Contact topology of Gorenstein toric isolated singularities

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(I) Contact Manifolds (co-oriented)

- (M^{2n+1}, ξ), $\xi \subset TM$ s.t. $\xi = \operatorname{Ker}(\alpha)$ with $\alpha \wedge (d\alpha)^n =$ vol. form
  $\uparrow$ contact form

  Note: $f\alpha$ is also contact form, $\forall\, 0 < f \in C^\infty(M)$.

- Symplectization of $(M, \xi = \ker\alpha)$

  $$S(M)^{2n+2} \underset{\text{symp}}{\subset} (T^*M, d\lambda),\ S(M) := \bigcup_{p \in M} S_p(M) \text{ where}$$

  $$S_p(M) = \{\beta \in T_p^*M : \beta = e^t\alpha,\ t \in \mathbb{R}\} \subset T_p^*M \setminus \{0\}$$

  Note that $S(M) \underset{\alpha}{\cong} M \times \underset{\underset{t}{\Psi}}{\mathbb{R}}$, $\quad \omega = d(e^t\alpha)$

- Examples

  1) $S^{2n+1} \subset \mathbb{R}^{2(n+1)} \cong \mathbb{C}^{n+1}$, $\xi_{std} := (TS^{2n+1} \cap iTS^{2n+1})$

     $$S(S^{2n+1}) \cong (\mathbb{R}^{2(n+1)} \setminus \{0\}, \omega_{std})$$

  2) $M = \partial$ (pseudo-convex nbhd of isolated singularity in $\mathbb{C}$-manifold)

- Gray's Stability Theorem

  (M, ξ_t) contact $\forall t \in [0,1] \Rightarrow \exists\, \varphi_t : M \circlearrowleft$ s.t.

  $$(\varphi_t)_*(\xi_0) = \xi_t,\ \forall t \in [0,1].$$

  Hence, contact structures have no local invariants.

II (Cylindrical) Contact Homology

$(M,\xi) \rightsquigarrow$ pick contact form α

$\rightsquigarrow$ Reeb vector field R_α defined by

$$R_\alpha \lrcorner\, d\alpha \equiv 0 \quad \text{and} \quad \alpha(R_\alpha) \equiv 1$$

Assume:

(i) α non-degenerate, i.e. all closed R_α-orbits are non-degenerate.

(ii) For any closed R_α-orbit γ have well-defined degree $\deg(\gamma) \in \mathbb{Z}$, e.g. assume $c_1(\xi) = 0$.

(iii) $\deg(\gamma) \in 2\mathbb{Z}$, $\forall$ closed R_α-orbit γ.

Then [Eliashberg-Hofer and Pardon]

$HC_*(M,\xi) :=$ graded $\mathbb{Q}$-vector space freely generated by all closed R_α-orbits, is a contact invariant (i.e. independent of choice of any such α).

- Mean Euler Characteristic (van Koert '05)

$$\chi(M,\xi) := \lim_{N\to\infty} \frac{1}{2N} \sum_{j=-N}^{N} \dim_{\mathbb{Q}} HC_{2j}(M,\xi).$$

III (Good) Gorenstein Toric Contact Manifolds

- $(M^{2n+1}, \xi) \circlearrowleft \mathbb{T}^{n+1}$, i.e. $S(M)^{2(n+1)} \circlearrowleft \mathbb{T}^{n+1}$ is a toric symplectic cone.
- Arise as contact reductions of $(S^{2d-1} \subset \mathbb{C}^d, \xi_{std}) \circlearrowleft \mathbb{T}^d$ by $\quad \mathbb{K} := \ker(\beta : \mathbb{T}^d \longrightarrow \mathbb{T}^{n+1})$.
- Determined by $\nu_j \in \mathbb{Z}^{n+1}$, $j = 1, \dots, d$, which are also the defining normals of moment cone $\equiv$ image of moment map $\mu : S(M)^{2n+2} \longrightarrow \mathcal{L}ie^* \mathbb{T}^{n+1} \cong \mathbb{R}^{n+1}$.
- Gorenstein, i.e. $c_1(\xi) = 0$, implies w.l.o.g. that

$$\nu_j := (v_j, 1) \in \mathbb{Z}^{n+1} \text{ with } v_j \in \mathbb{Z}^n, \; j = 1, \dots, d.$$

- Smoothness requires that each facet of

$$D := \operatorname{conv}(v_1, \dots, v_d) \subset \mathbb{R}^n$$

is $\operatorname{Aff}(n, \mathbb{Z})$ - equivalent to $\operatorname{conv}(e_1, \dots, e_n)$ with $\{e_1, \dots, e_n\}$ = standard $\mathbb{Z}$-basis of $\mathbb{Z}^n \subset \mathbb{R}^n$,

i.e. $\quad D \subset \mathbb{R}^n$ is a toric diagram.

Notation: $D \rightsquigarrow (M_D, \xi_D)$.

- Examples

a) $D = \operatorname{conv}(\vec{0}, e_1, \dots, e_{n-1}, e_n) \subset \mathbb{R}^n \rightsquigarrow (S^{2n+1}, \xi_{std})$

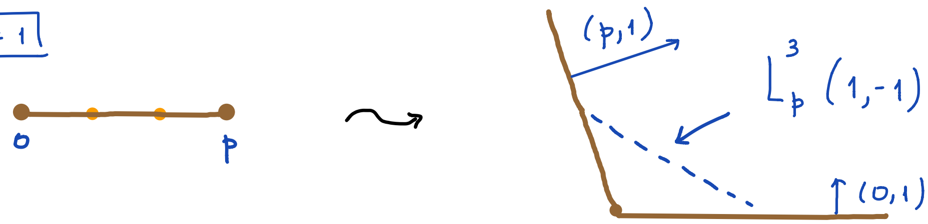
b) $D = \mathrm{conv}(\vec{0}, e_1, \dots, e_{n-1}, v) \subset \mathbb{R}^n$ with

$$v = (\alpha_1, \dots, \alpha_{n-1}, p) \in \mathbb{Z}^{n-1} \times \mathbb{N} \subset \mathbb{Z}^n$$

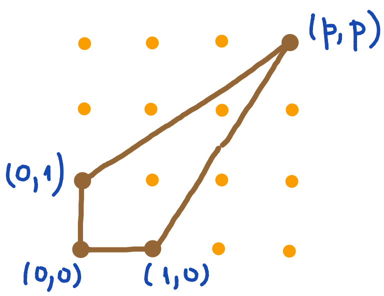
$$\Downarrow \quad (\alpha_1, p) = \dots = (\alpha_{n-1}, p) = (\alpha_n, p) = 1$$

$$L_p^{2n+1}\Big(1, -\alpha_1, \dots, -\alpha_{n-1}, \underbrace{\Big(\sum_{j=1}^{n-1} \alpha_j\Big) - 1}_{=\alpha_n}\Big)$$

$\boxed{n=1}$



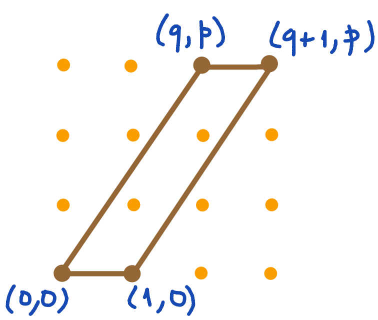
c)



$(S^2 \times S^3, \xi_p)$, $p \in \mathbb{N}$

($p = 1$ gives unit co-sphere bundle of S^3, i.e. conifold)

d) [Moreira]



$(p,q) = 1$

unit co-sphere bundle of $L_p^3(1,q)$

IV $HC_*(M_D, \xi_D)$

- $D \subset \mathbb{R}^n \rightsquigarrow (M_D^{2n+1}, \xi_D) \circlearrowleft \mathbb{T}^{n+1}$

  $\rightsquigarrow$ toric vector fields $R_\nu \in \mathfrak{X}(M, \xi)$ parametrized by $\nu \in \mathbb{L}ie(\mathbb{T}^{n+1}) \cong \mathbb{R}^{n+1}$

- [Martelli - Sparks - Yau '06]

  $\mathrm{int}(D) \subset \mathbb{R}^n$ parametrizes (normalized) toric Reeb vector fields:

  $$\mathfrak{r} = (r_1, \dots, r_n) \in \mathrm{int}(D) \rightsquigarrow \nu = (r_1, \dots, r_n, 1) \in \mathbb{L}ie(\mathbb{T}^{n+1})$$
  $$\rightsquigarrow R_\nu$$

- [A. - Macarini]

  R_ν non-degenerate $\Longleftrightarrow$ r_j's irrational and $\mathbb{Q}$-independent

  In that case:

  (i) simple closed R_ν-orbits $\gamma_1, \dots, \gamma_m$ $\underset{1:1}{\longleftrightarrow}$ facets of $D \subset \mathbb{R}^n$ $F_1, \dots, F_m$

  (ii) $\deg(\gamma_j^\ell) \in 2\mathbb{Z}$, $j = 1, \dots, n$, $\ell \in \mathbb{N}$

  (iii) Computable $HC_*(M_D, \xi_D)$.

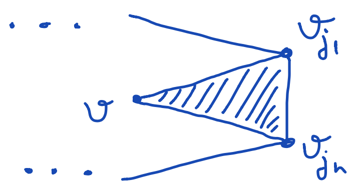
(V) Application 1

Thm. [A.–Macarini '16]

$$\chi(M_D, \xi_D) = \frac{n!\, \mathrm{Vol}(D)}{2}$$

"Proof":

(i) D



$F_j = \mathrm{conv}(v_{j1}, \dots, v_{jn})$

$\updownarrow$ 1:1

γ_j

$$\frac{n!}{2} \mathrm{Vol}\big(\mathrm{conv}(v, F_j)\big) = \frac{1}{\Delta(\gamma_j)}, \quad \text{where}$$

$$\Delta(\gamma_j) \equiv \text{mean index of } \gamma_j := \lim_{\ell \to \infty} \frac{\deg(\gamma_j^{\ell})}{\ell}$$

(ii) Resonance Relation [Ginzburg - Kerman '10]

$$\Rightarrow \sum_{j=1}^{m} \frac{1}{\Delta(\gamma_j)} = \chi(M_D, \xi_D).$$

Q.E.D.

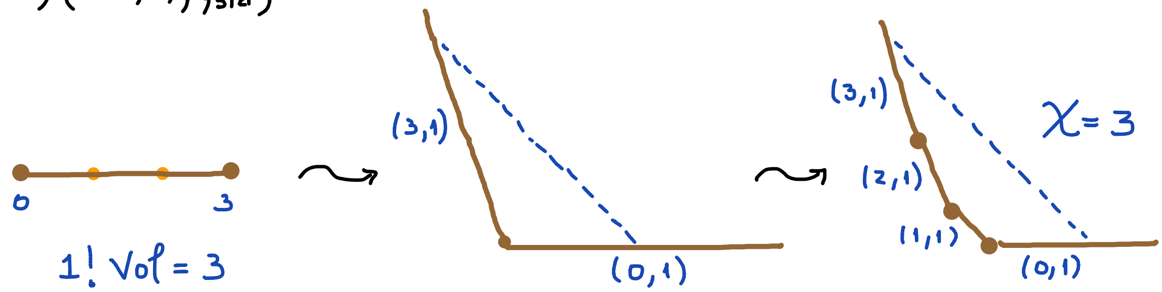
Cor.: $2\chi(M_D, \xi_D)$ = Euler characteristic of any $c_1 = 0$ toric symplectic filling

"Proof": Batyrev – Dais '96:

$n!\, \mathrm{Vol}(D)$ = Euler characteristic of any crepant toric smooth resolution of the Gorenstein toric isolated singularity determined by D. Q.E.D.

Examples :

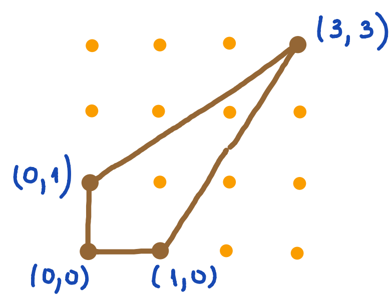
a) $(L(3,2), \xi_{std})$



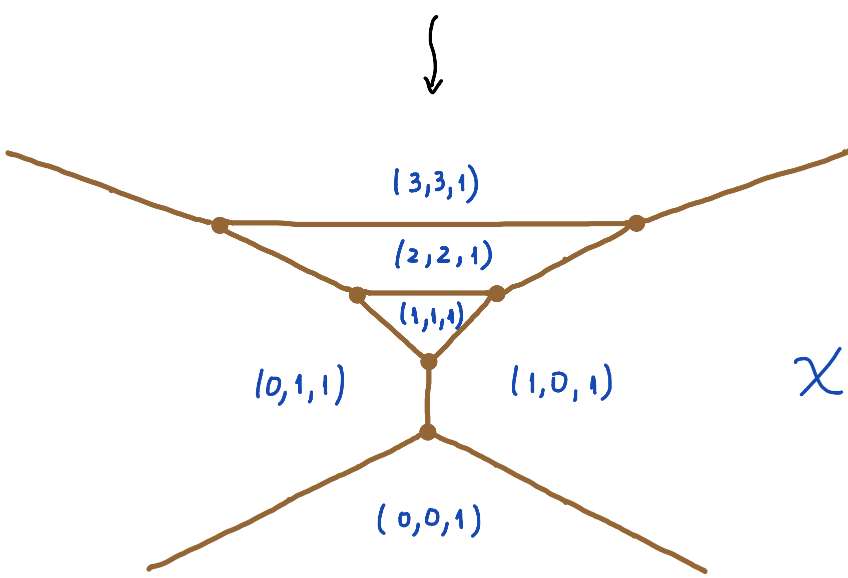
$1!\,\mathrm{Vol} = 3$

$\chi = 3$

b) $(S^2 \times S^3, \xi_3)$



$2!\,\mathrm{Vol} = 6$



$\chi = 6$

(VI) Application 2 [A. – Macarini – Moreira '17]

- Gorenstein Lens spaces L of dim $2n+1$

$$\updownarrow 1:1$$

toric diagrams $D \subset \mathbb{R}^n$ with $n+1$ vertices

$$\updownarrow 1:1$$

links of Gorenstein cyclic quotient isolated singularities

- $\pi_1(L)$ = finite cyclic, say of order p

$$HC_*(L,\xi) = \underbrace{HC_*^1 \oplus \cdots \oplus HC_*^{p-1} \oplus HC_*^0}_{\text{explicitely computable (e.g. using Mathematica)}}$$

- Current outcomes

(i) Inequivalent contact structures on diffeomorphic Lens spaces, e.g.

$L_5^{11}(1,-2,-2,1,1,1)$ & $L_5^{11}(1,-2,2,-1,1,-1)$

HC	0	2	4	6	8	10
γ	0	1	1	1	1	1
γ^2	0	1	1	1	1	1
γ^3	0	0	0	1	1	1
γ^4	0	0	0	1	1	1
$1 \equiv \gamma^5$	0	0	0	0	0	1
Total rank	0	2	2	4	4	5

HC	0	2	4	6	8	10
γ	0	0	1	1	1	1
γ^2	0	0	1	1	1	1
γ^3	0	0	1	1	1	1
γ^4	0	0	1	1	1	1
$1 \equiv \gamma^5$	0	0	0	0	0	1
Total rank	0	0	4	4	4	5

(ii) If $n>1$ is a power of 2, then any contactomorphism of a Gorenstein Lens space L^{2n+1} is trivial on $\pi_1(L)$. This is not necessarily true for diffeomorphisms of L, e.g. inverse homomorphism can always be realized by a diffeomorphism of L. [Hsiang - Jahren]