Geography of symplectic fillings in dimension 4

Tian-Jun Li

University of Minnesota

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A symplectic manifold is a manifold (M^{2n}, ω) where M is a smooth, oriented manifold, and ω is a closed 2-form such that ω^n is the volume form compatible with the given orientation, called a symplectic form.

Cohomological invariants:
$[\omega] \in H^2(M; \mathbb{R})$ and $c_i(M, \omega) \in H^{2i}(M; \mathbb{Z})$

An almost complex structure J is an automorphism of TM with $J^2 = -id$. J is tamed by ω if $\omega(v, Jv) > 0$ for any nonzero v. The space $\mathcal{J}_\omega$ of $\omega-$tamed J is connected. Thus we can define the symplectic Chern classes:
$c_i(M, \omega) = c_i(TM, J)$ for any $\omega-$tamed J.

$K_\omega = -c_1(M, \omega)$ is called the symplectic canonical class.

Submanifolds—symplectic, Lagrangian, contact

Symplectic submanifolds
Donaldson: If M is closed and the class $[\omega]$ is closed, there are symplectic hypersurfaces Poincare dual to some high multiple of $[\omega]$.
Such a hypersurface is called a Donaldson hypersurface.
A consequence is that any closed symplectic manifold has symplectic submanifolds of arbitrary codimension.

Lagrangian submanifolds

Hypersurfaces of contact type

Contact manifolds

Closed (cooriented) contact $(2n-1)$-manifold (Y, ξ) with contact 1-form α

- $\alpha^{n-1} \wedge d\alpha > 0$ (compatible with chosen orientation of Y)
- $\xi = \ker(\alpha)$ hyperplane distribution

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Example. The standard contact structure on S^3, (S^3, ξ_{std}).
$\alpha_0 = (x_1 dy_1 - y_1 dx_1) + (x_2 dy_2 - y_2 dx_2)$
$\xi = TS^3 \cap J(TS^3)$
plane field of complex tangencies, the $J-$invariant subspace.

Liouville vector field

Symplectic $2n$-manifold (X, ω): $\omega^{2n} > 0$ (compatible with chosen orientation of X)

A vector field V on (X, ω) is called a Liouville vector field if $\mathcal{L}_V \omega = \omega$.

Notice that for a Liouville vector field V, by Cartan's formula, the $1-$form $\beta = \iota_V \omega$ is a primitive of ω, namely, $d\beta = \omega$.

Suppose V is defined near ∂X and transversal to ∂X, then $\beta = \iota_V \omega$ defines a contact $1-$form on ∂X.

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Cohomology invariants:
(ω, β) defines a class in the relative cohomology $H^2(X, \partial X)$.
$K_\omega \in H^2(X)$ may not have a lift in the relative cohomology.

Contact boundary–convex and concave

(X, ω) is a symplectic $2n-$manifold with contact boundary (Y, ξ) if

- there is a transversal Liouville vector field V (ie. $\mathcal{L}_V \omega = \omega$) defined near ∂X
- $(\partial X, \ker(\iota_V(\omega)))$ contactomorphic to (Y, ξ)

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If the Liouville vector field points **outward**, then (X, ω) is said to have convex boundary and is called a convex symplectic manifold.

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For a hypersurface of contact type in a closed manifold, one side is convex, one side is concave.

Conversely, given a pair of convex and concave manifolds with common boundary (Y, ξ), they glue together to a closed manifold.

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Many (Y,ξ) do not admit symplectic fillings. For instance, overtwisted (Y,ξ) are not fillable.
(Y,ξ) is called overtwisted if there is an embedded disk $D\subset Y$ such that $\xi_p = T_pD$ for any $p\in\partial D$.
Every 3-manifold Y admits overtwisted contact structures.

Exact fillings and Stein fillings

An exact filling is a symplectic filling such that ω is exact and there is a primitive restricts to the boundary being the contact one-form. It is equivalent that there is a global outward Liouville vector field.

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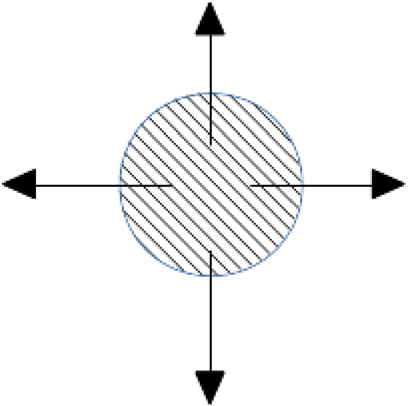
A Stein manifold is a complex manifold (X, J) with a proper function $\phi : W \to [0, \infty)$ such that $dJ(d\phi)$ is a Kähler form. A domain of the form $W = \phi^{-1}([0, t])$ for a regular value t of ϕ is called a Stein domain.
A Stein filling of (Y, ξ) is a Stein domain (W, J, ϕ) which has Y as its boundary and ξ as the set of complex tangencies to Y.
$\nabla\phi$ is a global Liouville field.
Stein fillings are 'holomorphic' exact fillings.

Filling example I

(B^4, ω_{std}) is a symplectic filling of (S^3, ξ_{std}) with radially Liouville vector field pointing outward.



(S^3, ξ_{std}): $\alpha_0 = (x_1 dy_1 - y_1 dx_1) + (x_2 dy_2 - y_2 dx_2)$
$\xi = TS^3 \cap J(TS^3)$
plane field of complex tangencies, the $J-$invariant subspace.
$\omega = dx_i \wedge dy_i$
$V = x_i \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial y_i}$
$\iota_V \omega = \alpha_0$
This filling is Stein (exact).

More Filling examples

Disk cotangent bundle $(D^*\Sigma_g, \omega_{can})$ is a symplectic filling of the unit cotangent bundle $(S^*\Sigma_g, \xi_{can})$ with fiberwise radially outward pointing Liouville vector field.

Locally, for $q_i \in \Sigma_g$ and $(q_i, p_i) \in D^*\Sigma_g$

- $\omega_{can} = dp_i \wedge dq_i$
- $\alpha_{can} = p_i \wedge dq_i$
- $V = p_i \partial_{p_i}$

Question

Question: Can one classify symplectic fillings/cappings of a contact manifold (Y, ξ) ?

- Up to homotopy type? homeomorphism? diffeomorphism? symplectic deformation equivalence?
- Finitely many? Infinitely many?

$$\text{Stein fillings} \subset \text{exact fillings} \subset \text{symplectic fillings}$$

Geography

Ozbagci-Stipsicz, Smith: Some (Y, ξ) admits infinitely many symplectic (even Stein) fillings.

Baykur and Van Horn-Morris: There are infinite families of contact 3-manifolds, where each contact 3-manifold admits a Stein filling whose Euler characteristic is larger and signature is smaller than any two given numbers.
For a general contact 3-manifold, the Geography needs to be understood first.

Theorem (L-Mak)

For any contact 3-manifold (Y, ξ), the set of integers

$$\{2\chi(N) + 3\sigma(N) \in \mathbb{Z} | (N, \omega) \text{ a minimal symplectic filling of } (Y, \xi)\}$$

is bounded from below. Moreover, the lower bound can be explicitly calculated given a maximal symplectic cap.

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This is proved by constructing maximal symplectic caps.
The case of Stein fillings was established by Stipsicz (2002).

Given a contact manifold (Y, ξ), a concave manifold with (Y, ξ) as boundary is called a symplectic cap of (Y, ξ).
Symplectic caps and symplectic fillings of (Y, ξ) glue to closed symplectic manifolds.

Eynyre-Honda: Symplectic caps always exist.

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Identify/construct various types of caps, motivated by the theory of closed symplectic 4-manifolds, to constrain (the geography of) symplectic fillings:
• Maximal caps ($K_\omega \cdot K_\omega \geq 0$ for $\kappa^s \geq 0$ symplectic 4-manifolds)
• Uniruled caps (Smooth classification of symplectic uniruled 4-manifolds)
• Calabi-Yau caps (Homological classification of symplectic Calabi-Yau surfaces)

Minimality in dimension 4

Let M be a closed, oriented smooth 4-manifold.
Let $\mathcal{E}_M$ be the set of cohomology classes whose Poincaré dual are represented by smoothly embedded spheres of self-intersection -1.
M is said to be (smoothly) minimal if $\mathcal{E}_M$ is the empty set.
Equivalently, M is minimal if it is not the connected sum of another manifold with $\overline{\mathbb{CP}^2}$.

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Suppose ω is a symplectic form compatible with the orientation.
(M, ω) is said to be (symplectically) minimal if $\mathcal{E}_\omega$ is empty, where

$$\mathcal{E}_\omega = \{E \in \mathcal{E}_M | \ E \text{ is represented by an embedded } \omega\text{−symplectic sphere}\}.$$

We say that (N, τ) is a minimal model of (M, ω) if (N, τ) is minimal and (M, ω) is a symplectic blow up of (N, σ).

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A basic fact proved using SW theory is: $\mathcal{E}_\omega$ is empty if and only if $\mathcal{E}_M$ is empty. In other words, (M, ω) is symplectically minimal if and only if M is smoothly minimal.

Kodaira dimension type invariants

Roughly speaking, a Kodaira dimension type invariant on a class of $n-$dimensional manifolds
is a numerical invariant taking values in the finite set

$$\{-\infty, 0, 1, \cdots, \lfloor \frac{n}{2} \rfloor\},$$

where $\lfloor x \rfloor$ is the largest integer bounded by x.

Holomorphic Kodaira dimension κ^h

Let us first recall the original Kodaira dimension in complex geometry.

Definition

Suppose (M, J) is a complex manifold of real dimension $2m$. The holomorphic Kodaira dimension $\kappa^h(M, J)$ is defined as follows:

$$\kappa^h(M,J) = \begin{cases} -\infty & \text{if } P_l(M,J) = 0 \text{ for all } l \geq 1, \\ 0 & \text{if } P_l(M,J) \in \{0,1\}, \text{ but } \not\equiv 0 \text{ for all } l \geq 1, \\ k & \text{if } P_l(M,J) \sim c l^k;\ c > 0. \end{cases}$$

Here $P_l(M, J)$ is the $l-$th plurigenus of the complex manifold (M, J) defined by $P_l(M, J) = h^0(\mathcal{K}_J^{\otimes l})$, with $\mathcal{K}_J$ the canonical bundle of (M, J).

Definition of κ^s for minimal (M, ω)

For a minimal symplectic 4−manifold (M^4, ω) with symplectic canonical class K_ω, the Kodaira dimension of (M^4, ω) is defined in the following way:

$$\kappa^s(M^4, \omega) = \begin{cases} -\infty & \text{if } K_\omega \cdot [\omega] < 0 \text{ or } & K_\omega \cdot K_\omega < 0, \\ 0 & \text{if } K_\omega \cdot [\omega] = 0 \text{ and } & K_\omega \cdot K_\omega = 0, \\ 1 & \text{if } K_\omega \cdot [\omega] > 0 \text{ and } & K_\omega \cdot K_\omega = 0, \\ 2 & \text{if } K_\omega \cdot [\omega] > 0 \text{ and } & K_\omega \cdot K_\omega > 0. \end{cases}$$

Here K_ω is defined as the first Chern class of the cotangent bundle for any almost complex structure compatible with ω.

κ^s well defined via Taubes symplectic SW theory

κ^s is well defined since there doesn't exist minimal (M, ω) with

$$K_\omega \cdot [\omega] = 0, \quad K_\omega \cdot K_\omega > 0.$$

Properties:

- κ^s is independent of ω, so it is an oriented diffeomorphism invariant of M.
- Liu: $\kappa^s(M) = -\infty$ if and only if M is $\mathbb{CP}^2, S^2 \times S^2$ or an S^2-bundle over a Riemann surface of positive genus.

General (M, ω)

The Kodaira dimension of a non-minimal manifold is defined to be that of any of its minimal models. $\kappa^s(M, \omega)$ is defined for any (M, ω) since

- Minimal models always exist
- Minimal model almost unique up to diffeomorphisms. If (M, ω) has non-diffeomorphic minimal models, then these minimal models have $\kappa^s = -\infty$.
- Diffeomorphic minimal models have the same κ^s.

Basic property:

- κ^s is an oriented diffeomorphism invariant of M.
- Dorfmeister+Zhang: $\kappa^s = \kappa^h$ whenever both are defined, eg. the Kodaira-Thurston manifolds.
- $\kappa^s = 2$ manifolds are the symplectic 4-manifolds of general type introduced by LeBrun.
  Question (LeBrun): Yamabe invariant of M is negative equivalent to M general type?

Definition

Let (X, ω) be a closed symplectic four manifold and D be a (connected) smooth symplectic surface in X. Then D is called **maximal** if any symplectic exceptional class in (X, ω) pairs positively with $[D]$.

L-Zhang: There is a notion of relative Kod dimension for a maximal surface F with positive genus, by replacing K_ω by $K_\omega + [F]$. It is analogous to the Log Kod dim.

Constraints on the adjoint class

Lemma

Suppose F is maximal and has positive genus
If $\kappa^s(M, \omega) \geq 0$, *then*

$$(K_\omega + [F]) \cdot [\omega] > 0, \quad (K_\omega + [F])^2 \geq 0.$$

If $\kappa^s(M, \omega) = -\infty$ *and* $(K_\omega + [F])^2 > 0$, *then* $(K_\omega + [F]) \cdot [\omega] > 0$.

Constraints on the adjoint class

Lemma

Suppose F is maximal and has positive genus
If $\kappa^s(M,\omega)\geq 0$, then

$$(K_\omega+[F])\cdot[\omega]>0,\quad (K_\omega+[F])^2\geq 0.$$

If $\kappa^s(M,\omega)=-\infty$ and $(K_\omega+[F])^2>0$, then $(K_\omega+[F])\cdot[\omega]>0$.

When $\kappa^s(M,\omega)=-\infty$, as $b^+(M)=1$ in this case, the statement follows from the light cone lemma and $[\omega]^2\geq 0$.
Consequently, $\kappa(M,\omega,F)$ is well defined since it is impossible to have

$$(K_\omega+[F])\cdot[\omega]=0 \quad \text{and} \quad (K_\omega+[F])^2>0.$$

Proposition

$\kappa(M, \omega, F) \geq \kappa(M, \omega)$.
If F is not empty, then $\kappa(M, \omega, F) = -\infty$ *if and only if M is an* S^2-*bundle and F is a section.*
$\kappa(M, \omega, F) = 0$ *if and only if* $\kappa(M) = -\infty$ *and F is an anti-canonical surface.*

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A maximal symplectic surface gives a explicit lower bound $K_\omega\cdot K_\omega$.
Suppose F is a maximal surface of genus $g\geq 1$. Then

$$K_\omega\cdot K_\omega\geq\begin{cases}-K_\omega\cdot[F] & \text{if } \kappa(M)\geq 0\\ -K_\omega\cdot[F]+(2-2g) & \text{if } \kappa(M)=-\infty \text{ and } \kappa(M,\omega,F)\geq 0\\ 8-8g & \text{if } \kappa(M,\omega,F)=-\infty\end{cases}$$

Maximal cap

Definition

Let (P, ω_P) be a concave symplectic manifold and D be a smooth symplectic surface in P. Then D is called **maximal** if, for any minimal symplectic filling (N, ω_N) of ∂P, D is maximal in $(N \cup_{\partial P} P, \omega)$.
A cap is called maximal if it admits a maximal surface.

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Proposition

Let (P, ω) be a maximal cap with D as a maximal symplectic surface with genus $g > 0$. Then there is a lower bound on $(2\chi + 3\sigma)(N)$ of any minimal strong symplectic filling N of $Y = \partial P$ given by

$$(2\chi + 3\sigma)(N) \geq \min\{c_1(P) \cdot D + 2 - 2g, \quad 8 - 8g\} - (2\chi + 3\sigma)(P)$$

Donaldson hypersurface

The primary sources of maximal caps are Donaldson caps.

Definition

Let (P, ω_P, α_P) be a concave symplectic pair with rational period. A **closed** symplectic hypersurface D is called a Donaldson hypersurface of (P, ω_P, α_P) if it is Lefschetz dual to an integral multiple of $\frac{1}{2\pi}[(\omega_P, \alpha_P)]$. We will often just say that D is a Donaldson hypersurface of (P, ω_P).
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A cap with a Donaldson hypersurface is called a Donaldson cap.

A concave symplectic pair is (P, ω_P, α_P) where (P, ω_P) is a concave symplectic manifold and α_P is a contact one form on ∂P induced by an inward pointing Liouville vector field. Such a pair is called of rational period if $\frac{1}{2\pi}[(\omega_P, \alpha_P)] \in H^2(P, \partial P; \mathbb{Q})$.

Donaldson cap

Question

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Observation: Any contact 3-manifold admits a Donaldson cap. Furthermore we can assume the Donaldson cap to have arbitrarily large b^+.

Sketch of proof

By Etnyre-Honda, there exists a Stein fillable contact 3-manifold (Y_2, ξ_2) such that (Y, ξ) is exact (Stein) cobordant to (Y_2, ξ_2). Denote the exact cobordism by (SC, τ).

Let (N, ω_N) be a Stein filling of (Y_2, ξ_2). By Lisca-Matic, (N, ω_N) embeds into a minimal surface X of general type. In fact, inspecting their argument, we see that there is an affine surface A in X such that $N \subset A$ and X is the projective compactification of A. The divisor $D := X \backslash A$ is ample, and by Hironaka's resolution of singularities we can assume that it is a simple normal crossing divisor.

In particular, we can smooth D out to a smooth symplectic Donaldson hypersurface in $P := X \backslash N$. By gluing P with (SC, τ), we get a Donaldson cap of (Y, ξ).

Lemma

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Lemma

Let (P, ω_P, α_P) be a concave symplectic pair and (N, ω_N) a symplectic filling of (Y, ξ). Then any symplectic exceptional class in $(N \cup_Y P, \omega)$ which admits no embedded symplectic representative in (N, ω_N) pairs positively with $PD[(\omega_P, \alpha_P)]$

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Corollary

Any contact 3-manifold admits a maximal cap, which can be assumed to have arbitrarily large b^+.

The bound of $2\chi + 3\sigma$ for arbitrary (Y, ξ)

Corollary

For any contact 3-manifold (Y, ξ), the set of integers

$$\{2\chi(N) + 3\sigma(N) \in \mathbb{Z} | (N, \omega) \text{ a minimal symplectic filling of } (Y, \xi)\}$$

is bounded from below. Moreover, the lower bound can be explicitly calculated given a maximal cap.

'Minimal' maximal cap

Corollary

For any co-oriented contact 3-manifold (Y, ξ), there exists a symplectic cap (P, ω_P) of (Y, ξ) such that for any minimal strong symplectic filling (N, ω_N) of (Y, ξ), the glued symplectic manifold $(N \cup P, \omega)$ is minimal. In particular, any minimal convex symplectic 4-manifold embeds into a minimal closed symplectic 4-manifold.

Proof: Let (P, ω_P) be a maximal cap of (Y, ξ) with a genus $g \geq 1$ maximal surface D. Denote the self-intersection number of D as s. We consider a symplectic four tours (T^4, ω) with product symplectic form. One can easily find a symplectic surface D' of genus g in (T^4, ω). By adjunction, $[D']^2 \geq 0$.

We can perform $[D']^2 + s$ symplectic blow-ups along D' to get a symplectic surface D" of genus g and self-intersection $-s$ in (X', ω'). Notice that D" is maximal in (X', ω').

We now perform Gompf's symplectic sum surgery between (P, ω_P) and (X', ω') along D and D", which results in another symplectic cap (P', ω'_P) of (Y, ξ).

Now, for any minimal symplectic filling (N, ω_N) of (Y, ξ), the glued symplectic manifold $N \cup P'$ can also be obtained as performing symplectic sum surgery between $N \cup P$ and X'.

Since D and D" are maximal in $N \cup P$ and X', respectively. The minimality theorem of Usher implies that $N \cup P'$ is minimal.

Uniruled/Calabi-Yau concave manifolds

For a concave manifold (P, ω_P) with contact boundary $(\partial P, \ker(\alpha_P))$, we say that (P, ω_P, α_P) is

- Uniruled if $c_1(P, \omega_P) \cdot [(\omega_P, \alpha_P)] > 0$
- CY if $c_1(P, \omega_P)$ is torsion

• If a Uniruled concave manifold embed in a closed manifold, then the closed manifold has $\kappa^s = -\infty$
• If a Calabi-Yau concave manifold embeds in a closed manifold with exact complement, then the closed manifold has $\kappa^s = -\infty$ or 0

Examples

- Any planar contact 3-manifold admits a uniruled cap but not vice-versa [This is the main class of contact 3-manifolds that good obstructions (homological) to fillings are known]
- all known contact manifolds that admit finitely many filling up to diffeomorphism admit uniruled caps (eg. includes S^*S^2 and S^*T^2)
- Ohta-Ono: Some singularities admit CY caps

Rigidity from uniruled caps

Theorem (L-Mak-Yasui)

If (Y, ξ) admits a uniruled cap P, then there are uniform bounds (only depends on P) on the Betti numbers of all the minimal strong fillings of (Y, ξ).

The uniform bounds can be made explicit for a large class of contact manifolds and recover several known results from the literature, for example,
Wand: For a planar contact manifold (Y, ξ), $e(N) + \sigma(N)$ is constant for any minimal strong filling N of (Y, ξ).

Rigidity from CY caps

Theorem (L-Mak-Yasui)

If (Y, ξ) admits a Calabi-Yau cap P, then there are uniform bounds (only depends on P) on the Betti numbers of all the **exact** *fillings of (Y, ξ).*

Recall: an exact filling is a symplectic filling such that ω is exact and there is a primitive restricting to the boundary being the contact one-form.

This result relies on the homology classification of $\kappa = 0$ manifolds.

$\kappa = 0 - -$Betti number bounds

Theorem (L, Bauer)

$b^+(M) \leq 3$ *if* $\kappa(M, \omega) = 0$

Consequently,

- $b_1(M) \leq 4$
- Euler number ≥ 0
- Symplectic Noether type inequality

$$b^+ \leq 3 + |comp(K_\omega)|$$

  holds when $\kappa = 0$
- $vb_1(M) \leq 4$, where $vb_1(M)$ is the supremum of $b_1(\tilde{M})$ among all finite covers $\tilde{M}$.

$\kappa = 0 - -$Homology types

- If M is minimal, then it has the same $\mathbb{Q}-$cohomology ring as K3, Enriques surface or a T^2-bundle over T^2

In fact, $\mathbb{Z}-$homology K3, $\mathbb{Z}-$homology Enriques

The following table list possible homological invariants of $\kappa = 0$ manifolds:

b_1	b_2	b^+	χ	σ	known manifolds
0	22	3	24	-16	K3
0	10	1	12	-8	Enriques surface
4	6	3	0	0	4-torus
3	4	2	0	0	T^2-bundles over T^2
2	2	1	0	0	T^2-bundles over T^2

Smith-Thomas-Yau: simply connected non-Kahler CY 3-fold
Fine-Panov: flexible in higher dimension

Sketch of proof

(N, ω_N),filling; (P, ω_P), uniruled/CY cap;
(X, ω_X),glued closed symplectic manifold

1. $c_1(X) \cdot [\omega_X] = c_1(N) \cdot [(\omega_N, \alpha_N)] + c_1(P) \cdot [(\omega_P, \alpha_P)]$
2. when we do the gluing, there is a choice to shrink $c_1(N) \cdot [(\omega_N, \alpha_N)]$ to be arbitrarily small; blow-down increases $c_1 \cdot [\omega]$
3. Hence X is uniruled if P is. X is minimal SCY or non-minimal uniruled when P is CY and N is exact
4. If X is minimal SCY, the the bounds follow from the homology classification of SCY.
5. If X is uniruled, bound base genus from P [explained in the following slides]
6. bound the number of exceptional spheres from P [explained in the following slides]

Bounding base genus

1. $[(\omega_P, \alpha_P)]^2 = \int_P \omega_P^2 - \int_{\partial P} \omega_P \wedge \alpha_P > 0$ because Stoke's orientation is different from contact orientation!
2. there is a closed surface $\Sigma \subset P$ with $[\Sigma]^2 > 0$
3. the base genus of X is bounded by the genus of Σ and hence this bound only depends on P

Bounding exceptional spheres

1. By slightly perturbing $[(\omega_P, \alpha_P)]$, we can assume Σ is Lefschetz dual to $c[(\omega_P, \alpha_P)]$ for some $c > 0$
2. we can also assume $[\Sigma]^2 \geq g(\Sigma) - 1$ by taking a even larger c
3. any exceptional spheres in X has non-zero GW invariants
4. neck-stretch along Y
5. in neck-stretch limit, we have $[u_\infty] \cdot [(\omega_P, \alpha_P)] = \int_{\Sigma_{u_\infty}} u_\infty^* \omega_P - \int_{\partial \Sigma_{u_\infty}} u_\infty^* \alpha_P$ because u_∞ asymptotic to Reeb orbits along ∂P, where $[u_\infty]$ is the relative homology of top building.

Hence, any excpetional curves pairs $[\Sigma]$ positively in X.

Bounding exceptional spheres II

1. Based on SW theory, BH Li-L showed $[\Sigma]^2 \geq g(\Sigma) - 1$ implies that $[\Sigma]$ has a closed embedded symplectic representative Σ_{symp} in X.
2. Since any exceptional curve intersect Σ_{symp}, Σ_{symp} is called maximal
3. In this case, a result of L-Zhang reads $(-c_1(X, \omega_X) + [\Sigma_{symp}])^2 > 0$
4. By adjunction, the genus and self-intersection of Σ_{symp} gives a lower bound on $c_1(X, \omega_X)^2$
5. since symplectic representative minimize genus in this case, the genus and self-intersection of Σ gives a lower bound on $c_1(X, \omega_X)^2$
6. lower bound of $c_1(X, \omega_X)^2$ together with base genus bound on (X, ω_X) bounds the number of exceptional curves.

Contact Kod dim

Since every contact 3-manifold admits a symplectic cap, we can introduce the Kodaira dimension for any contact 3-manifold (Y, ξ) as follows.

$$Kod(Y, \xi) = \begin{cases} -\infty & \text{if it admits a uniruled cap} \\ 0 & \text{if it admits a CY cap but no uniruled cap} \\ 1 & \text{if it does not admit CY cap or uniruled cap} \end{cases}$$

A comprehensive geography picture for various fillings.
Arbitrary (Y, ξ): a lower bound on $2\chi + 3\sigma$ for symplectic fillings.
(Y, ξ) with $Kod = 0$: bounds on the Betti numbers for exact fillings.
(Y, ξ) with $Kod = -\infty$: bounds on the Betti numbers for symplectic fillings.

Cotangent bundles

Conjecture

Any exact symplectic filling of $S^*\Sigma_g$ is diffeomorphic to $D^*\Sigma_g$.

The conjecture true for $g = 0, 1$ by McDuff and Wendl.

Theorem (L-Mak-Yasui)

Any exact symplectic filling of $S^*\Sigma_g$ has the same integral homology and intersection form as $D^*\Sigma_g$. Moreover, it has vanishing first Chern class.

VH Morris-Sivek: all Stein fillings are simple homotopic to $D^*\Sigma_g$.

Sketch of proof

1. there is a K3 admiting a fibration with Lagrangian torus fibers and a (-2)-Lagranian section
2. By resolving a 'comb configuration', we have Lagrangian Σ_g
3. Complement of Weinstein nhbd is a CY cap P
4. Intersection form of P is given by $-2E_8 \oplus 2H \oplus (2-2g) \oplus (0)^{2g}$
5. after gluing any exact filling N, we get an integral homology $K3$
6. by homology LES and intersection form argument, $H_2(N) = \mathbb{Z}$ and the intersection form of N is $(k^2(2g-2))$ for some k
7. by more homological argument, $k = 1$ and $H_1(N) = \mathbb{Z}^{2g}$

Proposition

A convex symplectic 4-manifold (N, ω_N) *is symplectically minimal if and only if it is smoothly minimal.*

Corollary

Let (N, ω_N) *be a convex symplectic manifold. If there is a smooth* -1 *sphere in* N*, there is a symplectic* -1 *sphere homologous to it up to sign. Moreover, the classes of symplectic* -1 *spheres are pairwise orthogonal.*

Unknown whether the Proposition is true for concave symplectic 4-manifolds.

Removing a ball in a rational 4-manifold with more than two blow-ups gives a counterexample of the Corollary for concave symplectic 4-manifolds.

Symplectic cobordisms

Using the corollary we obtain a restriction for exact self cobordisms of fillable contact 3-manifolds.

Corollary

Suppose (Y, ξ) *is a strongly fillable contact 3-manifold. Then the set*

$$\{2\chi(W) + 3\sigma(W) \in \mathbb{Z} | (W, \omega) \text{ is a self exact cobordism of } (Y, \xi)\}$$

is bounded below by 0*. In particular, if it is also bounded above, then the set is* $\{0\}$.

Future programs on concave manifolds

In contrast, concave symplectic manifolds receive relatively little attention. We propose that they deserve more serious study.
Observation : Concave symplectic manifolds seem to resemble closed symplectic manifolds.
Concave symplectic 4-manifold with (S^3, ξ_{std}) canonically corresponds to closed symplectic 4-manifolds

- Gromov-Witten invariants, Seiberg-Witten invariants
- Donaldson hypersurfaces should exist in (most) concave symplectic 4-manifolds.
- Kodaira dim
- At least, systematic investigation of concave manifolds lead to deeper understanding of symplectic fillings.