Propagation of singular behavior in UE and GUE sums

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Random matrix theory and strongly correlated systems

University of Warwick, UK, 21 March 2016

1. Unitary Ensemble plus GUE

Unitary ensemble (UE)

M is random $n \times n$ Hermitian matrix from UE

$$\frac{1}{Z_n} e^{-n \operatorname{Tr} V(M)} dM$$

- **Eigenvalues have joint p.d.f.**

$$\frac{1}{\tilde{Z}_n} \Delta_n(x)^2 \prod_{j=1}^{n} e^{-nV(x_j)}, \qquad \Delta_n(x) = \prod_{1 \leq i < j \leq n} (x_j - x_i)$$

- **Equilibrium measure μ_V is the minimizer of**

$$\iint \log \frac{1}{|s-t|} d\mu(s) d\mu(t) + \int V(t) d\mu(t)$$

- **μ_V is limiting eigenvalue distribution as $n \to \infty$**

GUE matrix

H **is** $n \times n$ **(scaled) GUE matrix with distribution**

$$\frac{1}{Z_n} e^{-\frac{n}{2} \operatorname{Tr} H^2} dH$$

- **Eigenvalues of** $\sqrt{\tau} H$ **follow semi-circle law with variance** τ

$$d\lambda_\tau(s) = \frac{1}{2\pi\tau} \sqrt{4\tau - s^2}\, ds, \qquad s \in [-2\sqrt{\tau}, 2\sqrt{\tau}].$$

Sum of UE and GUE

We are interested in eigenvalues of

$$X = M + \sqrt{\tau} H$$

with M from a unitary ensemble, H from a (scaled) GUE, independent from M, and $\tau > 0$.

- Eigenvalues are determinantal point process
- Interpretation as non-intersecting paths
- Free probability:

  $$\mu_V \boxplus \lambda_\tau$$

  is limiting distribution of eigenvalues as $n \to \infty$

Determinantal point process

Brézin and Hikami (1998), Zinn-Justin (1998), Johansson (2001)

- **If M is fixed with eigenvalues $a_1, \ldots, a_n$, then eigenvalues of $M + \sqrt{\tau}H$ have joint density**

$$\propto \frac{1}{\Delta_n(a)} \cdot \Delta_n(x) \cdot \det \left[e^{-\frac{n}{2\tau}(x_k - a_j)^2} \right]_{j,k=1}^{n}$$

- **If M is random from UE, then after averaging over $a_1, \ldots, a_n$,**

$$\propto \Delta_n(x) \cdot \det \left[\int_{-\infty}^{\infty} a^{j-1} e^{-\frac{n\tau}{2}(x_k - a)^2} e^{-nV(a)} da \right]_{j,k=1}^{n}$$

- **It is polynomial ensemble (special case of DPP)**

Non-intersecting paths

Johansson (2001), Bleher and Kuijlaars (2004):

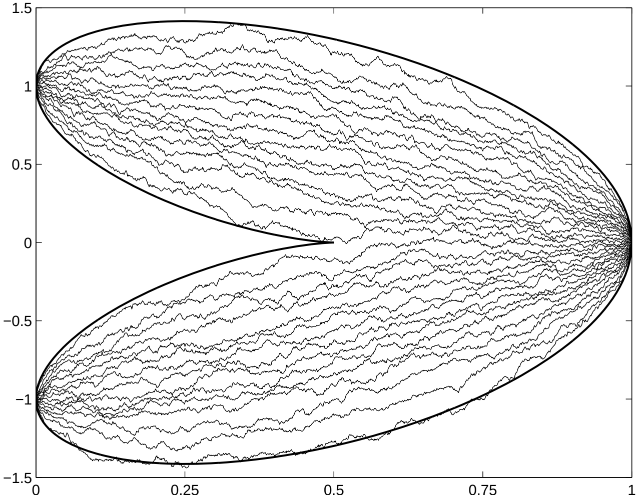
- **Non-intersecting Brownian bridges with starting positions $a_1, \ldots, a_n$ at time $t = 0$ and ending positions $b_1, \ldots, b_n$ at time $t = T$**
- **Joint density for particles at time $t \in (0, T)$:**

$$\propto \frac{1}{\Delta_n(a)\Delta_n(b)} \cdot \det\left[e^{-\frac{n}{2t}(a_j - x_k)^2}\right]_{j,k=1}^{n} \cdot \det\left[e^{-\frac{n}{2(T-t)}(x_j - b_k)^2}\right]_{j,k=1}^{n}$$

- **In limit when all $b_k \to 0$**

$$\propto \frac{1}{\Delta_n(a)} \cdot \det\left[e^{-\frac{n}{2t}(a_j - x_k)^2}\right]_{j,k=1}^{n} \cdot \Delta_n(x) \cdot \prod_{j=1}^{n} e^{-\frac{n}{2(T-t)} x_j^2}$$

Figure



Picture if all $a_j \to \pm 1$.

Random starting points

- **If** a_j **are random eigenvalues of matrix from UE ensemble** $\frac{1}{Z_n} e^{-nV(M)} dM$ **then, after averaging over** $a_1, \ldots, a_n$**,**

$$\propto \Delta_n(x) \cdot \det \left[\int_{-\infty}^{\infty} a^{j-1} e^{-\frac{n}{2t}(x_k - a)^2} e^{-nV(a)} da \right]_{j,k=1}^{n} \cdot \prod_{j=1}^{n} e^{-\frac{n}{2(T-t)} x_j^2}$$

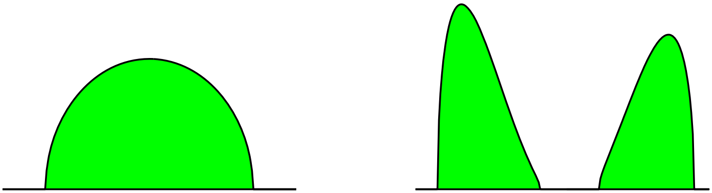
- **For** $T \to \infty$**, this is exactly the same as eigenvalues of** $M + \sqrt{t} H$**.**

2. Singular potential

Typical behavior of equilibrium measure

Suppose V is real analytic.

- μ_V is supported on finite union of intervals with density ψ_V
- Generically ψ_V has square root vanishing at endpoints, and is positive in the interior of each of the intervals.



Singular behavior

Singular interior point

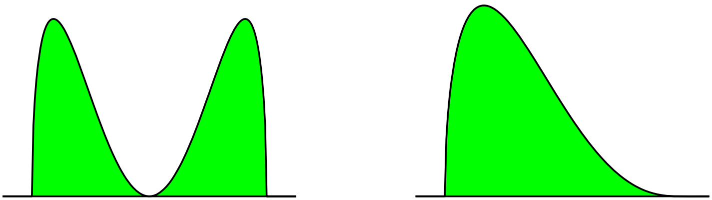
- ψ_V **vanishes at interior point** x^*

$$\psi_V(x) \sim (x - x^*)^\kappa, \qquad \kappa = 2k \text{ for integer } k \geq 1.$$

Singular edge point

- ψ_V **vanishes to higher order at edge point** x^*

$$\psi_V(x) \sim |x - x^*|^\kappa, \qquad \kappa = 2k + \tfrac{1}{2} \text{ for integer } k \geq 1.$$



Correlation kernels

Limiting correlation kernels

- **Sine kernel** at regular interior point
- **Airy kernel** at regular edge point
- **Painlevé kernels** at singular points

3. Results

(a) **Propagation of singular density for $\tau < \tau_{cr}$**
(b) **Propagation of correlation kernel**
(c) **Density at critical τ_{cr}**

4 Propagation of singular density

Propagation of singular density

Suppose μ_V **has density**

$$\psi_V(s) = c_0^{\kappa+1}|s - x^*|^\kappa\, h(s), \qquad h(x^*) = 1,$$

where h **is real analytic at** x^***. Define**

$$\tau < \tau_{cr} = \left[\int \frac{d\mu_V(s)}{(s - x^*)^2}\right]^{-1} \text{ and } x_\tau^* = x^* - \tau \int \frac{d\mu_V(s)}{s - x^*}$$

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Theorem

$\mu_\tau = \mu_V \boxplus \lambda_\tau$ **has density** ψ_τ **satisfying**

$$\psi_\tau(s) = c_\tau^{\kappa+1}|s - x^*_\tau|^\kappa h_\tau(s), \quad c_\tau = \frac{\tau_{cr}}{\tau_{cr} - \tau} c_0, \quad h_\tau(x^*) = 1$$

where h_τ **is real analytic at** x^*_τ

Propagation of singular density

The singularity at an interior point or edge point propagates in the model $M + \sqrt{\tau} H$.

- **Interpretation in terms of non-intersecting Brownian paths**
- **Connection with two-matrix model**

Duits (2014)

About the proof

Biane (1997) describes how to calculate the density of

$$\mu_\tau = \mu_V \boxplus \lambda_\tau$$

The mapping

$$w \mapsto w + \tau \int \frac{d\mu_V(s)}{w - s}$$

has an inverse $z \mapsto F_\tau(z)$ that extends continuously and injectively to $\overline{\mathbb{C}^+}$. Then

$$\psi_\tau(x) = -\frac{1}{\pi} \operatorname{Im} \left(\int \frac{d\mu_V(s)}{F_\tau(x) - s} \right), \qquad x \in \mathbb{R}.$$

About the proof

$$\psi_\tau(x) = -\frac{1}{\pi} \operatorname{Im} \left(\int \frac{d\mu_V(s)}{F_\tau(x) - s} \right), \qquad x \in \mathbb{R}.$$

- **Expand** $w + \tau \int \frac{d\mu_V(s)}{w-s}$ **around** $w = x^*$
- **Expand its inverse** $F_\tau(z)$ **around** $z = x_\tau^*$
- **Combine this to find first non-zero term in expansion of** $\psi_\tau(x)$ **around** $x = x_\tau^*$**.**

5 Propagation of correlation kernel

Propagation of correlation kernel

- M **is from unitary ensemble with eigenvalue correlation kernel** $K_n^M(x,y)$ **and scaling limit**

$$\lim_{n\to\infty} \frac{1}{c_0 n^\gamma} K_n^M\left(x^* + \frac{x}{c_0 n^\gamma}, x^* + \frac{y}{c_0 n^\gamma}\right) = \mathcal{K}_{crit,\kappa}(x,y)$$

with $\gamma = (\kappa+1)^{-1}$

- $X = M + \sqrt{\tau} H$ **has correlation kernel** $K_n^X(x,y)$

Propagation of correlation kernel

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  with $\gamma = (\kappa+1)^{-1}$
- $X = M + \sqrt{\tau} H$ has correlation kernel $K_n^X(x,y)$

Theorem

Under these conditions

$$\lim_{n\to\infty} \frac{e^{-H_n(x)+H_n(y)}}{c_\tau n^\gamma} K_n^X\left(x_\tau^* + \frac{x}{c_\tau n^\gamma}, x_\tau^* + \frac{y}{c_\tau n^\gamma}\right) = \mathcal{K}_{crit,\kappa}(x,y)$$

for certain function H_n

About the proof

For $X = M + \sqrt{\tau} H$**,**

$$K_n^X(x,y) = \frac{n}{2\pi i \tau} \int_{x^*-i\infty}^{x^*+i\infty} ds \int_{-\infty}^{\infty} dt \, K_n^M(s,t)$$

$$\times \, e^{\frac{n}{2}(V(s)-V(t))} e^{\frac{n}{2\tau}((s-x)^2-(t-y)^2)}$$

Claeys-K-Wang (2015)

- **Proof is basically a steepest descent analysis of this double integral.**

Local part and rest

Separate

$$K_n^X(x,y) = K_{n,loc}^X(x,y) + K_{n,rest}^X(x,y)$$

with

$$K_{n,loc}^X(x,y) = \frac{n}{2\pi i\tau}\int_{x^*-iRn^{-\gamma}}^{x^*+iRn^{-\gamma}} ds \int_{x^*-Rn^{-\gamma}}^{x^*+Rn^{-\gamma}} dt\, K_n^M(s,t)$$

$$\times\; e^{\frac{n}{2}(V(s)-V(t))} e^{\frac{n}{2\tau}((s-x)^2-(t-y)^2)}$$

- R **is large, but fixed constant, independent of** n**, but it will depend on** x **and** y**.**

Analysis of local part

After change of variables

$$\frac{1}{c_\tau n^\gamma} K^X_{n,loc}\left(x^*_\tau + \frac{x}{c_\tau n^\gamma}, x^*_\tau + \frac{y}{c_\tau n^\gamma}\right)$$
$$= \frac{n^{1-2\gamma}}{2\pi i c_0 c_\tau} \int_{x^* - ic_0 R}^{x^* + ic_0 R} ds \int_{x^* - c_0 R}^{x^* + c_0 R} dt\, e^{F_n(s,x) - F_n(t,y)}$$
$$\times \underbrace{\frac{1}{c_0 n^\gamma} K^M_n\left(x^* + \frac{s}{c_0 n^\gamma}, x^* + \frac{t}{c_0 n^\gamma}\right)}_{\to \mathcal{K}_{crit,\kappa}(s,t)}$$

with

$$F_n(s,x) = \frac{n}{2} V\left(x^* + \frac{s}{c_0 n^\gamma}\right) + \frac{n}{2\tau}\left(\frac{s}{c_0 n^\gamma} - \frac{\tau}{2} V'(x^*) - \frac{x}{c_\tau n^\gamma}\right)$$

Taylor expansion of F_n

Expand as $n \to \infty$

$$F_n(s,x) = \frac{n}{2}V(x^*) + \frac{\tau n}{8}V'(x^*)^2 + \frac{n^{1-\gamma}}{2c_\tau}V'(x^*)x + \frac{n^{1-2\gamma}}{4c_0c_\tau}V''(x^*)x^2 + \frac{n^{1-2\gamma}}{2\tau c_0 c_\tau}\left[(s-x)^2 + O(n^{-\gamma})\right]$$

- **No term $sn^{-\gamma}$ because of choice for x^*_τ**
- **Complete square $(s-x)^2$ because of choice for c_τ**
- **Saddle point equation $\frac{\partial F_n}{\partial s} = 0$ gives the saddle**

$$x + O(n^{-\gamma})$$

Analysis of rest

$$K^X_{n,rest}(x,y) = K^X_n(x,y) - K^X_{n,loc}(x,y)$$

Scaled version

$$e^{-H_n(x)+H_n(y)} K^X_{n,rest}\left(x^*_\tau + \frac{x}{c_\tau n^\gamma}, x^*_\tau + \frac{y}{c_\tau n^\gamma}\right)$$

becomes $O(e^{-cn^{1-\gamma}})$ as $n \to \infty$ if R is large enough.

- **Proof uses deformation of t-contour into the complex plane, and uses bounds on orthogonal polynomials that come from steepest descent analysis of Riemann-Hilbert problem**

6 Density at critical τ_{cr}

Density at critical τ_{cr}

What about the density for critical $\tau_{cr} = \left[\frac{d\mu_V(s)}{(s-x^*)^2}\right]^{-1}$**?**

Density at critical τ_{cr}

What about the density for critical $\tau_{cr} = \left[\dfrac{d\mu_V(s)}{(s-x^*)^2} \right]^{-1}$**?**

We have results for

Singular interior point $\psi_V(s) \sim (s-x^*)^{2k}$ **as** $s \to x^*$

- $\kappa = 2k = 2$
- $\kappa = 2k \geq 4$ **and** $\displaystyle\int \frac{d\mu_V(s)}{(s-x^*)^3} = 0$
- $\kappa = 2k \geq 4$ **and** $\displaystyle\int \frac{d\mu_V(s)}{(s-x^*)^3} \neq 0$

Singular edge point $\psi_V(s) \sim (x^* - s)^{2k+\frac{1}{2}}$ **as** $s \to x^*-$

Density at critical τ_{cr}

Singular interior point with exponent $\kappa = 2k = 2$

Theorem

$$\psi_{\tau_{cr}}(s) = A_{\pm} \left| s - x^*_{\tau_{cr}} \right|^{1/2} + O(s - x^*_{\tau_{cr}}), \qquad \textbf{as } s \to x^*_{\tau_{cr}} \pm$$

where

$$\begin{pmatrix} A_- \\ A_+ \end{pmatrix} = \frac{1}{(\pi \tau_{cr} c_0)^{3/2} (1 + (\frac{1}{\pi} \fint \frac{h(s)}{s - x^*} ds)^2)^{1/4}} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$\theta = \frac{\pi}{4} + \arctan\left(\frac{1}{\pi} \fint \frac{h(s)}{s - x^*} ds \right)$$

Density at critical τ_{cr}

Singular interior point with exponent $\kappa = 2k \geq 4$.

Theorem

Suppose $\displaystyle\int \frac{d\mu_V(s)}{(s-x^*)^3} ds = 0.$ **Then**

$$\psi_{\tau_{cr}}(s) = A\left|s - x^*_{\tau_{cr}}\right|^{1/3} + O(|s - x^*_{\tau_{cr}}|^{2/3}), \qquad \textbf{as } s \to x^*_{\tau_{cr}}$$

where

$$A = \frac{\sqrt{3}}{2\pi\tau_{cr}^{4/3}\left(\displaystyle\int \frac{d\mu_V(s)}{(s-x^*)^4} ds\right)^{1/3}}$$

Density at critical τ_{cr}

Singular interior point with exponent $\kappa = 2k \geq 4$.

Theorem

Suppose $\displaystyle\int \frac{d\mu_V(s)}{(s-x^*)^3} ds > 0.$ **Then**

$$\psi_{\tau_{cr}}(s) = \begin{cases} A\left|s - x^*_{\tau_{cr}}\right|^{1/2} + O(s - x^*_{\tau_{cr}}), & \textbf{as } s \to x^*_{\tau_{cr}} + \\ B\left|s - x^*_{\tau_{cr}}\right|^{k-1/2} + O((s - x^*_{\tau_{cr}})^k) & \textbf{as } s \to x^*_{\tau_{cr}} - \end{cases}$$

where

$$A = \frac{1}{\pi \tau_{cr}^{3/2}\left(\int \frac{d\mu_V(s)}{(s-x^*)^3} ds\right)^{1/2}}, \quad B = \frac{c_0^{2k+1}}{2\tau_{cr}^{k+1/2}\left(\int \frac{d\mu_V(s)}{(s-x^*)^3} ds\right)^{k+1/2}}$$

Density at critical τ_{cr}

Singular right edge point with exponent $\kappa = 2k + \frac{1}{2}$**.**

Theorem

Suppose $\displaystyle\int \frac{d\mu_V(s)}{(x^* - s)^3} ds > 0.$ **Then**

$$\psi_{\tau_{cr}}(s) = A\left(x^*_{\tau_{cr}} - s\right)^{1/2} + O((s - x^*_{\tau_{cr}})^{3/4}) \quad \textbf{as } s \to x^*_{\tau_{cr}} -$$

where

$$A = \frac{1}{\tau_{cr}^{3/2} c_0^{1/4} \left(\int \frac{d\mu_V(s)}{(s - x^*)^3} ds\right)^{1/2}}$$

Speculations about correlation kernel

We have no results yet for the correlation kernels at critical τ_{cr}

- **Case $\kappa = 2k = 2$ leads to exponent $1/2$:**
  Tacnode kernel of Duits-Geudens (2013) !?

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- **Case** $\kappa = 2k \geq 4$ **with** $\int \frac{d\mu_V(s)}{(s - x^*)^3} = 0$ **leads to exponent** $1/3$**:** **Pearcey kernel !??**

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  Pearcey kernel !??

- **Case** $\kappa = 2k \geq 4$ **with** $\displaystyle\int \frac{d\mu_V(s)}{(s-x^*)^3} \leq 0$ **leads to two exponents** $1/2$ **and** $k - 1/2$**:**
  $\cdots$ **kernel ???**

Speculations about correlation kernel

We have no results yet for the correlation kernels at critical τ_{cr}

- **Case** $\kappa = 2k = 2$ **leads to exponent** $1/2$**:**
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- **Case** $\kappa = 2k \geq 4$ **with** $\int \frac{d\mu_V(s)}{(s - x^*)^3} = 0$ **leads to exponent** $1/3$**:**
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- **Case** $\kappa = 2k \geq 4$ **with** $\int \frac{d\mu_V(s)}{(s - x^*)^3} \leq 0$ **leads to two exponents** $1/2$ **and** $k - 1/2$**:**
  $\cdots$ **kernel ???**
- **Case** $\kappa = 2k + 1/2$ **leads to exponent** $1/2$**:**
  Airy kernel ??

Thank you for your attention