Metastability in a condensing zero-range process in the thermodynamic limit

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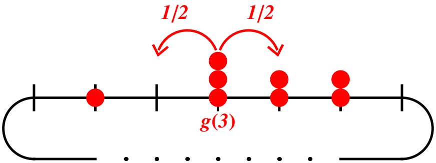
Zero-range process

Lattice: Λ of size L

State space: $X_L = \{0, 1, ..\}^\Lambda$

$$\boldsymbol{\eta} = (\eta_x)_{x\in\Lambda}$$



Jump rates: $p(x,y)\, g(\eta_x)$

choose $\quad g(k) = \left(\frac{k}{k-1}\right)^b \simeq 1 + \frac{b}{k}$ with $b > 0$

$g(0) = 0$, $g(1) = 1$

choose $\quad p(x,y) = \frac{1}{2}\delta_{y,x+1} + \frac{1}{2}\delta_{y,x-1}$

[Spitzer '70; Andjel '82; Evans '00]

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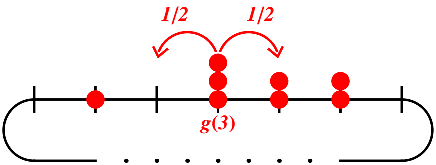

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Generator: $\mathcal{L}f(\boldsymbol{\eta}) = \sum_{x \in \Lambda_L} g(\eta_x)\left(\frac{1}{2} f(\boldsymbol{\eta}^{x,x+1}) + \frac{1}{2} f(\boldsymbol{\eta}^{x,x-1}) - f(\boldsymbol{\eta})\right)$

[Spitzer '70; Andjel '82; Evans '00]

Grand canonical invariant measures

- product measure ν_ϕ on X_L with marginals

$$\nu_\phi[\eta_x = k] = \frac{1}{z(\phi)} \frac{\phi^k}{g!(k)} \ ,$$

$\phi \leq \phi_c$ is radius of convergence of $z(\phi) = \sum_{k \geq 0} \phi^k / g!(k)$

- here $\quad g!(k) = \prod_{n=1}^{k} g(n) \propto k^b \quad$ and $\quad \phi_c = 1$

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- **density**

$$R(\phi) = \nu_\phi(\eta_x) = \frac{1}{z(\phi)} \sum_{k=0}^{\infty} k \frac{\phi^k}{g!(k)} = \frac{C}{z(\phi)} \sum_{k=0}^{\infty} k^{1-b} \phi^k\ , \quad \uparrow \text{ in } \phi$$

- **critical density** $\quad \rho_c := R(\phi_c) \in [0, \infty]$

  here $b > 2 \quad \Rightarrow \quad \rho_c < \infty \quad$ (Condensation)

Canonical measures and condensation

fixed number of particles N: $\mu_{L,N}[\,\cdot\,] = \nu_\phi[\,\cdot \mid \sum_x \eta_x = N]$

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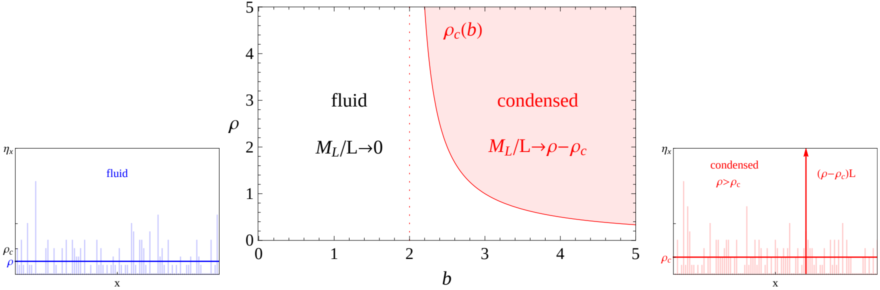
Equivalence of ensembles

In the thermodynamic limit $L, N \to \infty$, $N/L \to \rho$

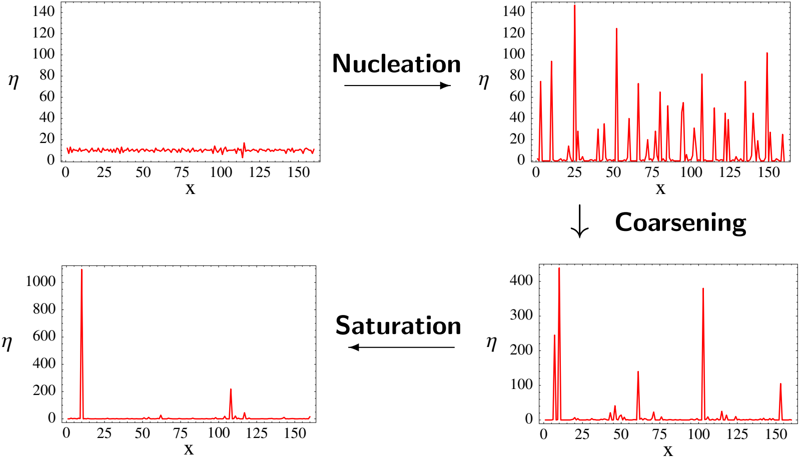
$$\mu_{L,N} \to \nu_\phi \quad \text{where} \quad \begin{cases} R(\phi) = \rho \ , \ \rho \le \rho_c \\ \phi = \phi_c \quad , \ \rho \ge \rho_c \end{cases} .$$



[Jeon, March, Pittel '00; Grosskinsky, Schütz, Spohn '03; Ferrari, Landim, Sisko '07; Armendáriz, L. '09]

Dynamics of condensation

ZRP with $g(k) \simeq 1 + b/k$



Metastability: dynamics of the condensate

Potential theoretic approach: Bovier, Gayrard, Eckhoff, Klein '01, '02,. . .
Martingale approach: Beltrán, Landim '10, '11, '15

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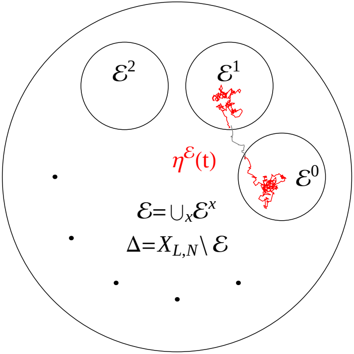
Trace process • metastable wells

$$\mathcal{E}^x := \{\eta_x \geq N - \rho_c L - \alpha_L,\, \eta_y \leq \beta_L,\, y \neq x\}\ ;$$



Trace process $\eta^{\mathcal{E}}$

- $\eta^{\mathcal{E}}$ is a Markov process on $\mathcal{E} = \cup_{x \in \Lambda} \mathcal{E}^x$ with generator $\mathcal{L}^{\mathcal{E}}$ and rates

$$r^{\mathcal{E}}(\eta, \xi) = r(\eta, \xi) + \sum_{\zeta \in \Delta} r(\eta, \zeta) \mathbb{P}_{\zeta}[T_{\mathcal{E}} = T_{\xi}]$$

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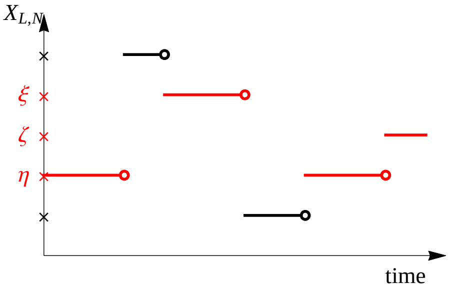

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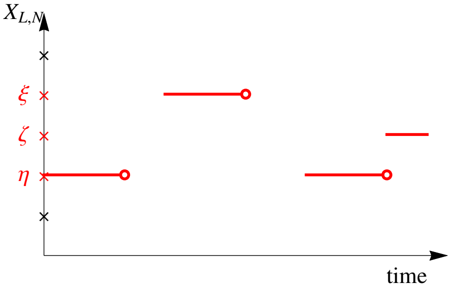

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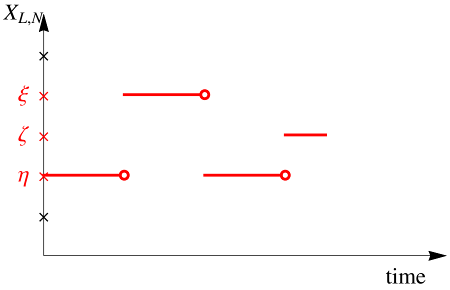

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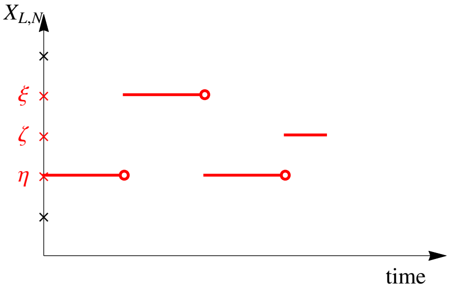
- invariant measure

$$\mu[\cdot] = \mu_{L,N}[\ \cdot \mid \mathcal{E}]$$



Main result

Theorem. [arXiv:1507.03797]

The ZRP with $b > 21$, as $L, N \to \infty$, $N/L \to \rho > \rho_c$,
exhibits metastability w.r.t. the rescaled condensate location

$$Y_t^L := \frac{1}{L} \sum_{x \in \Lambda} x \mathbb{1}_{\mathcal{E}^x}\big(\eta^{\mathcal{E}}(\theta_L t)\big) \in \mathbb{T} \quad \text{on the scale } \theta_L = L^{1+b} .$$

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For all initial conditions $\eta^L(0) \in \mathcal{E}^0$ we have $\big(Y_t^L : t \geq 0\big) \Rightarrow (Y_t : t \geq 0)$,
where $(Y_t : t \geq 0)$ is a Lévy-type process on $\mathbb{T}$ with $Y_0 = 0$ and generator

$$\mathcal{L}^{\mathbb{T}} f(u) = K_{b,\rho} \int_{\mathbb{T} \setminus \{0\}} \frac{f(v) - f(u)}{d(v,u)} \, dv \ ,$$

where $d(v,u) = |v-u|\big(1 - |v-u|\big)$ is the distance in $\mathbb{T}$. The amount of time spent outside wells is negligible.

$$\mathbb{E}_\eta \Big[\int_0^T \mathbb{1}_\Delta\big(\eta(t\theta_L)\big) dt \Big] \to 0.$$

Proof

- $\big(Y_t^L : t \geq 0\big)$ is **tight** on $D\big([0,T],\mathbb{T}\big)$
- identify limit points $\big(Y_t : t \geq 0\big)$ as solutions of the **martingale problem**

$$f(Y_t) - f(Y_0) - \int_0^t \mathcal{L}^{\mathbb{T}} f(Y_s)\, ds \quad \text{is a martingale .}$$

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Introduce **auxiliary process** $\mathcal{L}^\Lambda$ on Λ with averaged rates

$$r^\Lambda(x,y) = \frac{1}{\mu[\mathcal{E}^x]} \sum_{\eta\in\mathcal{E}^x,\, \xi\in\mathcal{E}^y} \mu[\eta]\, r^{\mathcal{E}}(\eta,\xi)\ , \quad \text{and write}$$

$$\int_0^t \Big(\mathcal{L}^{\mathbb{T}} f(Y_s^L) - \theta_L \mathcal{L}^{\mathcal{E}}(f\circ Y^L)(\eta^{\mathcal{E}}(\theta_L s))\Big) ds$$
$$= \int_0^t \Big(\mathcal{L}^{\mathbb{T}} f(Y_s^L) - \theta_L \mathcal{L}^\Lambda f(Y_s^L)\Big) ds + \theta_L \int_0^t \Big(\mathcal{L}^\Lambda f(Y_s^L) - \mathcal{L}^{\mathcal{E}}(f\circ Y^L)(\eta^{\mathcal{E}}(\theta_L s))\Big) ds$$

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1. central Lemma: **uniform bounds** on exit rates
2. Prove equilibration within wells on a scale $t_{\text{mix}} \ll \theta_L = L^{1+b}$
3. Prove convergence of averaged dynamics on the scale θ_L

1 – Coupling to a branching system of BD processes

$m = \lceil 2^b \rceil$ largest possible arrival rate for ZRP

$x \in \Lambda$, couple $\big(\eta_x(t) : t \geq 0\big)$ with a growing system of BD chains $\zeta_x^{\mathbf{k}}$,
indexed by the m-regular tree $\mathcal{R}_m$

- Each chain ζ_x has birth rate 1 and death rate $g(\zeta_x)$.
  Arrival events for $\eta_x(t)$ are used only for one of the coupled chains
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- At any time t, only m of the chains are coupled to $\eta_x(t)$, and the rest are evolving independently.
- Number of chains grows linearly with time
- $\max_{\mathbf{k}} \zeta_x^{\mathbf{k}}(t) \geq \eta_x(t)$ for all times $t \geq 0$.

Uniform exit rate bound: $$\sup_{\eta \in \mathcal{E}^x} \sum_{\xi \notin \mathcal{E}^x} r^{\mathcal{E}}(\eta, \xi) \leq C \, \frac{1}{L^5 (\log L)^2}$$

1 – Coupling to a branching system of BD processes

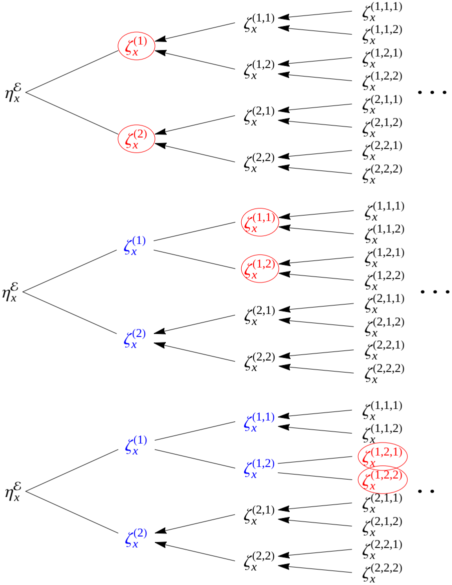
Example for $m = 2$

arrows $\rightarrow$: identical copies

coupled chains : red encircled

independent chains : in blue

- coupled at generation $n = 1$ (top)
- particle arrives at x (middle)
  chains in 1st gen. turn independent
  2 descendants on top coupled
- second particle arrives, etc.



2 – Equilibration within a well

Restricted process to a well $\mathcal{E}^x$ by ignoring jumps outside, $\mu^x = \mu[\,\cdot\,|\mathcal{E}^x]$

- bound on relaxation time t_{rel}, mixing time $t_{\mathsf{mix}}(\epsilon)$

$$t_{\mathsf{rel}} \leq CL^4 \quad \text{and} \quad t_{\mathsf{mix}}(\epsilon) \leq t_{\mathsf{rel}} \log\left(\frac{1}{\epsilon\mu_{\mathsf{min}}}\right) \leq CL^5 \log\left(1/\epsilon\right)$$

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- ergodic L^2 bound for functions with $\mu^x(h) = 0$, $x \in \Lambda$

$$\mathbb{E}_\mu \Big| \int_0^t h(\eta_u^{\mathcal{E}})\, du \Big|^2 \leq 24t\, t_{\mathsf{rel}} \sum_{x\in\Lambda} \mu\big[\mathcal{E}^x\big]\, \mu^x\big(h^2\big), \tag{1}$$

[J. Beltrán and C. Landim '15, *Martingale approach to metastability*]

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[J. Beltrán and C. Landim '15, *Martingale approach to metastability*]

- Apply (1) + 1. + bounds on $\sum_{y\neq x} r^\Lambda(x,y)$ from 2. to $h = r^{\mathcal{E}} - r^\Lambda$ to get

$$\sup_{\eta\in\mathcal{E}} \mathbb{E}_\eta \left| \theta_L \int_0^t \Big(\mathcal{L}^\Lambda f(Y_s^L) - \mathcal{L}^{\mathcal{E}}(f\circ Y^L)(\eta^{\mathcal{E}}(\theta_L s)) \Big) ds \right| \to 0$$

3 – Mean rates as capacities

$$\mu[\mathcal{E}^{A_1}]r^{\Lambda}(A_1, A_2) = \mu[\mathcal{E}^{A_1}]\,\frac{1}{|A_1|}\sum_{\substack{x\in A_1\\ y\in A_2}} r^{\Lambda}(x,y) \qquad A_1,\, A_2 \subset \Lambda$$

$$= \frac{1}{2}\Big(\mathrm{cap}\big(\mathcal{E}^{A_1}, \mathcal{E}\setminus\mathcal{E}^{A_1}\big) + \mathrm{cap}\big(\mathcal{E}^{A_2}, \mathcal{E}\setminus\mathcal{E}^{A_2}\big) - \mathrm{cap}\big(\mathcal{E}^{A_1\cup A_2}, \mathcal{E}\setminus\mathcal{E}^{A_1\cup A_2}\big)\Big)$$

[Bovier, den Hollander, *Metastability - a potential theoretic approach*]

Prove bounds

$$\theta_L\,\mathrm{cap}\big(\mathcal{E}^{A_1}, \mathcal{E}\setminus\mathcal{E}^{A_1}\big) \le K(b,\rho)\,(1+\bar{\epsilon}_L)\sum_{\substack{x\in A\\ y\notin A}} \mathrm{cap}_{\Lambda}(x,y)$$

$$\theta_L\,\mathrm{cap}\big(\mathcal{E}^{A_1}, \mathcal{E}\setminus\mathcal{E}^{A_1}\big) \ge K(b,\rho)\,(1-\underline{\epsilon}_L)\sum_{\substack{x\in A\\ y\notin A}} \mathrm{cap}_{\Lambda}(x,y)$$

where $\mathrm{cap}_{\Lambda}(x,y) = \frac{1}{|x-y|\,(L-|x-y|)}$ capacities of symmetric rw on Λ.

3 – Regularization

- Total exit rate from a well $\propto \log L$
- Upper and lower bounds for rates $r^{\Lambda}(x, y)$ do not match

see also [A. Bovier, R. Neukirch '14]

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- Coarse graining in Λ & Lipschitz test functions to regularize

$$\theta_L \mathcal{L}^{\Lambda} f(x) = \sum_{m=1}^{\bar{L}} r^{\Lambda}(V_0, V_m) \left(f\Big(\frac{x+\ell m}{L}\Big) - f\Big(\frac{x}{L}\Big) \right) + o(1)$$

with $|V_i| = \ell \propto \alpha_L \log^3 L \to \infty, \quad \bar{L} = L/\ell.$

($\to$ leads to choice of $\alpha_L = L^{1/2+5/(2b)}$)

- matching bounds from capacity representation for $r^{\Lambda}(V_0, V_m)$

$$\sup_{\eta \in \mathcal{E}} \mathbb{E}_{\eta} \Big| \int_0^t \Big(\mathcal{L}^{\mathbb{T}} f(Y_s^L) - \theta_L \mathcal{L}^{\Lambda} f(Y_s^L) \Big) ds \Big| \to 0$$

Thank you!