

*Einstein Metrics,*  
*Symplectic 4-Manifolds, &*  
*Smooth Topology*

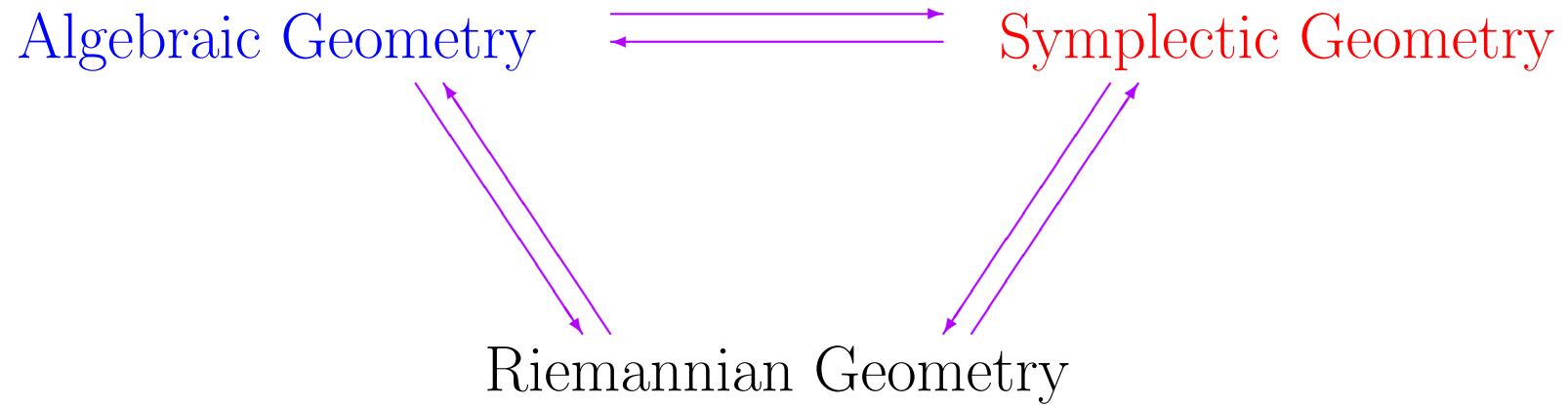
Claude LeBrun  
Stony Brook University

Symplectic and Algebraic Geometry  
University of Warwick, 2 August, 2017

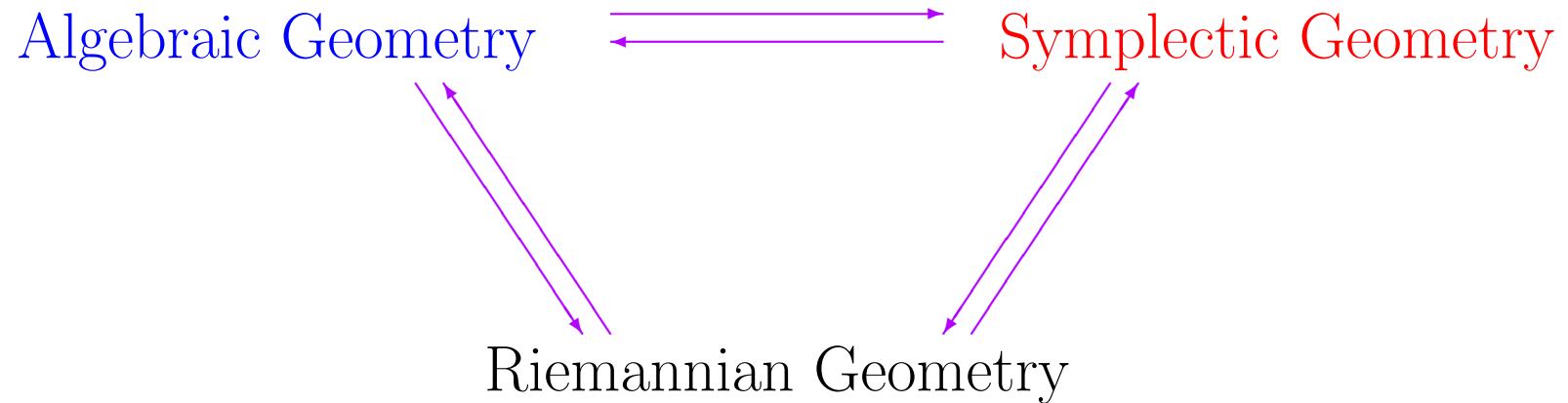
## Theme of this conference:

Algebraic Geometry  $\rightleftarrows$  Symplectic Geometry

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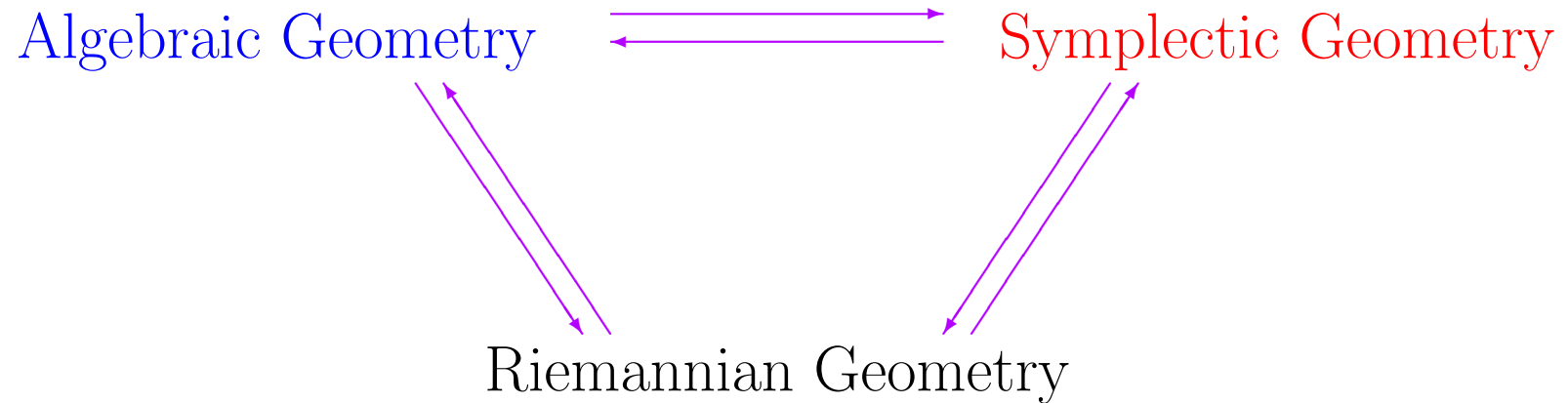


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Higher dimensions are demonstrably different.

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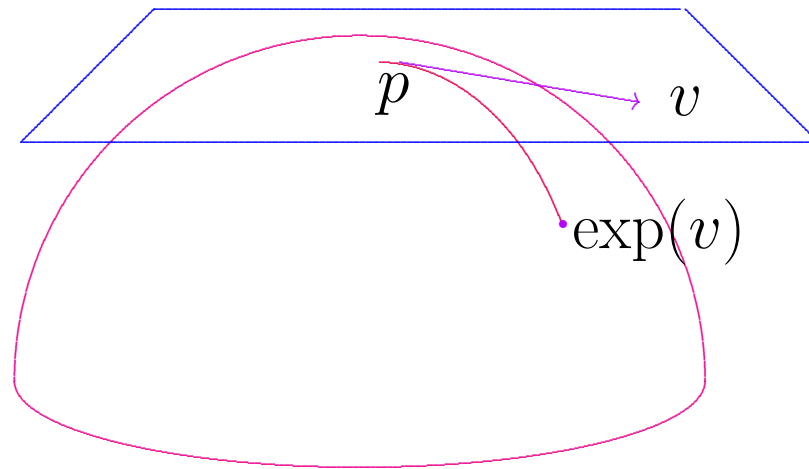
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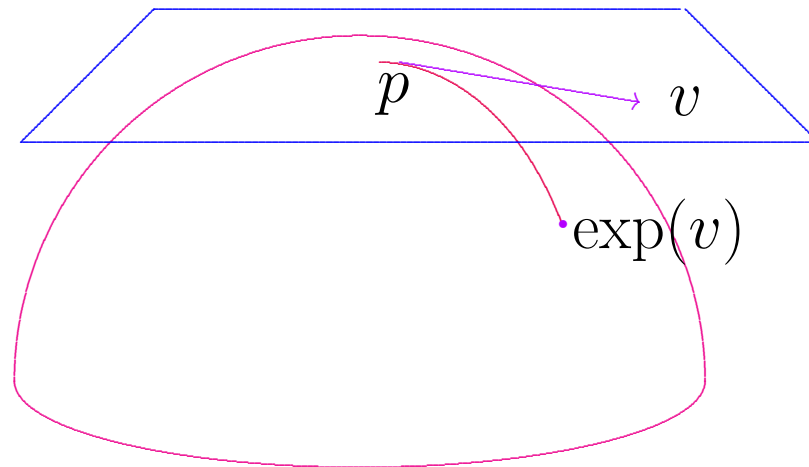




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Now choosing  $T_p M \xrightarrow{\cong} \mathbb{R}^n$  via some orthonormal  
basis gives us special coordinates on  $M$ .

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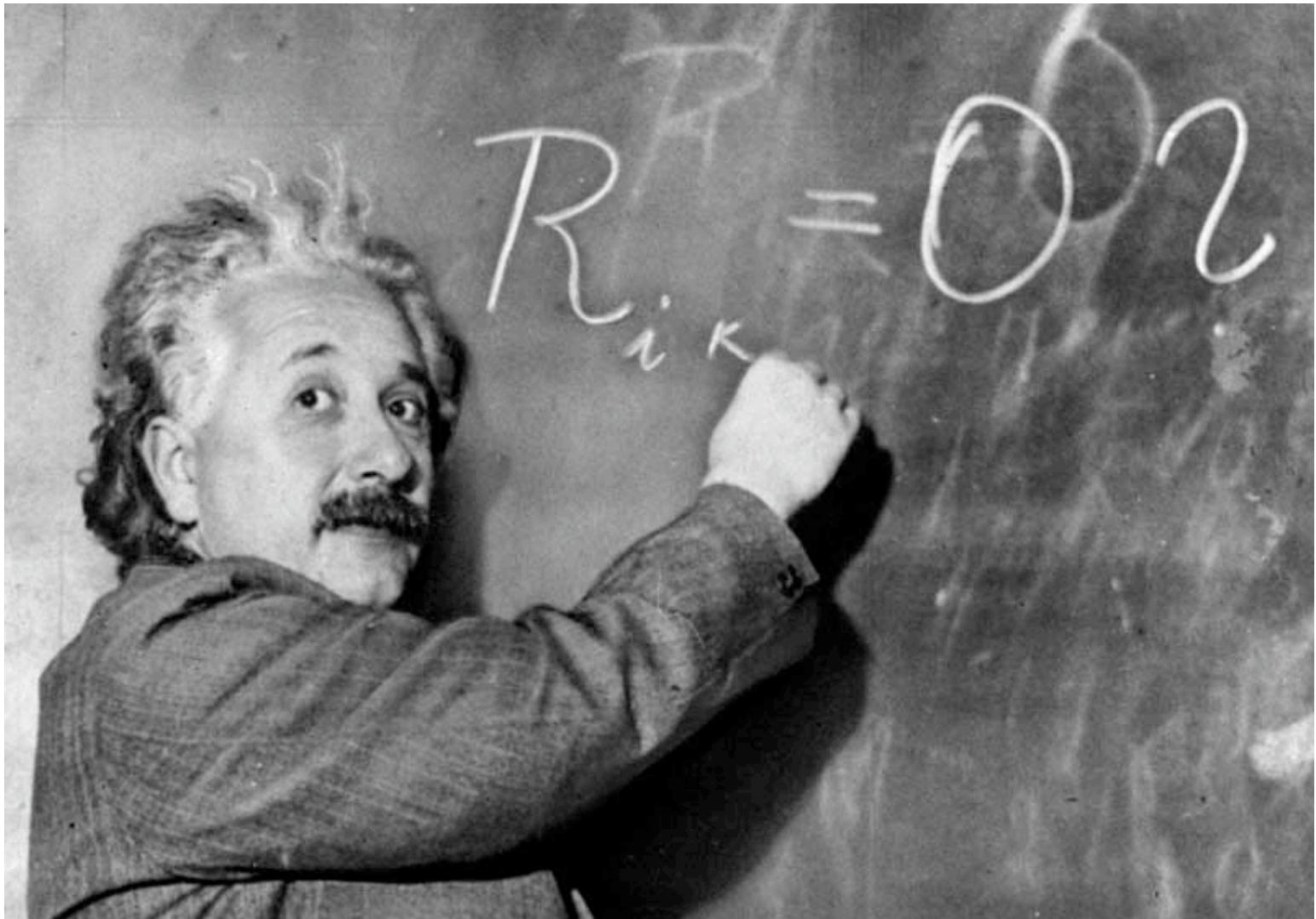
“... the greatest blunder of my life!”

— A. Einstein, to G. Gamow

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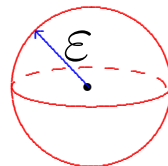
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$$\frac{\text{vol}_g(B_\varepsilon(p))}{c_n \varepsilon^n} = 1 - s \frac{\varepsilon^2}{6(n+2)} + O(\varepsilon^4)$$



## Variational Approach

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where  $V = \text{Vol}(M, g)$  inserted to make scale-invariant.



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Unique up to scale when  $s \leq 0$ .

“Yamabe metrics”

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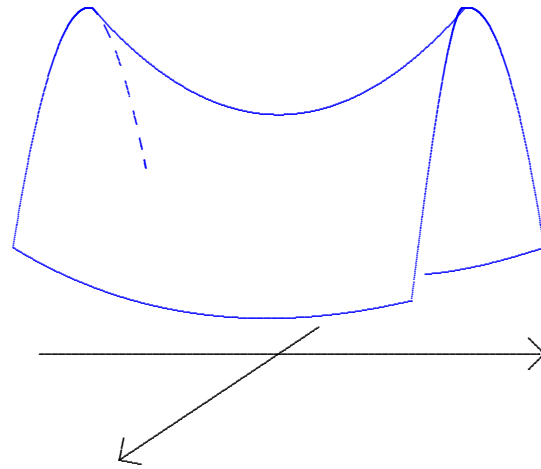
Schoen:

= only for round sphere.

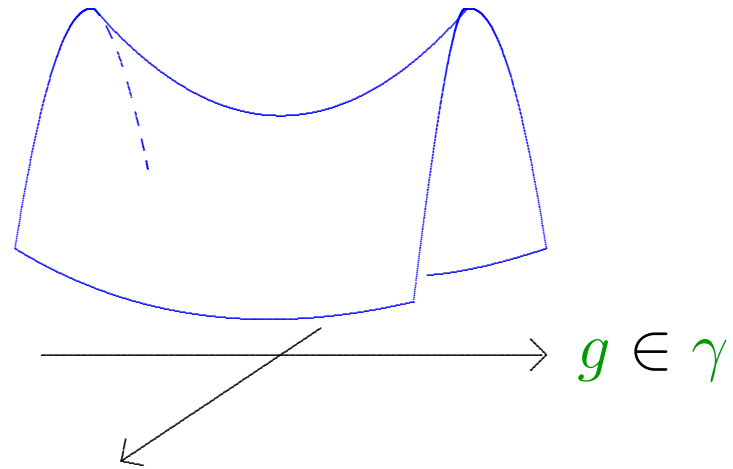
# Yamabe's Dream



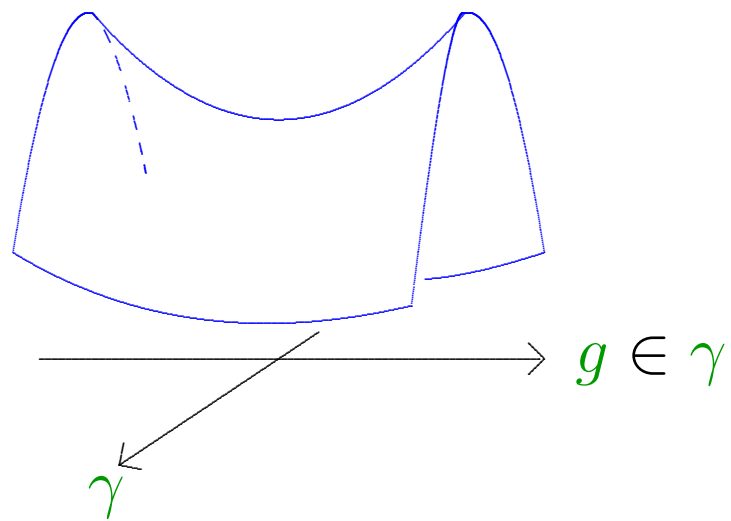
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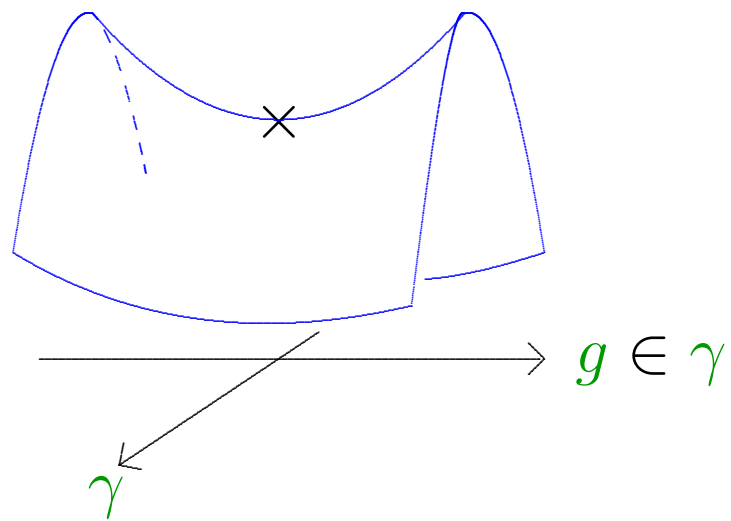
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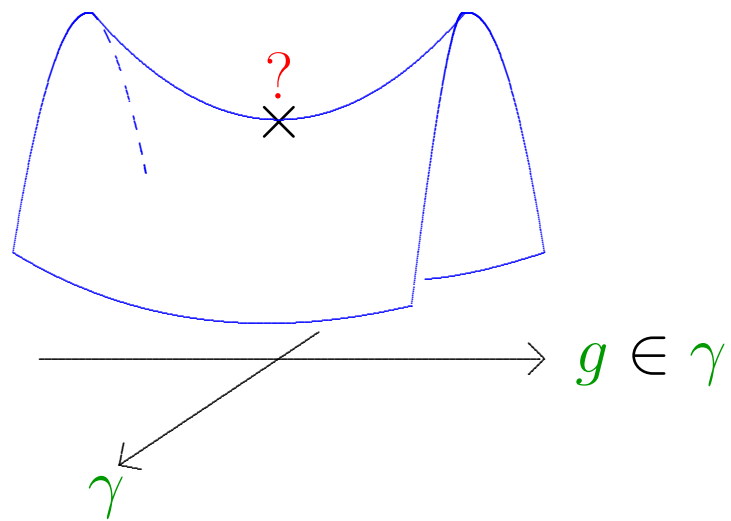
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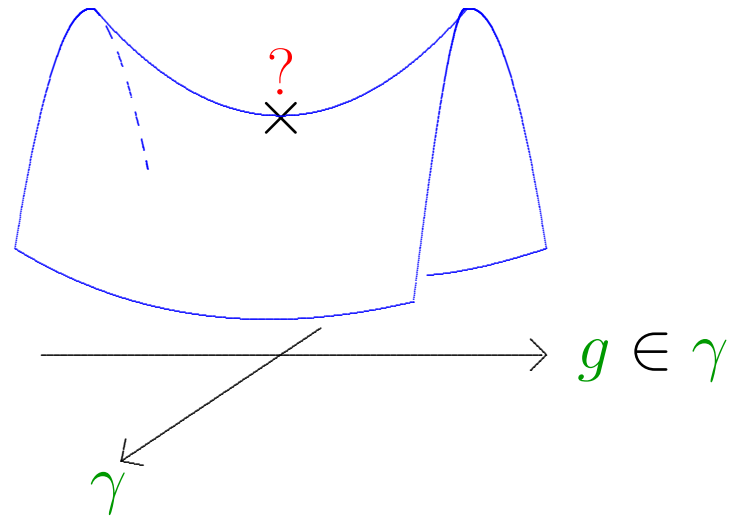
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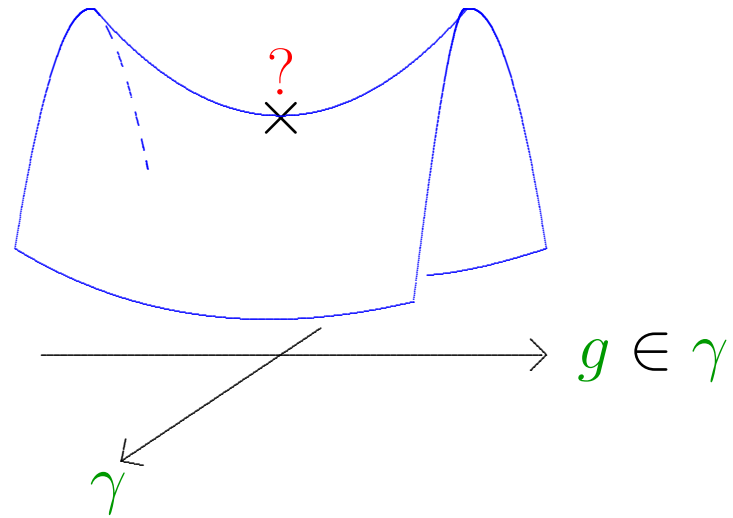


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Too good to be true! But ...

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This is a *diffeomorphism invariant* of  $M$ .

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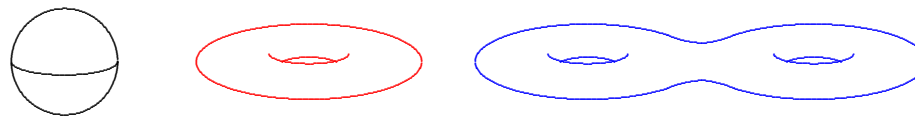
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Reminiscent of the situation for complex curves!



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But if  $\mathcal{Y}(M^4) = 0$ ,  $\exists g$  with  $s \equiv 0 \iff$

$\text{Kod}(M, J) = 0$  and  $M$  minimal.

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By contrast, ...

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Builds on:

**Theorem** (Gromov/Lawson). Let  $M^n$  be a simply connected  $n$ -manifold,  $n \geq 5$ . If  $M$  is *not spin*, then  $M$  carries a metric  $g$  with  $s > 0$ . That is,

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**Perelman**  $\implies$  similarly when  $n = 3$ .

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Convention:

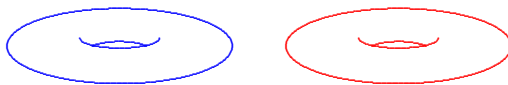
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Connected sum #:



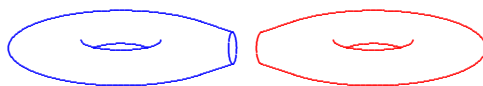


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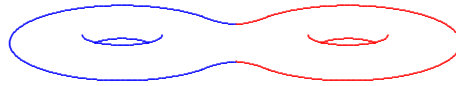


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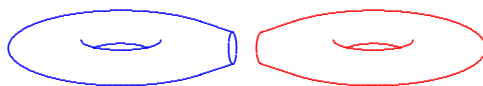


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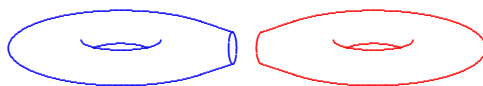


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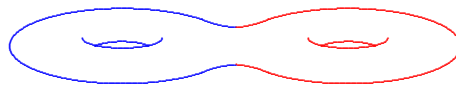


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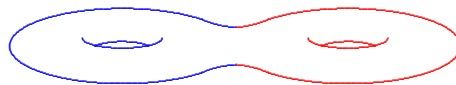


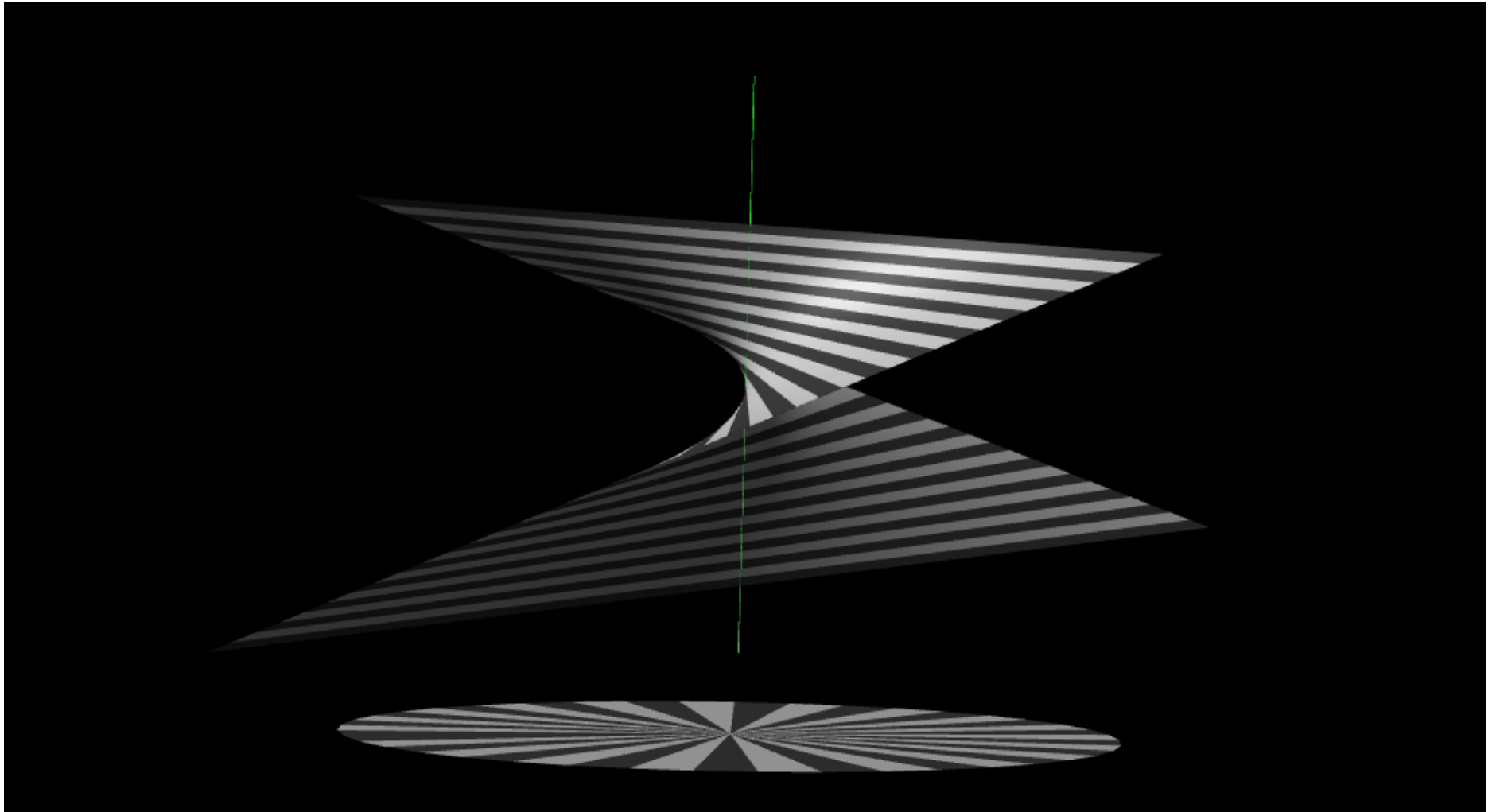
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**Ingredients:** Seiberg-Witten estimates for  $s$ ;  
Kähler-Einstein metrics; gravitational instantons.

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Same definition later rediscovered by [Tian-Jun Li](#), who extended it to define a notion of Kodaira dimension for any symplectic 4-manifold.

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**Şuvaina:** Equality can occur in non-complex case.

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Key case:  $\mathcal{Y}(M) = 0$ .

**Which 4-manifolds admit Einstein metrics?**

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What is the moduli space of such metrics?



# Symplectic 4-manifolds:

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Del Pezzo surfaces,

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**Existence:** Yau, Tian, Page, Chen-L-Weber, et al.

**No others:** Hitchin-Thorpe, Seiberg-Witten, ...

One tool: Hitchin-Thorpe Inequality:

$$(2\chi + 3\tau)(M) = \frac{1}{4\pi^2} \int_M \left( \frac{s^2}{24} + 2|W_+|^2 - \frac{|\mathring{r}|^2}{2} \right) d\mu_g$$

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In these cases, any Einstein metric is Kähler!

What about  $\lambda < 0$ ?

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**Proposition.** *If  $M^4$  admits both a symplectic form  $\omega$  and an (unrelated) Einstein metric  $g$ , it is either on the previous list, or else it is of general type.*

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Key point:  $\text{Kod}(M, \omega) \neq 1$ .

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Key point:  $\text{Kod}(M, \omega) \neq 1$ .

Again, Hitchin-Thorpe:

$c_1^2(M) > 0$  unless locally hyper-Kähler.

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**Theorem** (Aubin/Yau). *Compact complex manifold  $(M^{2m}, J)$  admits compatible Kähler-Einstein metric with  $s < 0 \iff \exists$  holomorphic embedding*

$$j : M \hookrightarrow \mathbb{C}\mathbb{P}_k$$

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**Remark.** When  $m = 2$ , such  $M$  are necessarily **minimal** complex surfaces of **general type**.

**Theorem (L '01).** *Let  $X$  be a minimal symplectic 4-manifold of general type, and let*

$$M = X \#_k \overline{\mathbb{C}P}_2.$$

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So being “very” non-minimal is an obstruction.

**Theorem (Curvature Estimates).** *For any Riemannian metric  $g$  on a symplectic manifold  $M$  with minimal model  $X$ . If  $\text{Kod}(M, \omega) \neq -\infty$ , then  $g$  satisfies the following curvature inequalities:*

$$\int_M s^2 d\mu_g \geq 32\pi^2 c_1^2(X)$$

$$\int_M \left( s - \sqrt{6}|W_+| \right)^2 d\mu_g \geq 72\pi^2 c_1^2(X).$$

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Play off basic classes against each other.



Let us now return to the  $\lambda > 0$  case.

**Theorem** (L '09). Suppose that  $M$  is a smooth compact oriented 4-manifold which admits a symplectic structure  $\omega$ . Then  $M$  also admits an Einstein metric  $g$  with  $\lambda \geq 0$  if and only if

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Definitive list ...

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Moduli space  $\mathcal{E}(M)$  completely understood.

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Above the line:

Know an Einstein metric on each manifold.

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Moduli space  $\mathcal{E}(M) \neq \emptyset$ . But is it connected?

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As Riemannian metrics, they satisfy a curious curvature condition. Namely, if  $\omega$  is a non-trivial self-dual harmonic 2-form, they satisfy

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everywhere on  $M$ . This scalar condition is a conformally invariant analog of the more familiar condition  $s > 0$ .

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on average.

$W_+(\omega, \omega)$  is non-trivially related to scalar curv  $s$ ,  
via Weitzenböck for harmonic self-dual 2-form  $\omega$ :

$$0 = \nabla^* \nabla \omega - 2W^+(\omega, \cdot) + \frac{s}{3} \omega$$

Taking inner product with  $\omega$  and integrating:

$$\int_M W_+(\omega, \omega) d\mu \geq \int_M \frac{s}{6} |\omega|^2 d\mu$$

In particular, an Einstein metric with  $\lambda > 0$  has

$$W_+(\omega, \omega) > 0$$

on average. But we will need this pointwise.

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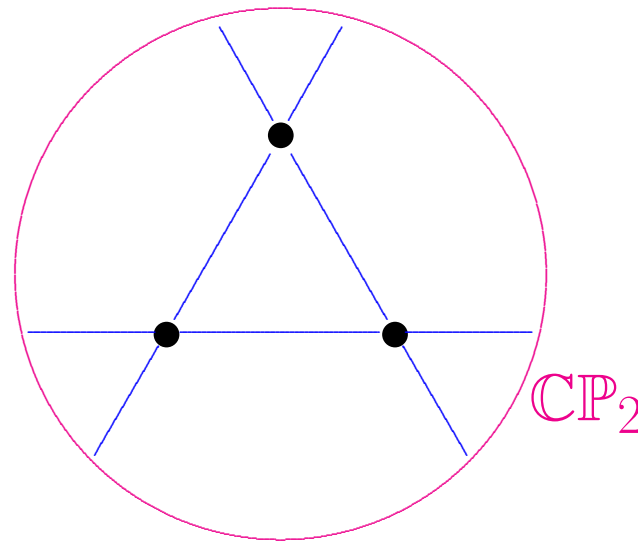
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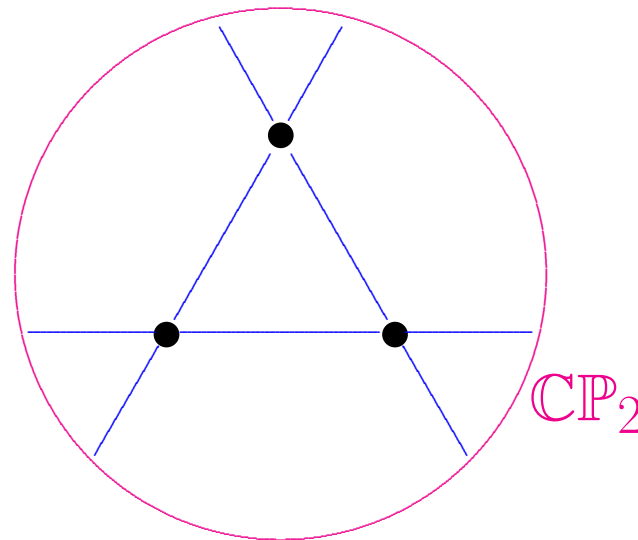
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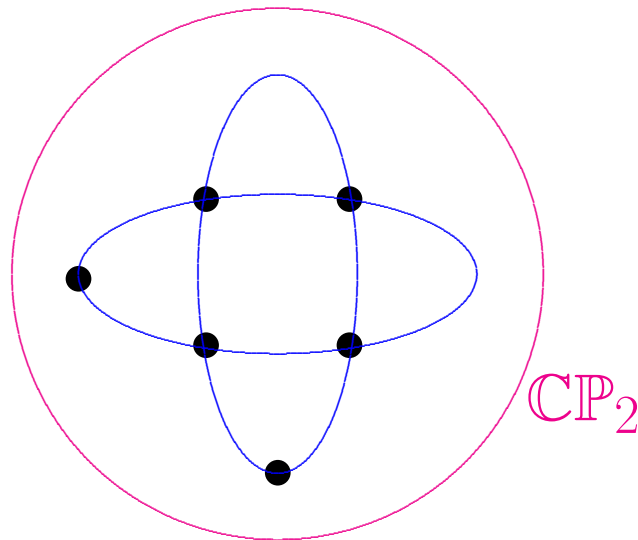


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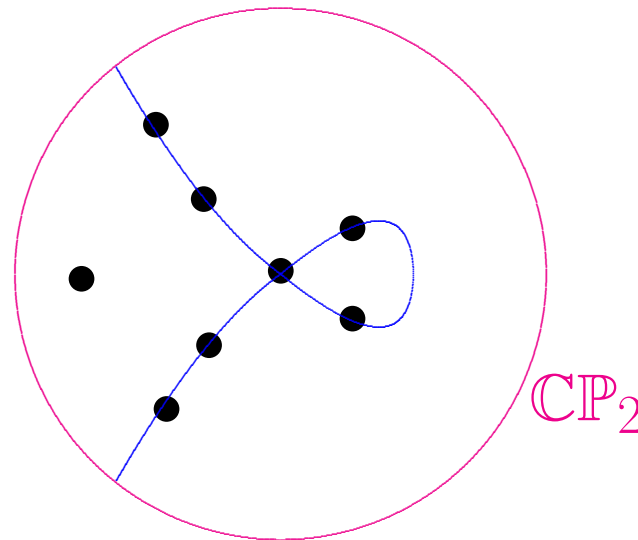


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No 3 on a line, no 6 on conic, no 8 on nodal cubic.

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Existence: Tian, Odaka-Spotti-Sun, Chen-L-Weber...

Uniqueness: Bando-Mabuchi, L '12...

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**Corollary.**

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**Corollary.**  $\mathcal{E}_{\omega}^+(M)$  is exactly one connected component of  $\mathcal{E}(M)$ .

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I hope that some of you will be intrigued enough to want to contribute something to the subject!