On spectral measures of beta ensembles

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Outline

1. **Motivation**
2. **Jacobi matrices**
3. **Gaussian beta ensembles**
4. **Wishart beta ensembles**

Random Jacobi matrices

- Random Jacobi matrix

$$J = \begin{pmatrix} a_1 & b_1 & & \\ b_1 & a_2 & b_2 & \\ & \ddots & \ddots & \ddots \\ & & b_{N-1} & a_N \end{pmatrix}, \begin{cases} \{a_i\}_{i=1}^N : \text{ real random variables,} \\ \{b_j\}_{j=1}^{N-1} : \text{ positive random variables.} \end{cases}$$

- Empirical distributions L_N

$$L_N = \frac{1}{N} \sum_{j=1}^N \delta_{\lambda_j}, \quad \Big(\{\lambda_1, \dots, \lambda_N\} \text{: (distint) eigenvalues of } J \Big).$$

- Asymptotic behaviour of L_N as $N \to \infty$, i.e., $L_N \stackrel{w}{\to} {}^\exists \mu_\infty$, or

$$\langle L_N, f \rangle = \frac{1}{N} \sum_{j=1}^N f(\lambda_j) \to \langle {}^\exists \mu_\infty, f \rangle \quad (f\text{: bounded continuous function}).$$

Gaussian beta ensembles

- Gaussian beta ensembles.

$$H_N^{(\beta)} = \frac{1}{\sqrt{\beta N}} \begin{pmatrix} \mathcal{N}(0,2) & \chi_{(N-1)\beta} & & \\ \chi_{(N-1)\beta} & \mathcal{N}(0,2) & \chi_{(N-2)\beta} & \\ & \ddots & \ddots & \ddots \\ & & \chi_\beta & \mathcal{N}(0,2) \end{pmatrix}.$$

- Joint probability density function of the eigenvalues

$$(\lambda_1, \dots, \lambda_N) \propto \prod_{i<j} |\lambda_i - \lambda_j|^\beta \exp(-\frac{\beta N}{4} \sum_{j=1}^N \lambda_j^2).$$

- Semi-circle law. $\mu_\infty = sc(x) = \sqrt{4 - x^2}/(2\pi), |x| \le 2,$

$$\frac{1}{N} \sum_{j=1}^N f(\lambda_j) \to \int_{-2}^2 f(x) \frac{\sqrt{4 - x^2}}{2\pi} dx \quad \text{almost surely.}$$

Example

- Gaussian beta ensembles (GβE). $\mu_\infty = sc$: semi-circle law;
- Wishart beta ensembles (WβE). $\mu_\infty = mp$: Marchenko-Pastur law.

- Fluctuations of eigenvalues around the limit:
  - GβE. (Johansson 98), for a 'nice' function f,

$$\sum_{j=1}^{N} \Big(f(\lambda_j) - \langle sc, f \rangle \Big) \xrightarrow{d} \mathcal{N}(0, a_f^2).$$

  - WβE. (Dumitriu & Edelman 2006), for polynomial p,

$$\sum_{j=1}^{N} \Big(p(\lambda_j) - \langle mp, p \rangle \Big) \xrightarrow{d} \mathcal{N}(0, \hat{a}_p^2).$$

Spectral measures

- μ_N: spectral measure of (J, e_1) if
$$\langle \mu_N, x^k \rangle = (J^k e_1, e_1) = J^k(1,1), k = 0, 1, \dots .$$
- $\mu_N = \sum_{j=1}^{N} q_j^2 \delta_{\lambda_j}$. 1-1 correspondence with Jacobi matrix.

Main resutls: Law of large numbers and central limit theorem.

- GβE.

$$LLN. \quad \langle \mu_N, f \rangle \to \langle sc, f \rangle, \quad (f\text{: bounded continuous function});$$
$$CLT. \quad \sqrt{N}(\langle \mu_N, p\rangle - \mathbf{E}[\langle \mu_N, p \rangle]) \xrightarrow{d} \mathcal{N}(0, \sigma_p^2), \quad (p\text{: polynomial}).$$

- WβE.

$$LLN. \quad \langle \mu_N, f \rangle \to \langle mp, f \rangle, \quad (f\text{: bounded continuous function});$$
$$CLT. \quad \sqrt{N}(\langle \mu_N, p\rangle - \mathbf{E}[\langle \mu_N, p \rangle]) \xrightarrow{d} \mathcal{N}(0, \widehat{\sigma}_p^2), \quad (p\text{: polynomial}).$$

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Spectral measures of Jacobi matrices

- Given a Jacobi matrix J, finite or infinite

$$J = \begin{pmatrix} a_1 & b_1 & & \\ b_1 & a_2 & b_2 & \\ & \ddots & \ddots & \ddots \end{pmatrix}, \quad a_i \in \mathbb{R}, b_i > 0.$$

- There is a measure μ on $\mathbb{R}$, called spectral measure of (J, e_1), s.t.

$$\langle \mu, x^k \rangle = (J^k e_1, e_1) = J^k(1,1), k = 0, 1, \dots$$

- Uniqueness is equivalent to the essential seft-adjointness of J on $\ell^2(\mathbb{N})$.

Jacobi matrices

μ: nontrivial prob. meas. on $\mathbb{R}$ s.t. $\int |x|^k d\mu(x) < \infty, k = 0, 1, \dots$.

- $\{1, x, x^2, \dots\}$ are independent in $L^2(\mathbb{R}, \mu)$.
- Define $\{P_n(x)\}_{n=0}^\infty$ as $\begin{cases} P_n(x) = x^n + \text{lower order}, \\ P_n \perp x^j, \quad j = 0, \dots, n-1. \end{cases}$
- $p_n := P_n / \|P_n\|_{L^2}$.

Theorem

(i) $xp_n(x) = b_{n+1}p_{n+1}(x) + a_{n+1}p_n(x) + b_n p_{n-1}(x), \quad n = 0, 1, \dots,$

where $b_{n+1} = \frac{\|P_n\|}{\|P_{n+1}\|}, a_{n+1} = \frac{\langle P_n, xP_n\rangle}{\|P_n\|^2}, P_{-1} \equiv 0.$

(ii) *Multiplication by x in the orthonormal set $\{p_j\}$ has the matrix*

$$J = \begin{pmatrix} a_1 & b_1 & & \\ b_1 & a_2 & b_2 & \\ & \ddots & \ddots & \ddots \end{pmatrix}$$

Jacobi matrices

- μ: trivial prob. meas.,i.e.,

$$\mu = \sum_{j=1}^{N} q_j^2 \delta_{\lambda_j}, \quad \begin{cases} \{\lambda_j\} : \text{distinct}, \\ \sum q_j^2 = 1, q_j > 0. \end{cases}$$

- $\{x^j\}_{j=0}^{N-1}$: independent in $L^2(\mathbb{R}, \mu)$. Define $P_0, \dots, P_{N-1}$. $p_n := P_n / \|P_n\|$;
-

$$J = \begin{pmatrix} a_1 & b_1 & & \\ b_1 & a_2 & b_2 & \\ & \ddots & \ddots & \ddots \\ & & b_{N-1} & a_N \end{pmatrix}$$

- $\{\lambda_j\}_{j=1}^{N}$: the eigenvalues of J, $\{v_j\}_{j=1}^{N}$: the corresponding normalized eigenvectors. Then

$$\mu = \sum_{j=1}^{N} |v_j(1)|^2 \delta_{\lambda_j} = \sum_{j=1}^{N} q_j^2 \delta_{\lambda_j}.$$

Jacobi matrix of semicircle distribution

- Semicircle distribution *sc* with density

$$sc(x) = \frac{1}{2\pi}\sqrt{4 - x^2}, \quad |x| \le 2.$$

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$$J = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & 1 & \\ & \ddots & \ddots & \ddots \end{pmatrix}.$$

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Gaussian orthogonal ensemble

GOE(N)

$$G_N^{(1)} = \begin{pmatrix} \mathcal{N}(0,2) & \mathcal{N}(0,1) & \mathcal{N}(0,1) & \dots & \mathcal{N}(0,1) \\ * & \mathcal{N}(0,2) & \mathcal{N}(0,1) & \dots & \mathcal{N}(0,1) \\ * & * & \mathcal{N}(0,2) & \dots & \mathcal{N}(0,1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \dots & \mathcal{N}(0,2) \end{pmatrix}$$

- Invariant under orthogonal conjugation,i.e.,

$$HG_N^{(1)}H^t \stackrel{d}{=} G_N^{(1)}, \quad H : \text{(deterministic) orthogonal matrix.}$$

- Joint distribution of the eigenvalues:

$$(\lambda_1, \dots, \lambda_N) \propto \prod_{i<j} |\lambda_i - \lambda_j| \exp(-\frac{1}{4} \sum_{j=1}^N \lambda_j^2).$$

Gaussian unitary ensemble

GUE(N)

$$G_N^{(2)} = \begin{pmatrix} \mathcal{N}(0,1) & \frac{\mathcal{N}(0,1)+i\mathcal{N}(0,1)}{\sqrt{2}} & \dots & \frac{\mathcal{N}(0,1)+i\mathcal{N}(0,1)}{\sqrt{2}} \\ * & \mathcal{N}(0,1) & \dots & \frac{\mathcal{N}(0,1)+i\mathcal{N}(0,1)}{\sqrt{2}} \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \dots & \mathcal{N}(0,1) \end{pmatrix}$$

- Invariant under unitary conjugation,i.e.,

$$UG_N^{(2)}U^* \overset{d}{=} G_N^{(2)}, \quad U: \text{(deterministic) unitary matrix.}$$

- Joint distribution of the eigenvalues:

$$(\lambda_1, \dots, \lambda_N) \propto \prod_{i<j} |\lambda_i - \lambda_j|^2 \exp\Big(-\frac{2}{4}\sum_{j=1}^N \lambda_j^2\Big).$$

GβE ($\beta > 0$)

- $$(\lambda_1, \dots, \lambda_N) \propto \prod_{i<j} |\lambda_i - \lambda_j|^{\beta} \exp(-\frac{\beta}{4} \sum_{j=1}^{N} \lambda_j^2).$$
- $\beta = 1$, GOE.
- $\beta = 2$, GUE.
- $\beta = 4$, GSE (Gaussian symplectic ensemble).

GOE, Jacobi/tridiagonal matrix model (Dumitriu & Edelman 2002)

- $$G_N^{(1)} = \begin{pmatrix} a_1 & x^t \\ x & B \end{pmatrix}, \quad \begin{cases} a_1 \sim \mathcal{N}(0,2), \\ x^t \sim (\mathcal{N}(0,1) \dots \mathcal{N}(0,1)), \\ B \overset{d}{=} G_{N-1}^{(1)}. \end{cases}$$
- H: $(N-1) \times (N-1)$ orthogonal matrix (depending only on x) s.t.
$$Hx = (\|x\|_2 0 \dots 0)^t = \|x\|_2 e_1.$$
- $$\begin{pmatrix} 1 & 0 \\ 0 & H \end{pmatrix} \begin{pmatrix} a_1 & x^t \\ x & B \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & H^t \end{pmatrix} = \begin{pmatrix} a_1 & \|x\|_2 e_1^t \\ \|x\|_2 e_1 & HBH^t \end{pmatrix}$$
$$\overset{d}{=} \begin{pmatrix} a_1 & \|x\|_2 & 0 & \dots & 0 \\ \|x\|_2 & & & & \\ 0 & & & & \\ \vdots & & & G_{N-1}^{(1)} & \\ 0 & & & & \end{pmatrix}$$

GOE, Jacobi/tridiagonal matrix model (Dumitriu & Edelman 2002)

- $a_1 \sim \mathcal{N}(0,2)$.
- $\|x\|_2 = \Big(\underbrace{\mathcal{N}(0,1)^2 + \cdots + \mathcal{N}(0,1)^2}_{N-1}\Big)^{1/2} \sim \chi_{N-1}$: chi distribution.
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$$G_N^{(1)} \overset{1^{st} step}{\rightsquigarrow} \begin{pmatrix} \mathcal{N}(0,2) & \chi_{N-1} & 0 & \dots & 0 \\ \chi_{N-1} & & & & \\ \vdots & & G_{N-1}^{(1)} & & \\ 0 & & & & \end{pmatrix}$$

$$\overset{finally}{\rightsquigarrow} \begin{pmatrix} \mathcal{N}(0,2) & \chi_{N-1} & & \\ \chi_{N-1} & \mathcal{N}(0,2) & \chi_{N-2} & \\ & \ddots & \ddots & \ddots \\ & & \chi_1 & \mathcal{N}(0,2) \end{pmatrix}$$

GOE, Jacobi/tridiagonal matrix model (Dumitriu & Edelman 2002)

- $^{\exists}H$: (random) orthogonal matrix s.t.

$$\begin{aligned}
&\begin{pmatrix} 1 & 0 \\ 0 & H \end{pmatrix} \left(G_N^{(1)} \right) \begin{pmatrix} 1 & 0 \\ 0 & H^t \end{pmatrix} \\
&=: J_N^{(1)} \overset{d}{=} \begin{pmatrix} \mathcal{N}(0,2) & \chi_{N-1} & & \\ \chi_{N-1} & \mathcal{N}(0,2) & \chi_{N-2} & \\ & \ddots & \ddots & \ddots \\ & & \chi_1 & \mathcal{N}(0,2) \end{pmatrix}
\end{aligned}$$

- The eigenvalues of $G_N^{(1)}$ are the same as those of $J_N^{(1)}$.
-

$$\left(G_N^{(1)} \right)^k (1,1) = \left(J_N^{(1)} \right)^k (1,1).$$

GβE, Jacobi/tridiagonal matrix model (Dumitriu & Edelman 2002)

- $H_N^{(\beta)} := \frac{1}{\sqrt{\beta N}} \begin{pmatrix} \mathcal{N}(0,2) & \chi_{(N-1)\beta} & & \\ \chi_{(N-1)\beta} & \mathcal{N}(0,2) & \chi_{(N-2)\beta} & \\ & \ddots & \ddots & \ddots \\ & & \chi_\beta & \mathcal{N}(0,2) \end{pmatrix}$
- The eigenvalues of $H_N^{(\beta)}$ have (scaled) GβE distribution,i.e.,

$$(\lambda_1, \dots, \lambda_N) \propto \prod_{i<j} |\lambda_i - \lambda_j|^{\beta} \exp(-\frac{\beta N}{4} \sum_{j=1}^{N} \lambda_j^2).$$

- $(q_1, \dots, q_N)$ is distributed as $(\chi_\beta, \dots, \chi_\beta)$ normalized to unit length, independent of $(\lambda_1, \dots, \lambda_N)$.

Empirical distributions: semi circle law & CLT (Johansson '98)

- Empirical distributions

$$L_N^{(\beta)} := \sum_{j=1}^{N} \frac{1}{N} \delta_{\lambda_j}, \quad (\lambda_1, \dots, \lambda_N): \text{ (scaled) G}\beta\text{E}.$$

- $f : \mathbb{R} \to \mathbb{R}$: bounded continuous function,

$$\langle L_N^{(\beta)}, f \rangle = \frac{1}{N} \sum_{j=1}^{N} f(\lambda_j) \to \langle sc, f \rangle \text{ a.s. as } N \to \infty.$$

- f: 'nice' function

$$\sum_{j=1}^{N} (f(\lambda_j) - \langle sc, f \rangle) \xrightarrow{d} \mathcal{N}(0, a_f^2),$$

where a_f^2 is a quadratic functional of f.

Spectral measures of GβE

- Empirical distributions

$$L_N^{(\beta)} := \sum_{j=1}^{N} \frac{1}{N} \delta_{\lambda_j}.$$

- Spectral measures

$$\mu_N^{(\beta)} = \sum_{j=1}^{N} q_j^2 \delta_{\lambda_j}.$$

  - $\mathbf{E}[q_j^2] = \frac{1}{N}$. $(q_1, \dots, q_N)$ independent of $(\lambda_1, \dots, \lambda_N)$.

Consequently, $\bar{L}_N = \bar{\mu}_N$, namely,

$$\mathbf{E}[\langle \mu_N, f \rangle] = \sum_{j=1}^{N} \mathbf{E}[q_j^2] \mathbf{E}[f(\lambda_j)] = \sum_{j=1}^{N} \frac{1}{N} \mathbf{E}[f(\lambda_j)] = \mathbf{E}[\langle L_N, f \rangle].$$

Semicircle law and CLT (Dette & Nagel (2012), D. 2015–)

Theorem

(i) *The spectral measures μ_N converges weakly to the semicircle distribution, almost surely, i.e.,*

$$\langle \mu_N, f \rangle \to \langle sc, f \rangle \text{ a.s. as } N \to \infty.$$

(ii) *For any polynomial p of positive degree,*

$$\sqrt{\beta N}\Big(\langle \mu_N, p \rangle - \mathbf{E}[]\Big) \xrightarrow{d} \mathcal{N}(0, \sigma_p^2) \text{ as } N \to \infty,$$

where $\sigma_p^2 > 0$ (does not depend on β).

- Recall that for a 'nice' function f,

$$\langle L_N^{(\beta)}, f \rangle \xrightarrow{a.s.} \langle sc, f \rangle \text{ as } N \to \infty.$$

$$N\left(\langle L_N^{(\beta)}, f \rangle - \langle sc, f \rangle\right) \xrightarrow{d} \mathcal{N}(0, a_f^2) \text{ as } N \to \infty.$$

Related results: spectral measures of Wigner matrices

- $\{\xi_{ii}\}_i$: i.i.d. $\mathbf{E}[\xi_{11}] = 0$; $\mathbf{E}[|\xi_{11}|^k] < \infty, k = 2, 3, \dots$.
- $\{\xi_{ij}\}_{i<j}$: i.i.d. $\mathbf{E}[\xi_{12}] = 0, \mathbf{E}[\xi_{12}^2] = 1$; $\mathbf{E}[|\xi_{11}|^k] < \infty, k = 3, 4, \dots$.
- X_N: symmetric matrix

$$X_N := \frac{1}{\sqrt{N}} \begin{pmatrix} \xi_{11} & \xi_{12} & \xi_{13} & \dots & \xi_{1N} \\ * & \xi_{22} & \xi_{23} & \dots & \xi_{2N} \\ * & * & \xi_{33} & \dots & \xi_{3N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \dots & \xi_{NN} \end{pmatrix}$$

  called a (real) Wigner matrix.
- ν_N: spectral measure, i.e., $\langle \nu_N, x^k \rangle = X_N^k(1, 1)$.
- ν_N converges weakly to the semicircle law.

(i) $k = 1$, $\langle \nu_N, x \rangle = X_N(1,1) = \frac{\xi_{11}}{\sqrt{N}}$. Therefore

$$\sqrt{N} \left(\langle \nu_N, x \rangle - \mathbf{E}[\langle \nu_N, x \rangle] \right) = \xi_{11} \xrightarrow{d} \xi_{11}.$$

In general, the limit distribution is not Gaussian.

(ii) $k = 2$, $\langle \nu_N, x^2 \rangle = X_N^2(1,1) = \sum_{i=1}^N \frac{1}{N} \xi_{1i}^2$. We consider

$$\begin{aligned} &\sqrt{N} \left(\langle \nu_N, x^2 \rangle - \mathbf{E}[\langle \nu_N, x^2 \rangle] \right) \\ &\quad = \frac{\xi_{11}^2 - \mathbf{E}[\xi_{11}^2]}{\sqrt{N}} + \frac{\sqrt{N-1}}{\sqrt{N}} \frac{(\xi_{12}^2 - 1) + (\xi_{13}^2 - 1) + \cdots + (\xi_{1N}^2 - 1)}{\sqrt{N-1}} \\ &\quad \xrightarrow{d} \mathcal{N}(0, \mathbf{E}[(\xi_{12}^2 - 1)^2]). \end{aligned}$$

The limit distribution is Gaussian!!

Theorem (Pizzo et al. (J. Stat. Phys. 2012), D. (Osaka J. Math. 2016))

Let $$S_{N,k} := \sqrt{N}\left(\langle \nu_N, x^k\rangle - \mathbf{E}[]\right).$$

(i) $k = 3, 5, \dots,$
$$S_{N,k} \xrightarrow{d} c_k Y + Z_k,$$
where c_k *is a constant,* $Y \sim \xi_{11}, Z_k \sim \mathcal{N}(0, a_k^2)$ *and* Y *and* Z_k *are independent.*

(ii) $k = 2, 4, \dots,$
$$S_{N,k} \xrightarrow{d} Z_k \sim \mathcal{N}(0, a_k^2).$$

(iii) *Multidimensional version. There are jointly Gaussian random variables* $\{Z_k\}$ *independent of* $Y \sim \xi_{11}$ *such that*
$$\{S_{N,k}\}_{k=1}^K \xrightarrow{d} (Y, Z_2, c_3 + Z_3, Z_4, \dots).$$

Chi distribution

What is $\chi_n, n > 0$?

- $n = 1, 2, \dots,$

$$\Big(\underbrace{\mathcal{N}(0,1)^2 + \cdots + \mathcal{N}(0,1)^2}_{n}\Big)^{1/2} \sim \chi_n.$$

- For $n > 0$,

$$p.d.f. = \frac{2}{\Gamma(\frac{n}{2})} x^{n-1} e^{-x^2}, \quad x > 0.$$

- $$\chi_n = \sqrt{2}\,\Gamma\left(\frac{n}{2}, 1\right)^{1/2},$$

  where $\Gamma(k, \theta)$ denotes the gamma distribution.

- As $n \to \infty$,

$$\chi_n - \sqrt{n} \xrightarrow{d} \mathcal{N}(0, \frac{1}{2}).$$

Decomposition of GβE matrix

$$\frac{1}{\sqrt{\beta N}}\begin{pmatrix} \mathcal{N}(0,2) & \chi_{(N-1)\beta} & & & \\ \chi_{(N-1)\beta} & \mathcal{N}(0,2) & \chi_{(N-2)\beta} & & \\ & \ddots & \ddots & \ddots & \\ & & & \chi_\beta & \mathcal{N}(0,2) \end{pmatrix}$$

$$\approx \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & 1 & \\ & \ddots & \ddots & \ddots \end{pmatrix} + \frac{1}{\sqrt{\beta N}}\begin{pmatrix} \mathcal{N}(0,2) & \mathcal{N}(0,\frac{1}{2}) & & \\ \mathcal{N}(0,\frac{1}{2}) & \mathcal{N}(0,2) & \mathcal{N}(0,\frac{1}{2}) & \\ & \ddots & \ddots & \ddots \end{pmatrix}$$

This implies the semi-circle law and the CLT for $\langle \mu_N, p \rangle$ with polynomial p.

- Variance formula:

$$\mathrm{Var}[\langle \mu_N, f\rangle] = \frac{\beta N}{\beta N + 2} \mathrm{Var}[\langle L_N, f\rangle] + \frac{2}{N\beta + 2} \left(\mathbf{E}[\langle \mu_N, f^2\rangle] - \mathbf{E}[\langle \mu_N, f\rangle]^2 \right).$$

- Consequently

$$N\beta \, \mathrm{Var}[\langle \mu_N, f\rangle] \to \langle sc, f^2\rangle - \langle sc, f\rangle^2,$$

provided that $\mathrm{Var}[\langle L_N, f\rangle] = o(N)$.

Theorem (D. 2016)

For a 'nice' function f,

$$\frac{\sqrt{N\beta}}{\sqrt{2}} \left(\langle \mu_N, f\rangle - \mathbf{E}[\langle \mu_N, f\rangle] \right) \xrightarrow{d} \mathcal{N}(0, \sigma_f^2),$$

where $\sigma_f^2 = \langle sc, f^2\rangle - \langle sc, f\rangle^2$.

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Jacobi matrices for WβE

- $m \in \mathbb{N}, n > m - 1,$

$$B_m = \begin{pmatrix} \chi_{\beta n} & & & \\ \chi_{\beta(m-1)} & \chi_{\beta(n-1)} & & \\ & \ddots & \ddots & \\ & & \chi_\beta & \chi_{\beta(n-m+1)} \end{pmatrix}, W_m = B_m B_m^t.$$

- Eigenvalues of W_m, $(a = \beta n/2, p = 1 + \beta(m-1)/2)$,

$$(\lambda_1, \dots, \lambda_m) \propto |\Delta(\lambda)|^\beta \prod_{i=1}^m \lambda_i^{a-p} \exp\left(-\frac{1}{2}\sum_{i=1}^m \lambda_i\right).$$

- $(q_1, \dots, q_m)$ is distributed as $(\chi_\beta, \dots, \chi_\beta)$ normalized to unit length, independent of $(\lambda_1, \dots, \lambda_m)$.

Decomposition of Wishart beta ensemble matrices

- $m, n \to \infty, m/n \to \gamma \in (0,1)$, $c_j \sim \chi_{\beta(n-j+1)}$, $d_j \sim \chi_{\beta(m-j)}$

$$\widetilde{W}_m = \frac{1}{\beta n} \begin{pmatrix} c_1^2 & c_1 d_1 & & \\ c_1 d_1 & c_2^2 + d_1^2 & c_2 d_2 & \\ & \ddots & \ddots & \ddots \\ & & c_{m-1} d_{m-1} & c_m^2 + d_{m-1}^2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \sqrt{\gamma} & & \\ \sqrt{\gamma} & 1+\gamma & \sqrt{\gamma} & \\ & \ddots & \ddots & \ddots \end{pmatrix} + \frac{1}{\sqrt{\beta n}} \begin{pmatrix} 2\bar{c}_1 & \sqrt{\gamma}\bar{c}_1 + \bar{d}_1 & & \\ \sqrt{\gamma}\bar{c}_1 + \bar{d}_1 & 2(\bar{c}_2 + \sqrt{\gamma}\bar{d}_1) & \sqrt{\gamma}\bar{c}_2 + \bar{d}_2 & \\ & \ddots & \ddots & \ddots \end{pmatrix},$$

- $\bar{c}_i, \bar{d}_j$: i.i.d. $\sim \mathcal{N}(0, \frac{1}{2})$.

$$\begin{cases} c_j \approx \sqrt{\beta n} + \bar{c}_j, \quad d_j \approx \sqrt{\beta n}\sqrt{\gamma} + \bar{d}_j, \\ c_j^2 \approx \beta n + 2\sqrt{\beta n}\bar{c}_j, \quad d_j^2 \approx \beta n \gamma + 2\sqrt{\beta n}\sqrt{\gamma}\bar{d}_j, \\ c_j d_j \approx \beta n \sqrt{\gamma} + \sqrt{\beta n}(\sqrt{\gamma}\bar{c}_j + \bar{d}_j). \end{cases}$$

- For more details, see arXiv:1601.01146

Thank you very much for your attention!