $\{m_j\}_{j=1}^{\infty}$ we have $f_0(x_{m_j}) = 0$, $j = 1, 2, \ldots$: in this case $\{(g_{m_j}, x_{m_j})\}$ would be a biorthogonal system, contradicting the fact that $\{g_{m_j}\}$ is almost overtotal on X.

- $\{n_{k_i}\}$ and $\delta > 0$ exist such that $|||f_{n_{k_i}}||| - |||f_0||| > \delta$, which forces $|||f_{n_{k_i}} - f_0||| > \delta$ for i big enough.
- By W.B. Johnson - H. Rosenthal (1972) some subsequence $\{f_{n_{k_{i_m}}} - f_0\}_{m=1}^{\infty}$ is a w^*-basic sequence.
- For $m = 1, 2, \ldots$ put $g_m = f_{n_{k_{i_m}}}$. For some sequence $\{x_m\}_{m=1}^{\infty}$ in X,

$$\{(g_m - f_0, x_m)\}_{m=1}^{\infty} \quad \text{is a biorthogonal system.}$$

- Only two cases must now be considered.
- 1) For some sequence $\{m_j\}_{j=1}^{\infty}$ we have $f_0(x_{m_j}) = 0$, $j = 1, 2, \ldots$: in this case $\{(g_{m_j}, x_{m_j})\}$ would be a biorthogonal system, contradicting the fact that $\{g_{m_j}\}$ is almost overtotal on X.
- 2) There exists q such that for any $m \geq q$ we have $f_0(x_m) \neq 0$. The almost overtotal sequence $\{g_{3j}\}_{j=q}^{\infty}$ annihilates the subspace $W = [\{f_0(x_{3j-1}) \cdot x_{3j-2} - f_0(x_{3j-2}) \cdot x_{3j-1}\}_{j=q}^{\infty}] \subset X$: being $\{x_m\}_{m=1}^{\infty}$ a linearly independent sequence, W is infinite-dimensional, a contradiction.

THANKS FOR YOUR ATTENTION!